Vortex lattices in rotating Bose Einstein condensates

Amandine Aftalion* and Xavier Blanc†
Laboratoire Jacques-Louis Lions, Université Paris 6,
175 rue du Chevaleret, 75013 Paris, France.

December 2, 2004

Abstract

The structure of the vortex lattice for a fast rotating condensate in a harmonic trap has been studied experimentally and numerically: it is an almost regular hexagonal lattice, with a distortion on the edges. In this paper, we provide rigorous proofs of results announced in [2]. We analyze the vortex pattern in the framework of the Gross Pitaveskii energy using wave functions in the Lowest Landau Level. We give an estimate of the energy in a class of distorted lattices and find the optimal distortion which provides a decay of the wave function similar to an inverted parabola.

Résumé

La structure des réseaux de vortex dans un condensat de Bose-Einstein en rotation rapide dans un piège harmonique a été l'objet de nombreuses études expérimentales et numériques. Il s'agit d'un réseau hexagonal régulier au centre du piège, distordu sur les bords. Dans cet article, nous démontrons rigoureusement les résultats annoncés dans [2]. Nous analysons la répartition des vortex en minimisant l'énergie de Gross-Pitaevskii dans le premier niveau de Landau. Nous prouvons une estimation de l'énergei pour une classe de distortion de réseaux et trouvons la distortion optimale. Cette dernière donne un comportement de la fonction d'onde comparable à une parabole inversée.

1 Introduction

One of the special features of Bose Einstein condensates (BEC), related to superfluidity, is the existence of quantized vortices. These vortices can be observed in different types of experiments, one of them being the equivalent of what is known for helium as the rotating bucket experiment [14]. When a normal fluid is rotated, the velocity field inside the fluid is governed by solid body rotation. In contradistinction, a quantum fluid such as a BEC, described by a macroscopic wave function, nucleates vortices. This has been observed experimentally recently, in particular in the ENS group [10, 22] but also in [1, 23]. When the rotational velocity is small, there are only a few vortices in the system [9]. Their three dimensional shape is of interest, as has been described in [3, 4] using tools developed by [7] for Ginzburg-Landau vortices.

When the velocity gets large, the size of the condensate and the number of vortices increase: a dense lattice is observed [1, 11, 15, 27], referred to as an Abrikosov lattice due to the analogy with superconductors. The description of the vortex lattice at high rotational velocity has been the focus

*email: aftalion@ann.jussieu.fr
†email: blanc@ann.jussieu.fr

of very recent papers in the condensed matter physics community, starting with the seminal paper of Ho [18] and very recently by Fischer and Baym [17], Baym and Pethick [6], Cooper, Komineas and Read [12], Watanabe, Baym and Pethick [31], Sheehy and Radzihovsky [28]. Our aim is to provide mathematical insight to the lattice pattern and distortion.

The mathematical interest of such a system can be related to homogeneous media since there are two scales emerging: the size of vortices (of order 1) and the size of the condensate (much larger). In this regime, vortices have approximately the same size as their mutual distance, which is very different from the lower rotation regime. Hence different mathematical tools need to be introduced relying on double scale convergence. Our aim is to understand the vortex patterns in this fast rotating regime using the minimization of the Gross Pitaevskii energy. This framework, known as the mean field Quantum Hall regime, is acceptable only if the number of vortices is much smaller than the number of atoms in the condensate, which is the case of the present experiments. Otherwise, one has to consider other models, as in [13, 29]. The reduction of the N body hamiltonian to the Gross Pitaevskii energy is an open question for this fast rotating regime. It has been derived only in the case of no rotation by Lieb, Seiringer and Yngvason [20].

We assume that the confinement of the trapping potential is much stronger along the z axis (which is always true for high values of the rotational velocity) and we are allowed to restrict to a two dimensional model [2]. Thus, we want to understand the shape of minimizers ψ of

$$\int_{\mathbb{R}^2} \frac{1}{2}|\nabla\psi - i\Omega \times r\psi|^2 + \frac{1}{2}(1-\Omega^2)r^2|\psi|^2 + \frac{1}{2}Na|\psi|^4. \tag{1.1}$$

under $\int_{\mathbb{R}^2}|\psi|^2 = 1$, where Ω is the rotational velocity. The contributions to the energy are the kinetic term $(1/2)|\nabla\psi|^2$, the term due to the potential trapping the atoms $(1/2)r^2|\psi|^2$ and the rotating term $-\Omega L_z$ where $L_z = i(y\partial_x - x\partial_y)$ is the angular momentum. The last term is due to atomic interactions, N being the number of atoms and a the reduced two dimensional scaterring length [24, 26]. In order for the trapping potential to remain stronger than the rotating force, we need to have $\Omega < 1$, so that the energy is bounded below. The first term in the energy is identical to the energy of a particle placed in a uniform magnetic field 2Ω. It is also reminiscent of type II superconductors near the second critical field H_{c2}. The eigenvalues and eigenfunctions for

$$\int_{\mathbb{R}^2} \frac{1}{2}|\nabla\psi - i\Omega \times r\psi|^2 \text{ under } \int_{\mathbb{R}^2}|\psi|^2 = 1. \tag{1.2}$$

are well known [21]. The first eigenvalue is Ω and the eigenfunctions are

$$P(z)e^{-\Omega|z|^2/2} \tag{1.3}$$

where P is a polynomial. This space is known as the lowest Landau level (LLL). The other eigenvalues are $(2k+1)\Omega$, $k \in \mathbb{N}$. Note that all along the paper, we will identify $z = x + iy$ with $r = (x, y)$.

We will see that as Ω approaches 1, the second and third term in the energy (1.1) produce a contribution of order $\sqrt{1-\Omega}$, which is much smaller than the gap between two eigenvalues: 2Ω. Thus, it is natural, as a first step, to restrict to the eigenfunctions of the first eigenvalue and find the minimizer of the energy in this reduced infinite dimensional space. Since we want to keep the same space as Ω varies, we actually restrict to

$$P(z)e^{-|z|^2/2} \text{ with } P(z) = \prod_{i=1}^{n}(z - z_i). \tag{1.4}$$

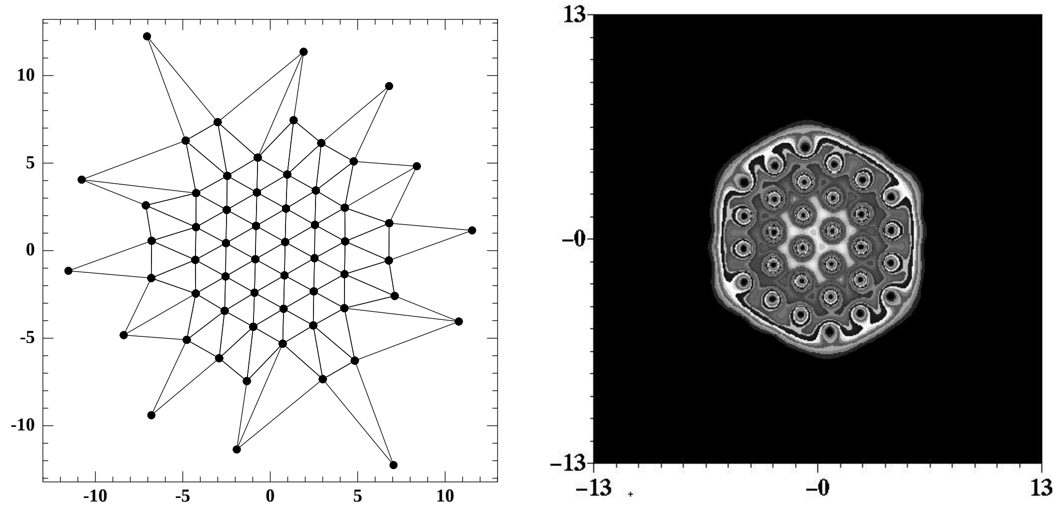

Figure 1: An example of (a): a configuration obtained by a conjugate gradient method applied to the energy as a function of the roots z_i. Here, $\Omega = 0.999$, $Na = 3$ and $n = 58$. (b): density plot

We minimize the energy on the polynomial P, that is on n, its degree, and z_i the location of the vortices. Our aim is to understand the optimal location of vortices. In [2], we have performed numerical computations, using a conjugate gradient on the z_i, which provide the following pattern for vortices (see Figure 1): in a central region, vortices are located on a regular triangular lattice, while the lattice is distorted towards the edges. The density plot of $|\psi|$ shows that the only visible vortices are the ones in the regular lattice part, the outer ones being in a region of very low density.

