A study of stability of reconstruction schemes for hyperbolic PDEs.

1 Introduction

This paper deals with the scalar initial value problem of first order in dimension 1 in space

$$\partial_t u(t,x) + \partial_x f(u)(t,x) = 0, \quad t \in \mathbb{R}^{+*}, x \in \mathbb{R}, \tag{1}$$

$$u(0,x) = u^0(x) \in L^\infty(\mathbb{R}), \tag{2}$$

where $f \in \mathcal{C}^1(\mathbb{R})$. Considering weak solutions of (1,2) allows multiple solutions. Therefore we restrict our study to the *entropy* solution of this problem, that is to say to a weak solution that satisfies the additional partial differential inequations

$$\partial_t S(u)(t,x) + \partial_x G(u)(t,x) \leq 0 \quad t \in \mathbb{R}^{+*}, x \in \mathbb{R}, \tag{3}$$

for every entropy-entropy flux pair (S,G), i.e. every pair of $C^1(\mathbb{R})$ functions (S,G) such that S is convex and $G' = S'f'$. It is known that there exists a unique entropy weak solution to (1): cf. [14, 7] for example, this solution belonging to $L^\infty((0,T)\times\mathbb{R})$ $\forall T \in \mathbb{R}^+$ and verifying furthermore $u(t,\cdot) \in BV(\mathbb{R})$ $\forall t \in \mathbb{R}^+$ if $u^0 \in BV(\mathbb{R})$. Let us recall that u is the entropy solution to (1,2) if and only if

$$\partial_t S_k(u)(t,x) + \partial_x G_k(u)(t,x) \leq 0 \quad \forall k \in \mathbb{R} \tag{4}$$

with $S_k(u) = |u-k|$ and $G_k(u) = \operatorname{sgn}(u-k)(f(u)-f(k))$.

We are here concerned with the numerical approximation of these entropy solutions in the standard framework of finite volume schemes.

Let $\Delta x \in \mathbb{R}$ and $\Delta t \in \mathbb{R}$ be given positive numbers, denoting respectively the space and time steps. We replace equation (1) by the discrete in time and space equation

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(f_{j+1/2}^n - f_{j-1/2}^n) \quad \forall n \in \mathbb{N}, \forall j \in \mathbb{Z},$$

with the numerical initial condition $(u_j^0)_{j\in\mathbb{Z}}$ given by

$$u_j^0 = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u^0(x)\,dx \quad \forall j \in \mathbb{Z}.$$

The terms $f_{j+1/2}^n$ ($j \in \mathbb{Z}, n \in \mathbb{N}$) are called the numerical fluxes and are to be computed in such a manner that the numerical approximation

$$\overline{u}_{\Delta x}^{\Delta t}(t,x) = \sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{Z}} u_j^n \mathbb{1}_{[n\Delta t,(n+1)\Delta t)}(t)\mathbb{1}_{[(j-1/2)\Delta x,(j+1/2)\Delta x)}(x)$$

converges towards the entropy solution to (1) as Δt and Δx tend to 0 (in a norm to be precised). In the following pages, we propose new conditions on the fluxes ensuring the convergence.

More precisely, this paper is an analysis of *reconstruction* schemes, whose spirit is to decompose the resolution, i.e. the computation of the fluxes $f_{j+1/2}^n$, in three steps:

- given the constant-in-cell solution at time step n, a reconstruction step consisting in reconstructing a new initial condition,
- the exact computation of the solution with this reconstructed condition,
- a projection of this solution on the mesh (to recover a constant-in-cell function).

The paper organizes as follows.

First (section 2), we state a theoretical result concerning entropy solutions: we show that the convolution of any entropy solution by the characteristic function of a bounded interval verifies a local maximum principle. This property seems to be new.

The result of this first step is then used, in section 3, to derive new conditions for a reconstruction scheme to be L^∞-decreasing and Total Variation Diminishing (TVD). These conditions are sufficient for a scheme to converge to a weak solution to (1,2). However, the limit solution is not necessarily towards the entropy solution.

Thus we focus in section 3.3 on numerical entropy inequalities. We give a general condition on the reconstruction for the global scheme to be consistent with entropy inequalities. This implies convergence towards the entropy solution. This condition is a consequence of an inequality of Hardy.

There is a wealth of literature on reconstruction schemes and entropy conditions. Usually, the focus is given on derivating second order or high order schemes. This is not the aim here: the present paper is only devoted to (weak) convergence conditions. Among the wide amount of studies, the reader can refer to the classical references [22], [23], to [12] for a general study of discrete entropy conditions, [8] for the geometric limiters theory (slope limiters), [11] for the limiter theory, [4] and [16] for the study of MUSCL schemes and entropy. One can read [3] for a precise study of links between geometric reconstruction and decrease of numerical entropy, [15] for high order approximation with entropy inequalities. We also also mention [18] for a general study of convergence and order.

2 Convolution of entropy solutions

We here state a preliminary result that will be used in section 3 for controlling the stability of reconstruction schemes. This result is a stability result concerning the convolution in space of the entropy solution to equation (1) with the characteristic function of any bounded interval.

Let $u \in L^\infty((0,+\infty)\times\mathbb{R})$ be the unique entropy solution to problem (1,2). Let us denote by $[u]_\delta : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ the convolution of u with the characteristic function of $[-\delta/2, \delta/2]$ normalized by a factor $1/\delta$:

$$[u]_\delta(t,x) = \frac{1}{\delta}\left(u(t,\cdot) * \mathbb{1}_{[-\delta/2,\delta/2]}(\cdot)\right)(x) = \frac{1}{\delta}\int_{x-\delta/2}^{x+\delta/2} u(t,y)\,dy.$$

Let us now define $c_1 = \min_{x\in\mathbb{R}} f'(u^0(x))$, $c_2 = \max_{x\in\mathbb{R}} f'(u^0(x))$. We denote by $\mathscr{C}_u(t,x)$ the dependence interval of x at time t:

$$\mathscr{C}_u(t,x) = [x - c_2 t, x - c_1 t].$$

Then, the following theorem holds.