In this paper, we want to understand rigorously the distortion of the lattice and are going to provide a rigorous proof of the results announced in [2]. When ψ is restricted to the lowest Landau level (1.4), the energy can be simplified since

$$\int_{\mathbb{R}^2} |\nabla|\psi||^2 = \Omega \text{ and } \int_{\mathbb{R}^2} \nabla|\psi|^2 = 2\int_{\mathbb{R}^2} |\nabla|\psi||^2 + (1-\Omega)^2 \int_{\mathbb{R}^2} r^2|\psi|^2 \tag{1.5}$$

so that for ψ of the form (1.4),

$$E(\psi) = E_{LLL}(\psi) = \Omega + \int_{\mathbb{R}^2} (1-\Omega) r^2 |\psi|^2 + \frac{Na}{2}|\psi|^4. \tag{1.6}$$

If we minimize bluntly the energy (1.6) with $\int |\psi|^2 = 1$, we find that the minimizer is

$$|\psi_{\min}(r)|^2 = \frac{2}{\pi R_0^2}\left(1 - \frac{r^2}{R_0^2}\right)_+, \quad R_0 = \left(\frac{2Na}{\pi(1-\Omega)}\right)^{1/4} \tag{1.7}$$

The reduced energy is

$$\epsilon_{\min} = E_{LLL}(\psi_{min}) - \Omega = \frac{2\sqrt{2}}{3}\sqrt{\frac{Na(1-\Omega)}{\pi}}\ . \tag{1.8}$$

This is a first indication of the main scales of the problem: the extension of the condensate is of size R_0, which is large when Ω approaches 1, while vortices will be located on a lattice of size of order 1. If ψ is of the form (1.4), it cannot match (1.7). Though, with an appropriate location of the zeroes z_i, the aim is to build a test function whose energy can be of the same order as (1.8) and whose decay approaches that of (1.7). Our main results, announced in [2], are the following: if the vortices are located on a regular lattice, the wave function decays like a gaussian and we provide a rigorous proof of the energy estimate obtained by Ho [18]. On the other hand, we are able to build distortions of the lattice which improve the energy and hence modify the decay of the wave function. This decay is similar to that of an inverted parabola, as already observed by [12, 27, 31]. The major role played by the outer distorted points is a new contribution of the authors [2], as will be explained below.

In what follows, ℓ will denote a regular hexagonal lattice, whose unit cell Q, centered at the points of the lattice has volume V. Moreover, we will identify complex numbers and vectors in $\mathbb{R}^2$ and in particular dz will denote the two dimensional Lebesgue measure $dx\, dy$. The symbol $\fint$ denotes the average of an ℓ-periodic function: $\fint f = \frac{1}{V}\int_Q f$.

Theorem 1.1 *Let ℓ be a regular hexagonal lattice, Q its unit cell and $V = |Q|$. Let*

$$\psi_R(z) = A_R \prod_{j\in\ell\cap B_R} (z-j)e^{-|z|^2/2} \tag{1.9}$$

with A_R chosen such that $\|\psi_R\|_{L^2(\mathbb{R}^2)} = 1$. Then as R tends to ∞,

$$|\psi_R(z)| \longrightarrow \psi(z) = \frac{1}{\sqrt{\pi}\sigma}\eta(z)e^{-|z|^2/(2\sigma^2)} \text{ in } L^p\left(\mathbb{R}^2, (1+|z|^2)dz\right) \text{ for all } p\geq 1, \tag{1.10}$$

where

$$\frac{1}{\sigma^2} = 1 - \frac{\pi}{V} \tag{1.11}$$

and η is a periodic function which vanishes at each point of ℓ. Moreover, η satisfies

$$-\Delta\left(\ln\eta\right) = 2\pi\delta_0 - \frac{2\pi}{V} \text{ in } Q,$$

with periodic boundary conditions. In addition, $\lim_{R\to+\infty} E_{LLL}(\psi_R) = E_{LLL}(\psi)$. As σ tends to infinity, then

$$E_{LLL}(\psi) - \Omega \sim (1-\Omega)\sigma^2 + \frac{1}{4}\frac{Nab}{\pi\sigma^2} \text{ where } b = \frac{\fint|\eta|^4}{(\fint|\eta|^2)^2}. \tag{1.12}$$

The main feature of the periodic lattice is to modify the decay of the gaussian from $e^{-|z|^2/2}$ to $e^{-|z|^2/2\sigma^2}$, where σ depends on the volume through (1.11). We need to choose the optimal σ in (1.12), which yields

$$\sigma^4(1-\Omega) = \frac{1}{4}\frac{Nab}{\pi} \tag{1.13}$$

This value of σ indeed satisfies $\sigma\to+\infty$ as Ω tends to 1. The volume condition (1.11) matched with the value of σ (1.13) implies

$$V = \pi\Big(1+\sqrt{(1-\Omega)\frac{4\pi}{Nab}}\Big). \tag{1.14}$$

This is close to the value predicted by solid body rotation arguments, π/Ω (see [18]), but different. The estimate of the energy is thus

$$E_{LLL}(\psi) - \Omega \underset{\Omega\to 1}{\sim} \sqrt{\frac{Nab}{\pi}(1-\Omega)} \tag{1.15}$$

This is to be compared to (1.8), which is better by a factor $\sqrt{8/9b}$, but is of the same magnitude, as $1-\Omega$ is small. Let us emphasize the presence of the coefficient b: it takes into account the averaged vortex contribution on each cell. As in the case of superconductors near H_{c_2}, for the Abrikosov lattice, the optimal lattice minimizing the ratio b is the hexagonal one [19]. An approximate value of b is 1.16. Note that our proof could hold with other lattices than the hexagonal one, as soon as the unit cell has some symmetries. The aim of the paper is to improve the numerical factor in front of the square root in (1.15).

The main observation is that modifying the location of the vortices from a regular lattice can change the decay of the wave function and hence improve the energy estimate.

Theorem 1.2 *There exists a sequence of functions ψ_Ω of the form (1.4), such that as Ω tends to 1,*

$$E_{LLL}(\psi_\Omega) - \Omega \sim \frac{2\sqrt{2}}{3}\sqrt{\frac{Nab}{\pi}(1-\Omega)}. \tag{1.16}$$

This is closer to the lower bound (1.8), than the regular lattice: the numerical factor is the same as in (1.8), except for the coefficient b, coming from the averaged vortex contribution.

Let us now explain the main ideas of the proof, announced in [2]. For the regular lattice, we split $\ln|\psi_R(z)|$ into $v_R(z) + w_R(z)$ with

$$v_R(z) = \sum_{j\in l\cap B_R} \ln|z-j| - \frac{1}{V}\int_Q \ln|z-y-j|\, dy \tag{1.17}$$

$$w_R(z) = \ln A_R - \frac{|z|^2}{2} + \frac{1}{V}\sum_{j\in l\cap B_R}\int_Q \ln|z-y-j|\, dy. \tag{1.18}$$

At this stage, we have just added and substracted the sum of the integrals. As R tends to ∞, we prove that v_R converges to a periodic series v and e^{w_R} to a gaussian with modified decay $1/\sigma^2$. The computation of the energy uses the double scale convergence [5] which allows to separate the integrals in v and the gaussian and get the contribution of b.