Theorem 1 The convoluted entropy solution $[u]_\delta$ verifies, $\forall t \in \mathbb{R}^+$, $\forall x \in \mathbb{R}$,

$$\min_{y\in\mathscr{C}_u(t,x)} [u]_\delta(0,y) \le [u]_\delta(t,x) \le \max_{y\in\mathscr{C}_u(t,x)} [u]_\delta(0,y) \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Corollary 1 The convoluted entropy solution $[u]_\delta$ verifies

$$\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \exists y(t,x) \in \mathscr{C}_u(t,x) \text{ such that } [u]_\delta(t,x) = [u]_\delta(0, y(t,x)).$$

This corollary is an immediate consequence of theorem 1 and of the fact that $[u]_\delta(t,\cdot)$ is a Lipschitz function (with Lipschitz constant $2\|u^0\|_{L^\infty}/\delta$).

Proof The present proof uses a parabolic regularization of (1) and decomposes in three main parts:

- we first show that the convoluted solution to the problem with parabolic regularizing satisfies a global maximum principle,
- we then deduce the same maximum principle for the convolution of the non-regularized entropy solution,
- we enforce the maximum principle by localizing it.

Regularized equation
Let $(\rho^\varepsilon)_{\varepsilon\in\mathbb{R}^{+\star}}$ be a $\mathcal{C}^\infty$-regularizing sequence. We define the regularized initial condition

$$u^{\varepsilon 0}(x) = \left(u^0 * \rho^\varepsilon\right)(x) = \int_{\mathbb{R}} u^0(y)\rho^\varepsilon(x-y)\,dy$$

and the regularized flux

$$f^\varepsilon(u) = \left(f * \rho^\varepsilon\right)(u) = \int_{\mathbb{R}} f(v)\rho^\varepsilon(u-v)\,dv.$$

For $\varepsilon > 0$ fixed, we consider the solution u^ε of the following parabolic problem

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \partial_x f^\varepsilon(u^\varepsilon)(t,x) = \varepsilon\partial^2_{x,x}u^\varepsilon(t,x), & t\in\mathbb{R}^+, x\in\mathbb{R}, \\ u^\varepsilon(0,x) = u^{\varepsilon 0}(x). \end{cases} \tag{5}$$

It is well-known that this problem has a unique solution and that this solution belongs to $\mathcal{C}^\infty(\mathbb{R}^+\times\mathbb{R})$ (see [7], e.g.).

We now introduce the convoluted regularized solution (for $\delta > 0$)

$$\left[u^\varepsilon\right]_\delta(t,x) = \frac{1}{\delta}\left(u^\varepsilon(t,\cdot) * \mathbb{1}_{[-\delta/2,\delta/2]}(\cdot)\right)(x),$$

that of course belongs to $\mathcal{C}^\infty(\mathbb{R}^+\times\mathbb{R})$. Let us show that $[u^\varepsilon]_\delta$ is the solution to a partial differential equation. Indeed, by convoluting relation (5) with $\mathbb{1}_{[-\delta/2,\delta/2]}$, we get

$$\frac{1}{\delta}\left[\mathbb{1}_{[-\delta/2,\delta/2]}(\cdot) * \left(\partial_t u^\varepsilon(t,\cdot) + \partial_x f^\varepsilon(u^\varepsilon)(t,\cdot)\right)\right](x) = \\ \frac{1}{\delta}\left[\mathbb{1}_{[-\delta/2,\delta/2]}(\cdot) * \left(\varepsilon\partial^2_{x,x}u^\varepsilon(t,\cdot)\right)\right](x),$$

which equivalently reads

$$\partial_t [u^\varepsilon]_\delta(t,x) + \frac{f^\varepsilon(u^\varepsilon(t,x+\delta/2)) - f^\varepsilon(u^\varepsilon(t,x-\delta/2))}{\delta} = \partial^2_{x,x}[u^\varepsilon]_\delta(t,x).$$

A key point is now to show that the above equation can be recast into an advection diffusion problem. Let us first remark that $\delta\partial_x[u^\varepsilon]_\delta(t,x) = u^\varepsilon(t,x+\delta/2) - u^\varepsilon(t,x-\delta/2)$, so that whenever $u^\varepsilon(t,x+\delta/2) \neq u^\varepsilon(t,x-\delta/2)$, we have

$$\frac{f^\varepsilon(u^\varepsilon(t,x+\delta/2)) - f^\varepsilon(u^\varepsilon(t,x-\delta/2))}{\delta} = \frac{f^\varepsilon(u^\varepsilon(t,x+\delta/2)) - f^\varepsilon(u^\varepsilon(t,x-\delta/2))}{u^\varepsilon(t,x+\delta/2) - u^\varepsilon(t,x-\delta/2)} \partial_x[u^\varepsilon]_\delta(t,x).$$