Let us be more precise about Theorem 1.2. We perform a general transformation of the lattice in the following way: for j in ℓ, a regular triangular lattice of unit cell with volume $V=\pi$, we define the transformed lattice ℓ'_R by

$$k \in \ell'_R \text{ if } k = \nu_R(|j|)\, j \text{ for } j \in \ell \cap B_R. \tag{1.19}$$

We assume that ν_R is close to 1 as Ω tends to 1, in the sense,

$$\nu_R^2(r) = 1 + \frac{f(r^2/R^2)}{R^2} + O\left(\frac{1}{R^4}\right) \text{ with } R = \left(\frac{2Nab}{\pi(1-\Omega)}\right)^{1/4}, \tag{1.20}$$

where $f(x)$ is a continuous function, such that for some γ, $f(\gamma)=\infty$ and $\int_0^\gamma f(s)\, ds = \infty$.

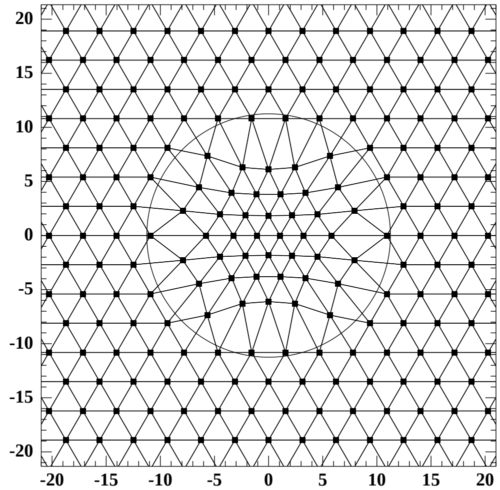

Figure 2: A plot of the distorted lattice defined by (1.21)

We would like to apply the same proof as for the regular lattice, using v_R and w_R for this distorted lattice. Contrary to the proof for the regular lattice, we cannot study the two limits $R \to \infty$ in (1.10) and $\sigma \to \infty$ in (1.12) separately, since now R is related to Ω through (1.20). Hence, the lattice has a finite extent at each R and we have to pass to the limit in the double scale convergence at the same time as the scale of the lattice. For technical reasons, we are unable to match w_R inside and outside the lattice and do the dominated convergence.

In order to circumvent this problem, we introduce an outer regular lattice, whose characteristic size tends to infinity in a last step. Let $\alpha \in (0, \gamma)$, R be related to Ω by (1.20), and

$$\lambda_R(r) = \begin{cases} \nu_R(r) & \text{if} \quad r \leq \alpha R, \\ \nu_{\alpha,R} = \nu_R(\alpha R) & \text{if} \quad r > \alpha R, \end{cases} \tag{1.21}$$

and $\ell'_R = \{\lambda_R(|j|)j, \quad j \in \ell\}$. We have plotted such a transformation in Figure 2. For fixed α, we let R tend to ∞, and study the limit of the wave functions vanishing at each point of ℓ'_R:

$$\psi_R(x) = A_R \prod_{j \in \ell} (x - \lambda_R(|j|)j)\, e^{-\frac{|x|^2}{2}}, \tag{1.22}$$

Since α is fixed, $\nu_R(\alpha R)$ tends to 1. We use similar ideas as in the regular lattice case and identify a double scale convergence to a periodic part on the one hand and a profile depending on the transformation f on the other hand, given by

$$|\psi(x)|^2 = e^{-F(|x|^2)} \mathbf{1}_{B_\alpha}(x) + e^{\alpha^2 f(\alpha^2) - F(\alpha^2) - f(\alpha^2)|x|^2} \mathbf{1}_{B_\alpha^c}(x), \tag{1.23}$$

where F is a primitive of f. The proof uses as a main tool that λ_R is close to the identity. As a final step only, once we have passed to the limit $\Omega \to 1$, we let α tend to γ, so that the exterior regular lattice has a unit volume cell which tends to infinity and the outer contribution disappears. We find an estimate for the energy:

$$E_{LLL}(\psi_\Omega) - \Omega \sim_{\Omega \to 1} \sqrt{\frac{2Nab(1-\Omega)}{\pi}} \int_0^\gamma \left(s e^{-F(s)} + \frac{1}{4} e^{-2F(s)} \right) ds, \tag{1.24}$$

where F is a primitive of f such that $\int_0^\gamma e^{-F(s)}\,ds = 1$.

We want to find which type of distortion f provides the optimal energy. The minimizer of (1.24) under $\int_0^\gamma e^{-F(s)}\,ds = 1$ is reached when

$$\gamma = 1 \text{ and } e^{-F(r^2)} = 2(1-r^2). \tag{1.25}$$

Thus, the decay of the wave function is asymptotically an inverted parabola. The corresponding value of f is $f(s) = 1/(1-s)$. The limiting value of the energy is (1.16).

Let us point out that the proof uses two lattices: an initial regular lattice and an image lattice obtained by (1.20). The points in the regular lattice which have an image are only those in the ball B_R since the other ones are sent to infinity ($\gamma = 1$). There are two regions in the initial lattice: the points sufficiently far away from the circle of radius R, for which ν_R is almost the identity, and the points close to the circle, at distance less than $\sqrt{R}$ for instance. For the first category of points, the image lattice is an almost regular lattice and the image points are inside the disk B_R. On the contrary, the points close to the circle are strongly modified by (1.20) and sent far away. This allows to understand better the distorted shape of Figure 1. It turns out that R is both the radius of the "horizon" for the initial lattice, but also the radius of the limiting inverted parabola, that is the envelope function for the image distorted lattice given by (1.25). The points which are not visible in the density profile, and are in the distorted region, have nevertheless a major contribution in creating the inverted parabola profile.

For each R, this analysis gives an estimate on the number of points in the distorted lattice, related to the number of points in a regular lattice of unit volume π, included in a ball of radius R. It is an open question though whether this number is indeed the number of vortices minimizing the energy.

If one wanted to get rid of the tool of the outer regular lattice in the proof, one would need to count the number of points in the lattice closest to the limiting circle of radius R and estimate the convergence of v_R and w_R due to the fact that these limiting points do not lie on a circle but on the edges of hexagons. We are not able to prove that the finite extension of the lattice (which is not the case with an outer regular lattice) does not create a boundary contribution in the energy. These boundary effects are more important than we expected, and are related to known problems about counting the number of point of a lattice in an annulus (see [8] and the references therein).

Our results deal with an upper bound for the energy. A natural question would be to get also the lower bound and prove Γ convergence type results. For the moment, the gap between the lower bound (1.8) and the upper bound (1.16) lies in the coefficient b. We believe that an optimal lower bound should match the upper bound, that is the optimal inverted parabola should have the coefficient b. Note that the only way to reproduce an inverted parabola in the space (1.4) is to use a lot of vortices, thus these vortices should provide a contribution in the energy. On the other hand, if one wants to compute the energy of the inverted parabola, the expression (1.6) is not suitable and one has to go back to the full Gross Pitaevskii energy. For the moment, we are unable to prove the lower bound: this would require to prove that a minimizing sequence has many zeroes and that these zeroes are located on an almost regular lattice. This seems very difficult and is probably related to similar difficulties in cristalization and sphere packing problems. Given our upper bound, one can deduce convergence of the minimizer of the Gross Pitaevskii energy to its projection on the space (1.4), but this is not optimal for the moment since a proper lower bound is missing.

Let us point out that other trapping potentials than r^2 can be dealt with these techniques. In [2], we have addressed the case of $r^2 + kr^4$ with k small, following recent experiments [10, 30].

According to the values of Ω, a giant vortex can be obtained.

The paper is organized as follows: in section 2, we study the regular lattice case and prove Theorem 1.1. Then, in section 3, we prove Theorem 1.2. Finally, section 3 is devoted to remarks concerning other trapping potentials.