Furthermore, if $u^\varepsilon(t,x+\delta/2) = u^\varepsilon(t,x-\delta/2)$, $\partial_x[u^\varepsilon]_\delta(t,x) = 0$. These considerations allow to define

$$v^\varepsilon_\delta(t,x) = \begin{cases} \dfrac{f^\varepsilon(u^\varepsilon(t,x+\delta/2)) - f^\varepsilon(u^\varepsilon(t,x-\delta/2))}{u^\varepsilon(t,x+\delta/2) - u^\varepsilon(t,x-\delta/2)} & \text{if } u^\varepsilon(t,x+\delta/2) \neq u^\varepsilon(t,x-\delta/2), \\ (f^\varepsilon)'(u^\varepsilon(t,x+\delta/2)) & \text{if } u^\varepsilon(t,x+\delta/2) = u^\varepsilon(t,x-\delta/2), \end{cases}$$

which acts as a velocity for an advection-diffusion equation. Indeed,

$$\begin{cases} \partial_t[u^\varepsilon]_\delta(t,x) + v^\varepsilon_\delta(t,x)\partial_x[u^\varepsilon]_\delta(t,x) = \partial^2_{x,x}[u^\varepsilon]_\delta(t,x), \\ [u^\varepsilon]_\delta(0,x) = \dfrac{1}{\delta}\left[u^{\varepsilon 0} * \mathbb{1}_{[-\delta/2,\delta/2]}\right](x). \end{cases}$$

Note that $\partial_x[u^\varepsilon]_\delta(t,x) = 0$ when $u^\varepsilon(t,x+\delta/2) = u^\varepsilon(t,x-\delta/2)$. It would therefore be possible to adopt other definitions for the velocity on such points. Indeed, the solution to the above advection-diffusion equation does not depend on $v^\varepsilon_\delta(t,x)$ at points (t,x) such that $u^\varepsilon(t,x+\delta/2) = u^\varepsilon(t,x-\delta/2))$. Nevertheless, the definition we use here plays a role in the regularity of the velocity, that we now study: we show that $v^\varepsilon_\delta \in \mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{R})$. Indeed, one can easily check that

$$v^\varepsilon_\delta(t,x) = \int_0^1 f^{\varepsilon\prime}(\theta u^\varepsilon(t,x+\delta/2)) + (1-\theta)u^\varepsilon(t,x-\delta/2))\,d\theta,$$

which states the regularity of v^ε_δ and shows furthermore that v^ε_δ is bounded over $\mathbb{R}^+ \times \mathbb{R}$. Let us now sum up what we know about $[u^\varepsilon]_\delta$:

- $[u^\varepsilon]_\delta \in \mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{R})$,

- $[u^\varepsilon]_\delta$ verifies the uniformly parabolic partial differential equation

$$\partial_t[u^\varepsilon]_\delta(t,x) + v^\varepsilon_\delta(t,x)\partial_x[u^\varepsilon]_\delta(t,x) = \partial^2_{x,x}[u^\varepsilon]_\delta(t,x)$$

  with given velocity $v^\varepsilon_\delta(t,x) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}$.

We can conclude (see [19] or [13] for example) that $[u^\varepsilon]_\delta$ verifies a *global* maximum principle, namely

$$\inf_{y\in\mathbb{R}}[u^\varepsilon]_\delta(0,y) \le [u^\varepsilon]_\delta(t,x) \le \sup_{y\in\mathbb{R}}[u^\varepsilon]_\delta(0,y) \quad \forall t \in \mathbb{R}^+.$$

Back to the non-regularized problem
First, $u^{\varepsilon 0}$ being a regularization of u^0, one has

$$\begin{aligned}\inf_{y\in\mathbb{R}}[u^\varepsilon]_\delta(0,y) &\ge \inf_{y\in\mathbb{R}}[u]_\delta(0,y),\\ \sup_{y\in\mathbb{R}}[u^\varepsilon]_\delta(0,y) &\le \sup_{y\in\mathbb{R}}[u]_\delta(0,y),\end{aligned}$$

so that

$$\inf_{y\in\mathbb{R}}[u]_\delta(0,y) \le [u^\varepsilon]_\delta(t,x) \le \sup_{y\in\mathbb{R}}[u]_\delta(0,y) \quad \forall t \in \mathbb{R}^+.$$

Recall (see [7] for example) that the entropy solution u to (1) is such that

$$\lim_{\varepsilon\to 0} u^\varepsilon = u \text{ in } \mathcal{C}\left([0,T], L^1_{loc}(\mathbb{R})\right)$$

$\forall T \in \mathbb{R}^+$.

Remark 1 The assumption $u^0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, commonly done, is not necessary.

A direct consequence is

$$\begin{aligned}|[u^\varepsilon]_\delta(t,x) - [u]_\delta(t,x)| &= \left|\int_{x-\delta/2}^{x+\delta/2} u^\varepsilon(t,y) - u(t,y)\,dy\right|\\ &\le \int_{x-\delta/2}^{x+\delta/2} |u^\varepsilon(t,y) - u(t,y)|\; dy\end{aligned}$$

We thus have, $\forall\delta > 0$:

$$\lim_{\varepsilon\to 0}[u^\varepsilon]_\delta(t,x) = [u]_\delta(t,x) \quad \forall (t,x) \in]0,\infty[\times\mathbb{R}.$$

We finally have the estimate

$$\inf_{y\in\mathbb{R}}[u]_\delta(0,y) \le [u]_\delta(t,x) \le \sup_{y\in\mathbb{R}}[u]_\delta(0,y),$$

which is a global maximum principle.

Local maximum principle

One important characteristic of entropy solutions of equation (1) is the finite speed of propagation. Due to it, the global maximum principle showed above should be local. To prove it, we propose the following procedure: let us consider the entropy solution $\tilde{u}$ of (1) with initial condition

$$\tilde{u}^0(y) = \begin{cases} u^0(y) & \text{if } y \in [x-c_2t-\frac{\delta}{2}, x-c_1t+\frac{\delta}{2}], \\ u^0\left(y-\delta-\delta\times E\left(\dfrac{y-(x-c_1t+\frac{\delta}{2})}{\delta}\right)\right) & \text{if } y > x-c_1t+\frac{\delta}{2}, \\ u^0\left(y+\delta+\delta\times E\left(\dfrac{x-c_2t-\frac{\delta}{2}-y}{\delta}\right)\right) & \text{if } y < x-c_2t-\frac{\delta}{2}, \end{cases}$$

where $E(s)$ denotes the integer part of $s\in\mathbb{R}$. So, $\tilde{u}^0 \in L^\infty(\mathbb{R})$ differs from u^0 only outside the interval of dependence of (x,t) and its normalized convolution with $\mathbb{1}_{[-\delta/2,\delta/2]}$, $[\tilde{u}^0]_\delta$, verifies

$$[\tilde{u}^0]_\delta(y) = \begin{cases} [u]_\delta(0,x-c_2t) & \text{if } y < x-c_2t, \\ [u]_\delta(0,y) & \text{if } y\in[x-c_2t,x-c_1t], \\ [u]_\delta(0,x-c_1t) & \text{if } y > x-c_1t \end{cases}$$

(see figure 1). We thus have $\sup_{y\in\mathbb{R}}[\tilde{u}^0]_\delta(y) = \max_{y\in[x-c_2t,x-c_1t]}[u]_\delta(0,y)$ and $\inf_{y\in\mathbb{R}}[\tilde{u}^0]_\delta(y) = \min_{y\in[x-c_2t,x-c_1t]}[u]_\delta(0,y)$. Finally, applying the global maximum principle to $[\tilde{u}]_\delta$, we have

$$\min_{y\in[x-c_2t,x-c_1t]}[u]_\delta(0,y) \le [\tilde{u}]_\delta(t,x) \le \max_{y\in[x-c_2t,x-c_1t]}[u]_\delta(0,y).$$

It remains to remark that

$$\begin{array}{l} \min_{x\in\mathbb{R}} f'(\tilde{u}^0) \ge c_1 = \min_{x\in\mathbb{R}} f'(u^0), \\ \max_{x\in\mathbb{R}} f'(\tilde{u}^0) \le c_2 = \max_{x\in\mathbb{R}} f'(u^0), \end{array}$$

so that the interval of dependence $\mathscr{C}_{\tilde{u}}(t,x)$ for $\tilde{u}^0$ is included in the one for u^0, $\mathscr{C}_u(t,x)$. The finite speed of propagation principle (see [20]) says that $u(t,x)$ only depends on $\{u^0(y) \text{ s.t. } y\in\mathscr{C}_u(t,x)\}$ and that $\tilde{u}(t,x)$ only depends on $\{\tilde{u}^0(y) \text{ s.t. } y\in\mathscr{C}_{\tilde{u}}(t,x)\}$, thus on $\{\tilde{u}^0(y) \text{ s.t. } y\in\mathscr{C}_u(t,x)\}$. Consequently,

$[u]_\delta(t,x)$ depends only on $\{u^0(y) \text{ s.t. } y \in \mathscr{C}_u(t,x)+[-\delta/2,\delta/2]\}$ and $[\tilde{u}]_\delta(t,x)$ depends only on $\{\tilde{u}^0(y) \text{ s.t. } y \in \mathscr{C}_u(t,x)+[-\delta/2,\delta/2]\}$. But $\tilde{u}^0 = u^0$ in the whole interval $\mathscr{C}_u(t,x)+[-\delta/2,\delta/2]$, so

$$[\tilde{u}]_\delta(t,x) = [u]_\delta(t,x)$$

which leads to

$$\min_{y\in[x-c_2t,x-c_1t]}[u]_\delta(0,y) \le [u]_\delta(t,x) \le \max_{y\in[x-c_2t,x-c_1t]}[u]_\delta(0,y).$$

This ends up the proof.

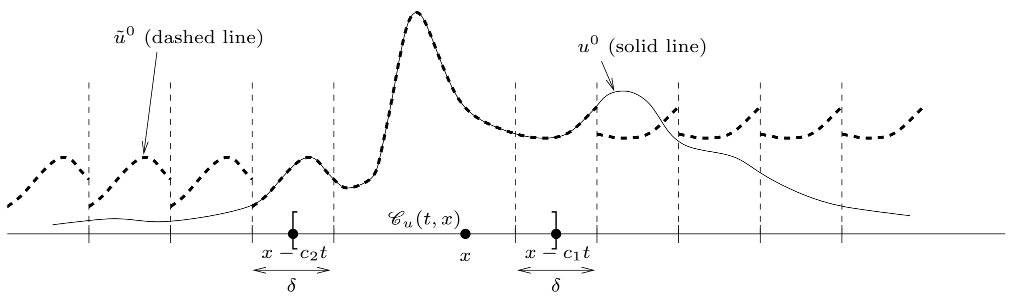

Figure 1: Construction of an initial data $\tilde{u}^0$ such that $\tilde{u}^0 = u^0$ in the whole interval $\mathscr{C}_u(t,x)+[-\delta/2,\delta/2]$ and $\sup_{y\in\mathbb{R}}[\tilde{u}^0]_\delta(y) = \max_{y\in[x-c_2t,x-c_1t]}[u]_\delta(0,y)$ and $\inf_{y\in\mathbb{R}}[\tilde{u}^0]_\delta(y) = \min_{y\in[x-c_2t,x-c_1t]}[u]_\delta(0,y)$.

Remark 2 Note that $\tilde{u}^0$ does not belong to $BV(\mathbb{R})$ nor to $L^1(\mathbb{R})$ even if u^0 does. This is the reason why we need the convergence of u^ε toward u without the integrability nor the boundedness of the total variation assumptions (cf. remark 1).