2 Regular lattice

In this section, we prove Theorem 1.1. We first need two technical lemmas:

Lemma 2.1 *Let ℓ be a lattice, and denote by Q its unit cell centered at 0. Let $Q_R = \bigcup_{k\in\ell\cap B_R} (Q+k)$ and for r in $\mathbb{R}^2$, let*

$$h_R(r) = \int_{Q_R} \left(\ln|r-r'| - \ln|r'|\right) dr'.$$

Then there exists $C>0$ and $R_0>0$ such that

$$\forall R \geq R_0, \quad h_R(r) \leq \left(\frac{\pi}{2} + \frac{C}{R}\right)|r|^2.$$

Proof: If Q_R was a ball, then the integral could be computed explicitely. Thus, we use a ball close to Q_R and estimate the difference. We separate the integral defining h_R into two parts:

$$h_R(r) = \int_{B_{R-a}} \left(\ln|r-r'| - \ln|r'|\right) dr' + \int_{Q_R\setminus B_{R-a}} \left(\ln|r-r'| - \ln|r'|\right) dr',$$

where $a>0$ is independent of R and such that $B_{R-a} \subset Q_R$. The first term is the radial solution of $\Delta u = \mathbf{1}_{B_{R-a}}$ such that $u(0)=0$. One easily computes this solution:

$$u(r) = \frac{\pi}{2}|r|^2 \mathbf{1}_{B_{R-a}} + \pi(R-a)^2\left(\frac{1}{2} + \ln\left(\frac{|r|}{R-a}\right)\right)\mathbf{1}_{B^c_{R-a}}.$$

Next, we consider the second term defining h_R and use the inequality $\ln(t) \leq \frac{1}{2}(t^2-1)$, valid for any $t>0$:

$$\begin{aligned}\int_{Q_R\setminus B_{R-a}} \left(\ln|r-r'| - \ln|r'|\right) dr' &\leq \int_{Q_R\setminus B_{R-a}} \frac{1}{2}\left(\frac{|r-r'|^2}{|r'|^2} - 1\right) dr' \\ &= |r|^2 \int_{Q_R\setminus B_{R-a}} \frac{dr'}{2|r'|^2} \leq C\frac{|r|^2}{R},\end{aligned}$$

the constant C being independent of R and r. Collecting both results, we infer

$$\begin{aligned}h_R(r) &\leq \frac{\pi}{2}|r|^2\mathbf{1}_{B_{R-a}}(r) + \pi(R-a)^2\left(\frac{1}{2} + \ln\left(\frac{|r|}{R-a}\right)\right)\mathbf{1}_{B^c_{R-a}}(r) + \frac{C}{R}|r|^2 \\ &\leq \left(\frac{\pi}{2} + \frac{C}{R}\right)|r|^2,\end{aligned}$$

using here again $\ln(t) \leq \frac{1}{2}(t^2-1)$. This gives the result. □

Lemma 2.2 *Let ℓ be the hexagonal lattice, and let Q be its elementary unit cell (i.e. the regular hexagon centered at 0). Let*

$$g(z) = \ln|z| - \frac{1}{|Q|}\int_Q \ln|z-y|\, dy. \tag{2.1}$$

Then we have, for some constant $C>0$,

$$\forall z \in B_1^c, \quad |g(z)| \leq \frac{C}{|z|^3}. \tag{2.2}$$

Hence, the function

$$v(x) = \sum_{j\in\ell} g(x-j) \tag{2.3}$$

is such that $e^{v(x)}$ exists, is continuous on $\mathbb{R}^2$, and ℓ-periodic.

Proof: We first point out that g is continuous on $\mathbb{R}^2 \setminus \{0\}$. Hence, we only need to show (2.2) on B_a for some $a>0$. We fix $a>0$ such that $Q \subset B_{\frac{a}{2}}$. For any $z \in B_a^c$ and any $y \in Q$, we have $\frac{|z-y|}{|z|} \geq \frac{|z|-\frac{a}{2}}{|z|} \geq \frac{1}{2}$. Hence,

$$\frac{|z-y|^2}{|z|^2} - 1 \geq -\frac{3}{4}.$$

For any $t > -\frac{3}{4}$, we have

$$t - \frac{t^2}{2} - |t|^3 \leq \ln(1+t) \leq t - \frac{t^2}{2} + \frac{t^3}{3}.$$

Hence, writing $g(z) = -\frac{1}{|Q|}\int_Q \frac{1}{2}\ln\left(1 - \frac{2y\cdot z}{|z|^2} + \frac{|y|^2}{|z|^2}\right) dy$, we infer

$$\frac{1}{2|Q|}\int_Q \left(-\frac{2y\cdot z}{|z|^2} + \frac{|y|^2}{|z|^2} - \frac{1}{2}\left(-\frac{2y\cdot z}{|z|^2} + \frac{|y|^2}{|z|^2}\right)^2 - \left|-\frac{2y\cdot z}{|z|^2} + \frac{|y|^2}{|z|^2}\right|^3\right) dy \leq -g(z)$$
$$\leq \frac{1}{2|Q|}\int_Q \left(-\frac{2y\cdot z}{|z|^2} + \frac{|y|^2}{|z|^2} - \frac{1}{2}\left(-\frac{2y\cdot z}{|z|^2} + \frac{|y|^2}{|z|^2}\right)^2 + \frac{1}{3}\left(-\frac{2y\cdot z}{|z|^2} + \frac{|y|^2}{|z|^2}\right)^3\right) dy.$$

Since Q is symmetric with respect to the origin, $\int_Q y\cdot z\, dy = 0$. In addition, Q is invariant under the rotation of angle $\pi/3$, so one easily shows that $\int_Q \left(|y|^2 - 2(y\cdot z)^2\right) dy = 0$. We thus have

$$\frac{1}{2|Q|}\int_Q \left(\frac{2y\cdot z|y|^2}{|z|^4} - \frac{|y|^4}{|z|^4} - \left|-\frac{2y\cdot z}{|z|^2} + \frac{|y|^2}{|z|^2}\right|^3\right) dy \leq g(z)$$
$$\leq \frac{1}{2|Q|}\int_Q \left(\frac{2y\cdot z|y|^2}{|z|^4} - \frac{|y|^4}{|z|^4} + \frac{1}{3}\left(-\frac{2y\cdot z}{|z|^2} + \frac{|y|^2}{|z|^2}\right)^3\right) dy.$$

Using $|y| \leq \frac{a}{2}$, we end up with

$$|g(z)| \leq \frac{1}{|Q|}\int_Q \left(\frac{|y|^3}{|z|^3} + \frac{|y|^4}{2|z|^4} + \frac{1}{2}\left|\frac{2|y|}{|z|} + \frac{|y|^2}{|z|^2}\right|^3\right) dy \leq \frac{\frac{a^3}{8} + \frac{a^3}{16} + 2a^3}{|z|^3}.$$

This ensures that the series (2.3) converges normally on any set of the form $\left(\bigcup_{j\in\ell} B_\varepsilon(j)\right)^c$, which implies that v exists, is ℓ-periodic, and continuous on $\mathbb{R}^2 \setminus \ell$. Near a point $k \in \ell$, we write

$$e^{v(x)} = |x-k| e^{-\frac{1}{|Q|}\int_Q \ln|z-y|\,dy} e^{\sum_{j\in\ell\setminus\{k\}} g(x-j)}$$

and the conclusion follows. □

Remark 2.3 *It is in this Lemma that we have used the symmetry properties of the lattice. It may be that another type of proof could allow to relax these hypotheses.*

Remark 2.4 *The estimate (2.2) is valid for a fixed hexagonal lattice ℓ_0. Now, $g = g_\ell$ depends on ℓ in the following way: if $\ell = \lambda\ell_0$, then $g_\ell(z) = g_{\ell_0}\left(\frac{z}{\lambda}\right)$. Hence, $|g_\ell(z)| \le \frac{C_0\lambda^3}{|z|^3}$ if $|z| \ge \lambda$, for some constant C_0 independent of ℓ.*

Proof of Theorem 1.1: Let $f_R(z) = \ln|\psi_R(z)|$. We split f_R into

$$f_R(z) = v_R(z) + w_R(z) \tag{2.4}$$

with

$$v_R(z) = \sum_{j\in l\cap B_R} \ln|z-j| - \frac{1}{V}\int_Q \ln|z-y-j|\,dy, \tag{2.5}$$

$$w_R(z) = \ln A_R - \frac{|z|^2}{2} + \frac{1}{V}\sum_{j\in l\cap B_R}\int_Q \ln|z-y-j|\,dy. \tag{2.6}$$