3 Convergence of reconstruction schemes

As mentioned in the introduction, we consider finite volume approximations of (1, 2) of the form

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(f_{j+1/2}^n - f_{j-1/2}^n) \quad \forall j\in\mathbb{Z}, \forall n\in\mathbb{N}, \tag{6}$$

where u_j^n is ought to represent the value of solution u in space cell $C_j = [(j-1/2)\Delta x, (j+1/2)\Delta x]$. The numerical initial condition u_j^0 is given by

$$u_j^0 = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u^0(x)\,dx \quad \forall j\in\mathbb{Z}. \tag{7}$$

We propose to compute the numerical fluxes $(f^n_{j+1/2})_{n\in\mathbb{N},j\in\mathbb{Z}}$ using a three-step procedure:

- given a constant-in-cell function, compute a reconstructed function that contains more details,
- compute the exact (entropy) solution at time Δt of (1) with the reconstructed function as initial condition,
- project this exact solution on the mesh in order to obtain a constant-in-cell function for the following time step.

Note that the last two steps are equivalent to compute the *fluxes* of the exact solution, which shows the finite volume form of the algorithm.

Each of these steps can be associated to an operator: we shall call $\mathcal{R}$, $\mathcal{E}$ and $\mathcal{P}$ respectively the reconstruction, the exact and the projection operators. Let us provide a more precise definition of them.

Definition 1 1 Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a constant-in-cell function.

$\mathcal{R}u : \mathbb{R} \longrightarrow \mathbb{R}$ denotes the reconstruction of u.

2 Let $t \in \mathbb{R}$ and $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a function in $L^\infty(\mathbb{R})$.

$\mathcal{E}(t)u : \mathbb{R} \longrightarrow \mathbb{R}$ denotes the exact entropy solution at time t of equation (1) with initial condition u.

3 Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a function in $L^\infty(\mathbb{R})$.

$\mathcal{P}u : \mathbb{R} \longrightarrow \mathbb{R}$ denotes the projection of u on the mesh:

$$\mathcal{P}u(x) = \sum_{j\in\mathbb{Z}} u_j \mathbb{1}_{[(j-1/2)\Delta x,(j+1/2)\Delta x)}(x)$$

with

$$u_j = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u(x)\,dx.$$

Let us now define the approximate solution $\overline{u}^n : \mathbb{R} \longrightarrow \mathbb{R}$ at time step n by

$$\overline{u}^n(x) = \sum_{j\in\mathbb{Z}} u^n_j \mathbb{1}_{[(j-1/2)\Delta x,(j+1/2)\Delta x)}(x).$$

The scheme is then defined by

$$\overline{u}^{n+1} = \mathcal{P}\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n. \tag{8}$$

For example, if $\mathcal{R}u = u$ for every constant-in-cell function u, the resulting scheme, that does not involve reconstruction, is the Godunov scheme. If $\mathcal{R}u$ is an affine-in-cell function for every constant-in-cell function u, the resulting scheme can be a MUSCL scheme and has been studied in [3]. Nevertheless we do not use such characterizations of the reconstruction form in the following.

The last two steps (exact computation and projection) are solved in a single step by taking as numerical fluxes in equation (6)

$$f^n_{j+1/2} = \frac{1}{\Delta t}\int_0^{\Delta t} f\left(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j+1/2)\Delta x)\right)\,ds \quad \forall n \in \mathbb{N}, \forall j \in \mathbb{Z}. \tag{9}$$

This shows the finite volume form of the reconstruction scheme.

In this paper, we will not insist on the second step of the algorithm, i.e. the exact computation of the solution with reconstructed initial condition. We assume it is possible to compute it and only focus on the reconstruction step (the projection step being classical).

The equivalent formulations (8) and (6, 7, 9) will be alternatively used.

Before beginning the numerical analysis, let us introduce the following notation:

$$\begin{cases} m = \inf_{j\in\mathbb{Z}} u^0_j, \\ M = \sup_{j\in\mathbb{Z}} u^0_j. \end{cases} \tag{10}$$

3.1 Conservativity

We consider only conservative reconstructions such that

$$\frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \mathcal{R}u(x)\,dx = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u(x)\,dx = \mathcal{P}u(j\Delta x). \tag{11}$$

The exact operator $\mathcal{E}(t)$ and the projection $\mathcal{P}$ being conservative, the whole scheme defined by (8) is consequently conservative.

3.2 L^∞-decreasing and decrease of the total variation

We say that a numerical scheme of the form (6, 7) is L^∞-decreasing if and only if $\forall u^0 \in L^\infty(\mathbb{R})$,

$$\sup_{j\in\mathbb{Z}} |u^{n+1}_j| \le \sup_{j\in\mathbb{Z}} |u^n_j| \quad \forall n \in \mathbb{N}.$$

It is said that a numerical scheme of the form (6, 7) is Total Variation Diminishing (TVD) if and only if $\forall u^0 \in BV(\mathbb{R})$, $\forall n \in \mathbb{N}$,

$$\sum_{j\in\mathbb{Z}} |u_{j+1}^{n+1} - u_j^{n+1}| \leq \sum_{j\in\mathbb{Z}} |u_{j+1}^{n} - u_j^{n}|.$$

In the sequel, the initial condition u^0 is supposed to belong to $BV(\mathbb{R})$.

Theorem 1 will help us exhibiting a new condition for the finite volume reconstruction scheme to be L^∞-decreasing and TVD.

The same notations as in the previous section for the convoluted solutions is used and Δx now shall play the role of δ:

$$[u]_{\Delta x} = \frac{1}{\Delta x}\mathbb{1}_{[-\Delta x/2, \Delta x/2]} * u$$

for any $u \in L^\infty(\mathbb{R})$.

For the sake of simplicity, we assume that there is no sonic point in the computational domain, i.e. for example that

$$f'(u) > 0 \quad \forall u \in [m, M]. \tag{12}$$

(cf. remark 3 for the case $f'(u) < 0$ $\forall u \in [m, M]$).

Proposition 1 Assume that (12) holds (no sonic point in $\mathbb{R}$).
Assume the CFL (Courant-Friedrichs-Levy) condition $\Delta t/\Delta x \leq 1/\max_{u\in[m,M]} |f'(u)|$ is fulfilled.
Assume the reconstructed solution $\mathcal{R}\overline{u}^n$ verifies, $\forall n \in \mathbb{N}$, $\forall j \in \mathbb{Z}$,

$$\min(u_{j-1}^n, u_j^n) \leq [\mathcal{R}\overline{u}^n]_{\Delta x}((j-\theta)\Delta x) \leq \max(u_{j-1}^n, u_j^n) \qquad \forall \theta \in [0,1]. \tag{13}$$

Then, the scheme given by (6, 7, 9), or (8) is L^∞-decreasing and TVD.

Remark that $\forall n \in \mathbb{N}$, $\forall j \in \mathbb{Z}$, $[\mathcal{R}\overline{u}^n]_{\Delta x}(j\Delta x) = u_j^n$. This result is not obvious because the constraint (13) does not bound the total variation of the reconstructed solution.

Proof First note that $\forall n \in \mathbb{N}$, $\forall j \in \mathbb{Z}$, $[\mathcal{R}\overline{u}^n]_{\Delta x}(j\Delta x) = u_j^n$. Thus condition (13) is equivalent to

$$\begin{aligned}\min([\mathcal{R}\overline{u}^n]_{\Delta x}((j-1)\Delta x), [\mathcal{R}\overline{u}^n]_{\Delta x}(j\Delta x))\\ \leq [\mathcal{R}\overline{u}^n]_{\Delta x}((j-\theta)\Delta x) \leq\\ \max([\mathcal{R}\overline{u}^n]_{\Delta x}((j-1)\Delta x), [\mathcal{R}\overline{u}^n]_{\Delta x}(j\Delta x)) \quad \forall \theta \in [0,1].\end{aligned}$$

We here use the formulation (8) of the scheme. Under the CFL condition, the interval of dependence of $(j\Delta x, \Delta t)$ is included in $[(j-1)\Delta x, j\Delta x]$ (recall that $f'(u) > 0 \ \forall u \in [m, M]$). Note that $[(j-1)\Delta x, j\Delta x]$ is the set of convex combinations of $(j-1)\Delta x$ and $j\Delta x$: $[(j-1)\Delta x, j\Delta x] = \{x \in \mathbb{R} \text{ s.t. } \exists \theta \in [0,1] \text{ s.t. } x = \theta(j-1)\Delta x + (1-\theta)j\Delta x\}$. Thus, a direct consequence of theorem 1 is that

$$\min_{\theta\in[0,1]} [\mathcal{R}\overline{u}^n]_{\Delta x}(\theta(j-1)\Delta x + (1-\theta)j\Delta x) \\ \leq [\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n]_{\Delta x}(j\Delta x) \leq \\ \max_{\theta\in[0,1]} [\mathcal{R}\overline{u}^n]_{\Delta x}(\theta(j-1)\Delta x + (1-\theta)j\Delta x),$$

and thus, by (13), we recover

$$\min(u_{j-1}^n, u_j^n) \leq [\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n]_{\Delta x}(j\Delta x) \leq \max(u_{j-1}^n, u_j^n)$$

arguing that $\theta(j-1)\Delta x + (1-\theta)j\Delta x = (j-\theta)\Delta x$. It remains to note that $[\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n]_{\Delta x}(j\Delta x) = \mathcal{P}\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n(j\Delta x) = u_j^{n+1}$ and we obtain

$$\min(u_{j-1}^n, u_j^n) \leq u_j^{n+1} \leq \max(u_{j-1}^n, u_j^n).$$

This is true for every $n \in \mathbb{N}$ and every $j \in \mathbb{Z}$, and, by a classical argument of incremental analysis of Harten (see [11]), the scheme is L^∞-decreasing and TVD.

The conclusion of this section is the following convergence result.

Theorem 2 Let us consider the scheme defined by (8) with constraints (11) and (13).
Assume (12) is verified.
Define the approximate solution as

$$\overline{u}_{\Delta x}^{\Delta t}(t,x) = \sum_{n\in\mathbb{N}} \sum_{j\in\mathbb{Z}} u_j^n \mathbb{1}_{[n\Delta t,(n+1)\Delta t[}(t) \mathbb{1}_{[(j-1/2)\Delta x,(j+1/2)\Delta x[}(x)$$

Then, for any sequences $(\Delta t_k)_{k\in\mathbb{N}}$, $(\Delta x_k)_{k\in\mathbb{N}}$ converging to 0 and that verify the CFL condition $\Delta t_k/\Delta x_k \leq \frac{1}{\max_{u\in[m,M]} |f'(u)|}$, there exists a sequence $\overline{u}_{\Delta x_{k_j}}^{\Delta t_{k_j}}(t,x)$ that converges in $L^\infty(]0,T[, L^1_{loc}(\mathbb{R})) \ \forall T \in \mathbb{R}^+$ and whose limit is a weak solution to (1, 2).

This is a classical consequence of conservativity, L^∞-stability and of the decreasing of the total variation: see [7] for example.

Remark 3 Any result of the present section remains valid replacing hypothesis (12) by

$$f'(u) < 0 \quad \forall u \in [m, M].$$

3.3 Numerical entropy inequalities

Theorem (2) does not imply entropy convergence, i.e. convergence toward the unique entropy solution to (1, 2). For this purpose, we need the scheme (that is to say: the reconstruction operator) to verify some entropy inequalities. This is the aim of this section. We first show a simple and constructive condition that ensures the existence of numerical entropy fluxes for *one strictly convex* entropy in section 3.3.1. This is sufficient to ensure entropy convergence in the case of a strictly convex flux f, but not in the general case. A more formal condition is then given to obtain the entropy convergence for any $f \in \mathcal{C}^1(\mathbb{R})$ in section 3.3.2.