Let v be given by (2.3). We have

$$v_R(z) - v(z) = \sum_{j\in\ell\cap B_R^c} g(z-j).$$

Hence, if $z \in B_R$, we deduce from Lemma 2.2 that

$$|v_R(z) - v(z)| \le \sum_{j\in\ell\cap B_R^c} \frac{C}{|z-j|^3}$$

for some constant C independent of R and z. One can thus find a constant C independent of R such that

$$\forall A \in (0,R), \quad \|v_R - v\|_{L^\infty(B_{R-A})} \le \frac{C}{A}. \tag{2.7}$$

In addition, we have, for any $z \in \mathbb{R}^2$, denoting by j_z the unique point of ℓ such that $|j_z - z| < 1$,

$$v_R(z) \le \ln|z-j_z| + C + \sum_{j\in\ell\setminus\{j_z\}} \frac{C}{|z-j|^3} \le \ln|z-j_z| + C,$$

for various constants C independent of z and R. Hence, e^{v_R} is bounded in $L^\infty(\mathbb{R}^2)$ independently of R. Next, using the inequality $|e^a - e^b| \le \frac{1}{2}(e^a+e^b)|a-b|$ and (2.7), we infer that e^{v_R} converges to e^v in $L^\infty_{\text{loc}}(\mathbb{R}^2)$.

Let us call $\tilde{w}_R(z) = w_R(z) - w_R(0) - \ln(A_R) + \frac{1}{2\sigma^2}|z|^2$. Applying Lemma 2.1, we have

$$\tilde{w}_R(z) \leq -\frac{|z|^2}{2} + \left(\frac{\pi}{2V} + \frac{C}{R}\right)|z|^2 + \frac{1}{2\sigma^2}|z|^2 = \frac{C}{R}|z|^2. \tag{2.8}$$

In addition, $\tilde{w}_R$ is a harmonic function in $Q_R = \bigcup_{j\in\ell\cap B_R}(Q+j)$ and vanishes at 0. Hence, using Harnack inequality, $\tilde{w}_R$ is bounded and we may extract convergence of $\tilde{w}_R$ in $L^\infty_{\text{loc}}(\mathbb{R}^2)$ to some $\tilde{w}$, which is harmonic non positive, and vanishes at 0. Applying Liouville theorem, we find that $\tilde{w} = 0$. Gathering all the previous results, we thus have, up to renormalization,

$$\psi_R \longrightarrow \psi \text{ almost everywhere in } \mathbb{R}^2,$$

where ψ is given by (1.10) and $\eta(z) = e^{v(z)}$. For R large enough,

$$|\psi_R(z)| \leq Ce^{-\frac{|z|^2}{4\sigma^2}}.$$

Hence, applying the dominated convergence theorem, we get (1.10).

Then we write the limiting energy, z being identified with a vector in $\mathbb{R}^2$:

$$\begin{aligned} E_{LLL}(\psi) &= \Omega + \int_{\mathbb{R}^2}\left((1-\Omega)|z|^2|\eta(z)|^2 e^{-\frac{|z|^2}{\sigma^2}} + \frac{Na}{2\sigma^2}|\eta(z)|^4 e^{-\frac{2|z|^2}{\sigma^2}}\right)\frac{dz}{\pi\sigma^2} \\ &= \Omega + \int_{\mathbb{R}^2}\left((1-\Omega)\sigma^2|\eta(\sigma\xi)|^2 e^{-|\xi|^2} + \frac{Na}{2\sigma^2}|\eta(\sigma\xi)|^4 e^{-2|\xi|^2}\right)d\xi. \end{aligned}$$

The function η is periodic, so $|\eta(\sigma\xi)|^2$ and $|\eta(\sigma\xi)|^4$ respectively converge L^∞-weak-* to $⨍|\eta|^2$ and $⨍|\eta|^4$ (see [5]). Hence, we find (1.12). ☐

3 Distorted lattice

In this section, we prove two theorems, which will imply Theorem 1.2. The first Theorem consists in studying a distorted lattice analogous to Figure 2 and find the limit of the wave function with an infinite number of vortices. The proof is similar to the regular lattice case, since only the central vortices are displaced from their regular location. The second Theorem consists in letting Ω tend to 1 and use the double scale convergence. The proof is more involve since this is precisely where the distortion of the lattice appears.

Theorem 3.1 *Let ℓ be a hexagonal lattice, and let Q be the regular hexagon of area π centered at zero. Let $\gamma > 0$ and let f be a positive Lipschitz continuous function defined in $[0,\gamma)$ such that*

$$\lim_{t\to\gamma} f(t) = +\infty \quad \textit{and} \quad \lim_{t\to\gamma}\int_0^t f(s)ds = \infty. \tag{3.1}$$

Let us define

$$\nu_R(t) = 1 + \frac{1}{2R^2}f\left(\frac{t^2}{R^2}\right) + O\left(\frac{1}{R^4}\right), \tag{3.2}$$

where $O\left(\frac{1}{R^4}\right)$ is uniform with respect to $t \in \mathbb{R}^+$. Let $\alpha \in (0,\gamma)$

$$\lambda_R(t) = \begin{cases} \nu_R(t) & \text{if} \quad t \leq \alpha R, \\ \nu_{\alpha,R} = \nu_R(\alpha R) & \text{if} \quad t > \alpha R. \end{cases} \tag{3.3}$$

For $R' > R$, we define

$$\psi_{R,R'}(x) = A_{R,R'} \prod_{j\in\ell\cap B_{R'}} (x - \lambda_R(|j|)j)\, e^{-\frac{|x|^2}{2}}, \tag{3.4}$$

where $A_{R,R'}$ is such that $\|\psi_{R,R'}\|_{L^2(\mathbb{R}^2)} = 1$. Then, we have the following convergence in $L^p(\mathbb{R}^2, (1+|x|^2)dx)$, for any $p < +\infty$:

$$\left|\psi_{R,R'}\right| \underset{R'\to+\infty}{\longrightarrow} A_R e^{v_R\left(\frac{x}{\nu_{\alpha,R}}\right) + w_R(x) + \left(\frac{1}{\nu_{\alpha,R}{}^2} - 1\right)\frac{|x|^2}{2}}, \tag{3.5}$$

where

$$w_R(x) = \sum_{j\in\ell\cap B_{\alpha R}} \frac{1}{\nu_{\alpha,R}^2|Q|} \int_{\nu_{\alpha,R}Q} \ln\left(\frac{|x-y-\nu_R(|j|)j|}{|x-y-\nu_{\alpha,R}j|}\right) dy, \tag{3.6}$$

and

$$v_R(y) = \sum_{j\in\ell} g\left(y - \frac{\lambda_R(|j|)}{\nu_{\alpha,R}} j\right), \tag{3.7}$$

the function g being defined by (2.1).

Then, we let Ω tend to 1, or equivalently R to infinity:

Theorem 3.2 *With the same definitions as in Theorem 3.1, we have*

$$\forall n \geq 1, \quad e^{nv_R(Rx)} \underset{R\to+\infty}{\overset{*}{\rightharpoonup}} \fint e^{nv} \quad in \quad L^\infty(\mathbb{R}^2), \tag{3.8}$$

$$e^{2w_R(Rx) + \left(\frac{1}{\lambda_R(\alpha R)^2} - 1\right) R^2|x|^2} \underset{R\to+\infty}{\longrightarrow} \rho(x) \tag{3.9}$$

in $L^p(\mathbb{R}^2, (1+|x|^2)dx)$, $\forall p \geq 1$, where, v is given by (2.3),

$$\rho(x) = e^{-F(|x|^2)} \mathbf{1}_{B_\alpha}(x) + e^{\alpha^2 f(\alpha^2) - F(\alpha^2) - f(\alpha^2)|x|^2} \mathbf{1}_{B_\alpha^c}(x), \tag{3.10}$$

and F is a primitive of f such that $\fint e^{2v} \int \rho = 1$.