3.3.1 Decrease of one entropy

Here is derived a sufficient condition on the reconstruction for the global scheme to be entropy consistent for one entropy. This is not sufficient for the scheme to be convergent toward the unique Kruzkov entropy solution in the general case, but it is well-known that it allows to select this solution when f is strictly convex, provided that the chosen entropy is strictly convex too. Let S be the entropy and G the associated entropy flux.

A very useful technique to prove the entropy convergence of a numerical solution is to exhibit some *discrete entropy fluxes*.

Definition 2 Let (S, G) be an entropy-entropy flux pair. It is said that scheme (6, 7, 9) has discrete entropy fluxes relatively to (S, G) if and only if $\forall (u_j^n)_{j\in\mathbb{Z}}\ \exists (G_{j+1/2}^n)_{j\in\mathbb{Z}}$ such that

- $G_{j+1/2}^n$ is consistent with G (in the classical sense of finite volume);
-
$$S_j^{n+1} \leq S_j^n - \frac{\Delta t}{\Delta x}\left(G_{j+1/2}^n - G_{j-/2}^n\right) \forall j \in \mathbb{Z}, n \in \mathbb{N} \tag{14}$$

$$\text{with} \quad S_j^n = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{R}\overline{u}^n(x))\,dx. \tag{15}$$

Remark 4 We here use the definition of [3], taking $S_j^n = 1/\Delta x \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{R}\overline{u}^n(x))\,dx$ instead of $S_j^n = S(u_j^n)$. On an algorithmic point of view, the schemes we will consider in the following shall make the use of both *the unknown* u_j^n and *the entropy associated to it* S_j^n. This is already the idea in [2].

It seems reasonable, as the exact resolution is used, to take the exact flux (similarly to eq. (9)) as entropy flux $G_{j+1/2}^n$,

$$G_{j+1/2}^n = \frac{1}{\Delta t}\int_0^{\Delta t} G\left(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j+1/2)\Delta x)\right)\, ds \quad \forall n\in\mathbb{N}, \forall j\in\mathbb{Z}. \quad (16)$$

Equation (14) acts like a constraint on the reconstruction procedure.

A stronger entropy inequality is also proposed in

Proposition 2 Assume the reconstructed solution $\mathcal{R}\overline{u}^n$ verifies, for any $n\in\mathbb{N}$ and any $j\in\mathbb{Z}$,

$$S_j^n \le \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^{n-1}(x))\, dx. \quad (17)$$

Then, the scheme given by (6, 7, 9), or (8) owns some discrete entropy fluxes relatively to (S, G).

Proof Because $\mathcal{E}(\Delta t)$ is the exact entropy operator, we get

$$\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n(x))\, dx - \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{R}\overline{u}^n(x))\, dx$$
$$+\left(\int_0^{\Delta t} G(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j+1/2)\Delta x))\, ds - \int_0^{\Delta t} G(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j-1/2)\Delta x))\, ds\right) \le 0,$$

so that

$$S_j^{n+1} \le \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^n(x))\, dx \le$$
$$S_j^n - \frac{1}{\Delta x}\left(\int_0^{\Delta t} G(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j+1/2)\Delta x))\, ds - \int_0^{\Delta t} G(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j-1/2)\Delta x))\, ds\right)$$

and the discrete entropy fluxes are given by

$$G_{j+1/2}^n = \frac{1}{\Delta t}\int_0^{\Delta t} G(\mathcal{E}(s)\mathcal{R}\overline{u}^n((j+1/2)\Delta x))\, ds \quad \forall n\in\mathbb{N}, j\in\mathbb{Z}.$$

Remark 5 The set of reconstructed solutions verifying (17) is not empty and contains the non-reconstructed solution, thanks to Jensen's inequality:

$$S(u_j^n) = S\left(\frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \mathcal{E}(\Delta t)\mathcal{R}\overline{u}^{n-1}(x)\,dx\right) \\ \leq \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} S(\mathcal{E}(\Delta t)\mathcal{R}\overline{u}^{n-1}(x))\,dx.$$

Thus the set of reconstructions verifying (14, 15, 16) contains the non-reconstructed solution.

The interest of the existence of numerical entropy fluxes stands in the following well-known theorem.

Theorem 3 Let us consider the scheme defined by (8) with constraints (11), (13) and (14) (conservativity, stability, existence of entropy fluxes).
Assume (12) is fulfilled.
Then, any weak solution to (1, 2) that is a limit point of approximate solutions as in theorem 2 verifies

$$\partial_t S(u) + \partial_x G(u) \leq 0.$$

3.3.2 Decrease of any entropy

We here propose a way to ensure the decrease of any Kruzkov entropy. By this we mean that

$$\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} |\mathcal{R}u(x) - k|\,dx \leq \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} |u(x) - k|\,dx \quad \forall k \in \mathbb{R}, \quad j \in \mathbb{Z}, \tag{18}$$

following entropy inequality (17). This section is essentially based on a link between the Kruzkov entropies and the theory of rearrangement, link which seems not to be commonly noticed in hyperbolic PDE analysis. We make the use of the decreasing (resp. increasing) rearrangement of a function on a bounded interval. Following [17], let us recall the

Definition 3 Let $f \in L^1(]a,b[,\mu)$ where μ is the Lebesgue measure on $]a,b[$. The *decreasing* rearrangement $f_\downarrow$ (resp. increasing rearrangement $f_\uparrow$) is defined by

$$f_\downarrow(x) = \sup\{y \text{ s.t. } \mu\{u \text{ s.t. } f(u) > y\} > x - a\} \quad x \in]a,b[$$

(resp. $f_\uparrow(x) = \inf\{y \text{ s.t. } \mu\{u \text{ s.t. } f(u) < y\} > x - a\}$).