Proof of Theorem 1.2: We let Ω tend to 1, R be given by (1.20), and we take a diagonal sequence in R'. Theorems 3.1, 3.2 and double scale convergence [5] provide the convergence of $\int |\psi_{R,R'}(Rz)|^2$ to $\fint e^{2v} \int \rho$, and similarly for the energy:

$$E_{LLL}(\psi_{R,R'}(Rz)) - \Omega \sim_{\Omega\to1} \sqrt{\frac{2Nab(1-\Omega)}{\pi}} \left(\fint e^{2v} \int_0^\infty s\rho(\sqrt{s})\, ds + \frac{1}{4}\fint e^{4v} \int_0^\infty \rho^2(\sqrt{s})\, ds\right), \tag{3.11}$$

where F is a primitive of f such that $\int_0^\gamma e^{-F(s)}\, ds = 1$ and $\fint e^{2v} = 1$. If one lets α tend to γ: the contribution to ρ in the outer part B_γ^c vanishes and the energy is given by (1.24).

We want to find which type of distortion f provides the optimal energy. The minimizer of (1.24) under $\int_0^\gamma e^{-F(s)}\, ds = 1$ is reached when

$$\gamma = 1 \text{ and } e^{-F(r^2)} = 2(1-r^2). \tag{3.12}$$

Thus, the decay of the wave function is asymptotically an inverted parabola. The corresponding value of f is $f(s) = 1/(1-s)$. The limiting value of the energy is (1.16).

Proof of Theorem 3.1: This proof is a mere adaptation of Section 2. Indeed, up to normalization by a constant, the function $\ln|\psi_{R,R'}|^2$ is equal to

$$\ln\left|\psi_{R,R'}(x)\right|^2 = 2\sum_{j\in\ell\cap B_{R'}}\Bigg(\ln|x-\lambda_R(|j|)j| \tag{3.13}$$

$$-\frac{1}{\nu_{\alpha,R}{}^2|Q|}\int_{\nu_{\alpha,R}Q}\ln|x-y-\lambda_R(|j|)j|dy\Bigg) \tag{3.14}$$

$$+\sum_{j\in\ell\cap B_{R'}}\frac{2}{\nu_{\alpha,R}{}^2|Q|}\int_{\nu_{\alpha,R}Q}\ln|x-y-\lambda_R(|j|)j|dy \tag{3.15}$$

$$-|x|^2. \tag{3.16}$$

The sum (3.13)-(3.14) may be written

$$\sum_{j\in\ell\cap B_{R'}} g\left(\frac{x}{\nu_{\alpha,R}}-\frac{\lambda_R(|j|)}{\nu_{\alpha,R}}j\right), \tag{3.17}$$

where g is defined by (2.1). Now, R being fixed, Lemma 2.2 ensures that the above sum converges as R' goes to infinity to $v_R\left(\frac{x}{\nu(\alpha R)}\right)$, where v_R is defined by (3.7). Moreover, the convergence of the exponential of (3.17) to $e^{v_R\left(\frac{x}{\nu(\alpha R)}\right)}$ is the same as in Theorem 1.1, that is, $L^\infty_{\rm loc}(\mathbb{R}^2)$. Next, the sum (3.15) is equal to

$$\begin{aligned}
&\sum_{j\in\ell\cap B_{R'}}\frac{2}{\nu_{\alpha,R}{}^2|Q|}\int_{\nu_{\alpha,R}Q}\ln|x-y-\lambda_R(|j|)j|dy\\
=\ &\sum_{j\in\ell\cap B_{\alpha R}}\frac{2}{\nu_{\alpha,R}{}^2|Q|}\int_{\nu_{\alpha,R}Q}\ln\frac{|x-y-\nu_R(|j|)j|}{|x-y-\nu_{\alpha,R}\ j|}dy\\
+\ &\sum_{j\in\ell\cap B_{R'}}\frac{2}{\nu_{\alpha,R}{}^2|Q|}\int_{\nu_{\alpha,R}Q}\ln|x-y-\nu_{\alpha,R}\ j|dy.
\end{aligned} \tag{3.18}$$

The first sum in the left-hand side of (3.18) is $w_R(x)$, while the second sum is the one appearing in (2.6), with $\nu(\alpha R)\ell$ replacing ℓ. Since this lattice is also a hexagonal one (with a different volume for its unit cell), the proof of its convergence applies, using Lemma 2.1. □

Proof of Theorem 3.2: For simplicity, we will give the proof in the case where the $O(1/R^4)$ is zero. We start with the proof of (3.9). We define $\varepsilon > 0$ depending on R such that, as R tends to infinity,

$$\left\{\begin{array}{l} R\varepsilon \longrightarrow +\infty,\\ R\varepsilon^2 \longrightarrow 0.\end{array}\right. \tag{3.19}$$

For instance, $\varepsilon = R^{-\frac{3}{4}}$ is a suitable choice. Writing

$$w_R(Rx) = \sum_{k\in\frac{\ell}{R}\cap B_\alpha}\frac{R^2}{\nu_{\alpha,R}{}^2|Q|}\int_{\frac{\nu_{\alpha,R}}{R}Q}\ln\left(\frac{|x-z-\nu_R(R|k|)k|}{|x-z-\nu_{\alpha,R}k|}\right)dz,$$

we split this sum into terms for which $|k-x|<\varepsilon$, and terms for which $|x-k|\geq\varepsilon$: in the first case, we use the inequality

$$\forall a,b>0,\quad |\ln a-\ln b|\leq \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)|b-a|,$$

and the fact that $|\nu_{\alpha,R}-\nu_R(|k|R)|\leq \frac{C}{R^2}$ for some constant C independent of R and x. Hence,

$$\begin{aligned}
&\left|\sum_{|k-x|<\varepsilon}\frac{R^2}{\nu_{\alpha,R}|Q|}\int_{\frac{\nu_{\alpha,R}}{R}Q}\ln\left(\frac{|x-z-\nu_R(R|k|)k|}{|x-z-\nu_{\alpha,R}k|}\right)dz\right|\\
\leq\ &\sum_{|k-x|<\varepsilon}\frac{R^2}{2\nu_{\alpha,R}|Q|}\int_{\frac{\nu_{\alpha,R}}{R}Q}\left(\frac{1}{|x-z-\nu_R(R|k|)k|}\right.\\
+\ &\left.\frac{1}{|x-z-\nu_{\alpha,R}k|}\right)|k||\nu_{\alpha,R}-\nu_R(|k|R)|dz\\
\leq\ &\frac{C}{R^2}\sum_{|k-x|<\varepsilon}\frac{R^2}{2\nu_{\alpha,R}|Q|}\int_{B_{\frac{3}{R}}}\frac{dy}{|y|}\leq C\#\left(\frac{\ell}{R}\cap B_\varepsilon(x)\right)\frac{1}{R}=CR\varepsilon^2,
\end{aligned}$$

which tends to zero as $R\to+\infty$. Next, we deal with $|k-x|\geq\varepsilon$, and denote the corresponding sum by $T_R(x)$:

$$T_R(x)=\sum_{|k-x|\geq\varepsilon}\frac{R^2}{\nu_{\alpha,R}{}^2|Q|}\int_{\frac{\nu_{\alpha,R}}{R}Q}\ln\left(\frac{|x-z-\nu_R(R|k|)k|}{|x-z-\nu_{\alpha,R}k|}\right)dz$$

Using the equality $\nu_R(R|k|)=1+\frac{f(|k|^2)}{2R^2}$, valid for any $|k|\leq\alpha$, we deduce:

$$T_R(x)=\sum_{|k-x|\geq\varepsilon}\frac{R^2}{\nu_{\alpha,R}{}^2|Q|}\int_{\frac{\nu_{\alpha,R}}{R}Q}\ln\left(\frac{|x-z-k-\frac{f(|k|^2)}{2R^2}k|}{|x-z-k-\frac{f(\alpha^2)}{2R^2}k|}\right)dz.$$

We have $|x-z-k|\geq|x-k|-|z|\geq\varepsilon-\frac{C}{R}=\varepsilon(1-\frac{C}{\varepsilon R})$ for $z\in\frac{\nu_{\alpha,R}}{R}Q$, so that for R large enough, we get $|x-z-k|\geq\frac{\varepsilon}{2}$. Hence, developping the quotient in the logarithm, we get

$$T_R(x)=\sum_{|k-x|\geq\varepsilon}\frac{R^2}{2\nu_{\alpha,R}{}^2|Q|}\int_{\frac{\nu_{\alpha,R}}{R}Q}\ln\left(\frac{1-\frac{f(|k|^2)}{R^2|x-z-k|^2}k\cdot(x-z-k)+O\left(\frac{1}{\varepsilon^2R^4}\right)}{1-\frac{f(\alpha^2)}{R^2|x-z-k|^2}k\cdot(x-z-k)+O\left(\frac{1}{\varepsilon^2R^4}\right)}\right)dz,$$

where the $O\left(\frac{1}{\varepsilon^2R^4}\right)$ are uniform with respect to x and k. Developping the logarithm, we thus find

$$T_R(x)=\sum_{|k-x|\geq\varepsilon}\left(\frac{R^2}{\nu_{\alpha,R}{}^2|Q|}\int_{\frac{\nu_{\alpha,R}}{R}Q}\frac{f(\alpha^2)-f(|k|^2)}{R^2}\frac{k\cdot(x-z-k)}{|x-z-k|^2}dz\right)+O\left(\frac{1}{\varepsilon^2R^2}\right).$$

Using the fact that f is smooth in $[0,\alpha]$, and recalling that the sum is a sum over the set $\frac{\ell}{R}\cap B_\alpha\cap B_\varepsilon(x)^c$, we find that it converges to the corresponding integral, namely

$$\lim_{R\to+\infty}T_R(x)=\frac{1}{|Q|}\int_{B_\alpha}\left(f(\alpha^2)-f(|y|^2)\right)\frac{y\cdot(x-y)}{|x-y|^2}dy.$$

We then point out that $\frac{x-y}{|x-y|^2}=-\nabla_y\ln|x-y|$, so that, integrating by parts, we have

$$\lim_{R\to+\infty}w_R(Rx)=\frac{1}{|Q|}\int_{B_\alpha}\mathrm{div}\left(f(\alpha^2)-f(|y|^2)y\right)\ln|x-y|dy.$$

This limit is a radially symmetric function, which solves the partial differential equation $\Delta u = \frac{2\pi}{|Q|}\mathrm{div}\left(f(\alpha^2) - f(|y|^2)y\right)$ in B_α, $\Delta u = 0$ elsewhere. For any primitive F of f, the function $\frac{1}{2}\left(f(\alpha^2)|y|^2 - F(|y|^2)\right)\mathbf{1}_{B_\alpha}(y) + \frac{1}{2}\left(\alpha^2 f(\alpha^2) - F(\alpha^2)\right)\mathbf{1}_{B_\alpha^c}(y)$ is such a solution, so we have (3.9) almost everywhere. In addition, the above proof allows to bound $2w_R(Rx) + \left(\frac{1}{\nu_{\alpha,R}^2} - 1\right)R^2|x|^2$ by $C - \frac{f(\alpha^2)}{4}|x|^2$ for some constant C independent of R and x, which allows to apply the dominated convergence theorem.

We now prove (3.8). We fix $n = 1$, the general proof following exactly the same pattern. It is sufficient to show that the following convergence holds for any measurable bounded set D:

$$\int_D e^{v_R(Rx)} \underset{R\to+\infty}{\longrightarrow} |D| \fint e^v. \tag{3.20}$$

Hence, we are going to prove that for any $a > 0$,

$$\left|e^{v_R(x)} - e^{v(x)}\right| \leq C\frac{1+\sqrt{|x|}}{R} \quad \text{for} \quad |x| \leq aR. \tag{3.21}$$

This, together with the fact that $e^{v(Rx)}$ converges in L^∞ weak-* to $\fint e^v$ (because e^v is continuous and periodic), will give (3.20). Let j_x the point of ℓ which is the closest to x. As R goes to infinity, $|(\lambda_R(|j_x|) - 1)j_x| = O\left(\frac{1}{R}\right)$ uniformly with respect to x since $|x| \leq aR$. Hence, for R large enough, $\lambda_R(|j_x|)j_x$ is the closest to x among all $\lambda_R(|j|)j$, $j \in \ell$. Hence, for $j \in \ell \setminus \{j_x\}$, we have, for some $\varepsilon > 0$,

$$\forall y \in Q, \quad |x - j - y| \geq \varepsilon \quad \text{and} \quad |x - j - \lambda_R(|j|)(j+y)| \geq \varepsilon. \tag{3.22}$$

We then isolate j_x in the sum defining v_R, and write

$$\left|e^{v_R(x)} - e^{v(x)}\right| \leq \left|e^{g_R(x-\lambda_R(|j_x|)j_x)} - e^{g_R(x-j_x)}\right| e^{\sum_{j\neq j_x} g_R(x-\lambda_R(|j|)j)} \tag{3.23}$$

$$+e^{g_R(x-j_x)}\left|e^{\sum_{j\neq j_x} g_R(x-\lambda_R(|j|)j)} - e^{\sum_{j\neq j_x} g_R(x-j)}\right|, \tag{3.24}$$

where $g_R(z) = g\left(\frac{z}{\nu_{\alpha,R}}\right)$. We first bound (3.23). For this purpose, we point out that, according to Lemma 2.2, one can find a constant C_ε such that

$$\forall z \in B_\varepsilon^c, \quad |g(z)| \leq \frac{C_\varepsilon}{|z|^3}. \tag{3.25}$$

Hence, the sum appearing in (3.23) may be bounded as follows:

$$\sum_{j\neq j_x} |g_R(x - \lambda_R(|j|)j)| \leq \sum_{j\neq j_x} \frac{C_\varepsilon {\nu_{\alpha,R}}^3}{|x - \lambda_R(|j|)j|^3},$$

which is bounded independently of R. Moreover, we have

$$\begin{aligned}
\left|e^{g_R(x-\lambda_R(|j_x|)j_x)} - e^{g_R(x-j_x)}\right| &\leq C\left||x - \lambda_R(|j_x|)j_x| - |x - j_x|\right| \\
&\quad + C\left|\int_Q \left(\ln\frac{|x-y-j_x|}{|x-y-\lambda_R(|j_x|)j_x|}\right)dy\right| \\
&\leq C\left|(1 - \lambda_R(|j_x|)j_x\right| \\
&\quad + C\int_Q\left(\frac{1}{|x-y-j_x|} + \frac{1}{|x-y-\lambda_R(|j_x|)j_x|}\right)|1-\lambda_R(|j_x|)||j_x|dy \\
&\leq C\frac{|x|}{R^2} + C\frac{|x|}{R^2}\int_{B_3}\frac{dy}{|y|} \leq C\frac{|x|}{R^2}
\end{aligned}$$

Hence, the left-hand side of (3.23) is bounded by $C\frac{|x|}{R^2}$. Next, we deal with (3.24). Since g is bounded from above, it is sufficient to show the following:

$$\left|\sum_{j\neq j_x} g_R(x-\lambda_R(|j|)j) - \sum_{j\neq j_x} g_R(x-j)\right| \leq C\frac{1+\sqrt{|x|}}{R} \tag{3.26}$$

In order to prove (3.26), we define $A>0$ depending on R and x, to be fixed later on, and distinguish in the above sum between terms for which $|j-j_x|\leq A$ and those for which $|j-j_x|>A$. We have:

$$\begin{aligned}\sum_{0<|j-j_x|\leq A} |g_R(x-\lambda_R(|j|)j) - g_R(x-j)| &\leq \|\nabla g\|_{L^\infty(B_\varepsilon^c)} \sum_{0<|j-j_x|\leq A} |j||\lambda_R(|j|)-1| \\ &\leq \frac{C}{R^2}\sum_{0<|j-j_x|\leq A} |j| \leq \frac{C}{R^2}A^2(|x|+A).\end{aligned}$$