Remark 6 It is said that two functions $f, g \in L^1(]a, b[, \mu)$ are *equimeasurable* if and only if $\mu\{u \text{ s.t. } f(u) > y\} = \mu\{u \text{ s.t. } g(u) > y\}\ \forall y \in \mathbb{R}$. The decreasing (resp. increasing) rearrangement $f_\downarrow$ (resp. $f_\uparrow$) is a decreasing (resp. increasing) function equimeasurable to f, i.e. $\mu\{u \text{ s.t. } f_\downarrow(u) > y\} = \mu\{u \text{ s.t. } f(u) > y\}\ \forall y \in \mathbb{R}$. Another straightforward property of these rearrangements is $\int_a^b f_\downarrow(x)\,dx = \int_a^b f_\uparrow(x)\,dx = \int_a^b f(x)\,dx$.

In the following, we consider the decreasing or increasing rearrangement of a function with $a = (j-1/2)\Delta x$ and $b = (j+1/2)\Delta x$.

Theorem 4 The reconstruction operation verifies (18) if and only if it is conservative (Eq. (11)) and

$$\int_{(j-1/2)\Delta x}^{y} (\mathcal{R}u)_\downarrow(x)\,dx \le \int_{(j-1/2)\Delta x}^{y} u_\downarrow(x)\,dx \qquad \forall y \in [(j-1/2)\Delta x, (j+1/2)\Delta x], \quad j \in \mathbb{Z}. \tag{19}$$

Symmetrically, the reconstruction operation verifies (18) if and only if it is conservative and

$$\int_{(j-1/2)\Delta x}^{y} (\mathcal{R}u)_\uparrow(x)\,dx \ge \int_{(j-1/2)\Delta x}^{y} u_\uparrow(x)\,dx \qquad \forall y \in [(j-1/2)\Delta x, (j+1/2)\Delta x], \quad j \in \mathbb{Z}.$$

Proof It is a direct consequence of a theorem by Hardy, Littlewood and Polya that can be found in [10] saying that if $f, g \in L^1(]a, b[, \mu)$,

$$\int_a^y f_\downarrow(x)\,dx \le \int_a^y g_\downarrow(x)\,dx \quad \forall y \in]a, b[$$
$$\text{and} \int_a^b f_\downarrow(x)\,dx = \int_a^b g_\downarrow(x)\,dx$$

if and only if

$$\int_a^b |f(x) - k|\,dx \le \int_a^b |g(x) - k|\,dx \quad \forall k \in \mathbb{R}.$$

Let us also mention [9], [6], [1] for developments.

Thus the family of inequations

$$\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} |\mathcal{R}u(x) - k|\,dx \le \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} |u(x) - k|\,dx \quad \forall k \in \mathbb{R}$$

is equivalent to

$$\int_{(j-1/2)\Delta x}^{y} \left(\mathcal{R}u\right)_{\downarrow}(x)\,dx \le \int_{(j-1/2)\Delta x}^{y} u_{\downarrow}(x)\,dx \quad \forall y \in [(j-1/2)\Delta x, (j+1/2)\Delta x]$$

$$\text{and } \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \left(\mathcal{R}u\right)_{\downarrow}(x)\,dx = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u_{\downarrow}(x)\,dx.$$

Now recall that

$$\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \left(\mathcal{R}u\right)_{\downarrow}(x)\,dx = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \mathcal{R}u(x)\,dx$$

and

$$\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u_{\downarrow}(x)\,dx = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u(x)\,dx.$$

Condition $\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \left(\mathcal{R}u\right)_{\downarrow}(x)\,dx = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u_{\downarrow}(x)\,dx$ thus writes $\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \mathcal{R}u(x)\,dx = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u(x)\,dx$ which is exactly the conservativity assumption (11).

The equivalence involving the increasing rearrangement can be shown on the same way or remarking that $f_{\uparrow} = -\left(-f\right)_{\downarrow}$.

Theorem 4 is of particular interest when the chosen reconstructed solution is either decreasing in $[(j-1/2)\Delta x, (j+1/2)\Delta x]$ (then, $\left(\mathcal{R}u\right)_{\downarrow} = \mathcal{R}u$) or increasing in $[(j-1/2)\Delta x, (j+1/2)\Delta x]$ (then, $\left(\mathcal{R}u\right)_{\uparrow} = \mathcal{R}u$).

Applying theorem 3 with all Kruzkov entropies leads to the convergence of approximate solutions toward the unique Kruzkov solution to (1, 2).

As a conclusion of this paper on general reconstruction operators, let us point out that

- condition (13) is weaker than the “no sawtooth” condition from [3] (indeed, saw-teeth are here allowed, the stability condition being required on the *convolution* of the saw-teeth),
- condition (19) is necessary and sufficient for the reconstruction to verify conservativity and the decrease of every Kruzkov entropy,
- it is proved in [17] that

$$\begin{array}{l} \int_a^b g(x)\,dx = \int_a^b f(x)\,dx, \\ \int_a^s g_{\downarrow}(x)\,dx \le \int_a^s f_{\downarrow}(x)\,dx \quad \forall s \in [a,b] \end{array}$$

  defines a partial ordering usually noted $g \prec f$, so that theorem 4 says that (18) defines a partial ordering between all integrable functions with a given integral.

4 Perspectives

In this paper, a new analysis of reconstruction schemes for scalar equations in dimension 1 is done. It leads to new convergence and entropy results. A forthcoming paper based on the present analysis exhibits a new kind of reconstruction schemes: discontinuous reconstruction schemes, that are based on a discontinuous reconstruction of the solution in each cell. This type of reconstruction is shown to lead to entropic non dissipative schemes that are able to compute shocks and rarefaction waves with a good accuracy in very long time. These schemes generalize the limited downwind scheme of [5] and the Ultra-Bee limiter (cf. [21]) in the non-linear case.

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