We have used here the fact that g is Lipschitz continuous in B_ε^c. Considering the case $|j-j_x|>A$, we have, using (3.25):

$$\sum_{A>|j-j_x|} |g_R(x-\lambda_R(|j|)j) - g_R(x-j)| \leq \sum_{A>|j-j_x|} \frac{C}{|x-j|^3} \leq \frac{C}{A}.$$

We thus may bound the left-hand side of (3.26) by $\frac{C}{A}+\frac{CA^2|x|}{R^2}+\frac{CA^3}{R^2}$. Choosing $A=\frac{\sqrt{R}}{1+|x|^{\frac{1}{4}}}$, we thus find (3.26), thereby concluding the proof of (3.21). □

4 Other trapping potentials

In the previous sections, we have studied a harmonic confinement, which is the case of most current experiments. As announced in [2], one could imagine a more general trapping potential, where in (1.1), $(1-\Omega^2)r^2/2$ is replaced by $(1-\Omega^2)r^2/2+W(r)$, and perform a similar analysis. Then, the limiting distribution replacing the inverted parabola should be

$$|\psi|^2 = \left(\frac{\mu-(1-\Omega)r^2-W(r)}{Nab}\right)_+ \tag{4.1}$$

where μ is such that $\int|\psi|^2=1$. There are two necessary conditions to apply our previous analysis: we need a small parameter (replacing $1-\Omega$) such that $E_{LLL}(\psi)-\Omega$ is small and the extent of the condensate (where $|\psi|^2$ is non zero) is large. The first condition is required so that the lowest Landau level is indeed a good approximation, and the second to apply the double scale convergence.

In recent experiments [10, 30], $W(r)=kr^4/4$. One can check that if $\Omega>\Omega_c=1+\sqrt{\Delta}$, where $\Delta=(3k^2Nab/8\pi)^{2/3}$, then the limiting distribution (4.1) has its support in an annulus of inner and outer radii $R_\pm=2(\Omega-1\pm\sqrt{\Delta})/k$. An interesting regime to study is when k is small and $\Omega-1=\alpha k^{2/3}$, with α such that $\Omega>\Omega_c$. Then the large scaling parameter replacing R is $k^{-1/6}$, which is the order of magnitude of $R_\pm$. The vortex lattice is located in the annulus (R_-,R_+) and is distorted towards the inner and outer edges, the inner disk corresponding to a giant vortex.

This approach does not allow to study the case where Ω is large and the annulus gets thin [16], since in that case, we are no longer in the setting to apply double scale convergence: there are few circles of vortices in the condensate.

Acknowledgements: This work was motivated by many fruitful discussions with Jean Dalibard and the joint paper [2]. We would like to thank him warmly for all his comments and suggestions.

References

[1] Abo-Shaeer JR, Raman C,Vogels JM, Ketterle W, Observation of Vortex Lattices in Bose-Einstein Condensates (2001) Science **292**, 476-479.

[2] Aftalion A, Blanc X, Dalibard J, Vortex patterns in a fast rotating Bose-Einstein condensate (2004) cond-mat/0410665. To appear in Phys. Rev. A

[3] Aftalion A, Jerrard RL, Properties of a single vortex solution in a rotating Bose Einstein condensate, (2003) C.R. Acad. Sci. Paris, Ser.I, 336.

[4] Aftalion A, Riviere T, Vortex energy and vortex bending for a rotating Bose-Einstein condensate, (2001) Phys. Rev. A **64**, 043611.

[5] Allaire G, Homogenization and two-scale convergence. SIAM J. Math. Anal. **23** 1482-1518 (1992).

[6] Baym G, Pethick CJ, Vortex core structure and global properties of rapidly rotating Bose-Einstein condensates, Phys. Rev. A, **69** (2004).

[7] Bethuel F, Brezis H, Helein F (1994) Ginzburg-Landau vortices, Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston, Inc., Boston, MA.

[8] Bleher, PM, Lebowitz JL, Energy-level statistics of model quantum systems: universality and scaling in a lattice-point problem. J. Statist. Phys. 74 (1994), no. 1-2, 167–217.

[9] Butts D, Rokhsar R, Predicted signatures of rotating Bose -Einstein condensates, (1999) Nature **397**, 327.

[10] Bretin V, Stock S, Seurin Y, Dalibard J, Fast Rotation of a Bose-Einstein Condensate Phys. Rev. Lett. **92**, 050403 (2004).

[11] Coddington et al, Experimental studies of equilibrium vortex properties in a Bose-condensed gas, cond-mat/0405240.

[12] Cooper NR, Komineas S, Read N,Vortex lattices in the lowest Landau level for confined Bose-Einstein condensates cond-mat/0404112.

[13] Cooper NR, Wilkin NK, Gunn JMF, Quantum Phases of Vortices in Rotating Bose-Einstein Condensates, Phys. Rev. Lett. **87**, 120405 (2001).

[14] Donnelly RJ, Quantized Vortices in Helium II, (Cambridge, 1991), Chaps. 4 and 5.

[15] Engels P et al, Observation of Long-lived Vortex Aggregates in Rapidly Rotating Bose-Einstein Condensates, Phys. Rev. Lett. **90**, 170405 (2003).

[16] Fetter AL, Jackson B, and Stringari S, cond-mat/0407119.

[17] Fischer UR, and Baym G, Vortex states of rapidly rotating dilute Bose-Einstein condensates, Phys. Rev. Lett. **90**, 140402 (2003).

[18] Ho TL, Bose-Einstein Condensates with Large Number of Vortices, Phys. Rev. Lett. **87** 060403 (2001).

[19] Kleiner WH, Roth LM, Autler SH, Bulk Solution of Ginzburg-Landau Equations for Type II Superconductors: Upper Critical Field Region, Phys. Rev. **133**, A1226, (1964).

[20] Lieb E, Seiringer R, Yngvason J, Bosons in a trap: A rigorous derivation of the Gross-Pitaevskii energy functional (2000) Phys. Rev. A **61**, 0436021.

[21] Lu K, Pan XB, Gauge invariant eigenvalue problems in R^2 and in R^2_+. Trans. Amer. Math. Soc. 352 (2000), no. 3, 1247–1276.

[22] Madison K, Chevy F, Bretin V, Dalibard J, Vortex Formation in a Stirred Bose-Einstein Condensate, (2000) Phys. Rev. Lett., **84**, 806.

[23] Matthews MR et al. Vortices in a Bose-Einstein Condensate, (1999) Phys. Rev. Lett., **83**, 2498.

[24] Olshanii M, Atomic Scattering in the Presence of an External Confinement and a Gas of Impenetrable Bosons, (1998) Phys.Rev.Lett **81**, 938-941.

[25] Raman C, Abo-Shaeer JR, Vogels JM, Xu K, Ketterle W, Vortex Nucleation in a Stirred Bose-Einstein Condensate (2001) Phys. Rev. Lett. **87**, 210402.

[26] Schnee K, Yngvason, Bosons in Disc-Shaped Traps: From 3D to 2D preprint (2004).

[27] V. Schweikhard, I. Coddington, P. Engels, V. P. Mogendorff, and E. A. Cornell, Rapidly Rotating Bose-Einstein Condensates in and near the Lowest Landau Level, Phys. Rev. Lett. **92**, 040404 (2004).

[28] Sheehy DE, Radzihovsky L, Vortices in Spatially Inhomogeneous Superfluids, (2004) cond-mat/0406205.

[29] Sinova J, Hanna CB, MacDonald AH, Quantum Melting and Absence of Bose-Einstein Condensation in Two-Dimensional Vortex Matter, Phys. Rev. Lett. **89**, 030403 (2002).

[30] Stock S, Bretin V, Chevy F, Dalibard J, Shape oscillation of a rotating Bose-Einstein condensate, Europhys.Lett. **65** 594 (2004).

[31] Watanabe G, Baym G, Pethick CJ, Landau levels and the Thomas-Fermi structure of rapidly rotating Bose-Einstein condensates, cond-mat/0403470 to appear in Phys. Rev. Lett.