The mortar spectral element method
in domains of operators

Part I: The divergence operator and Darcy’s equations

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Abstract: The mortar spectral element method is a domain decomposition technique that allows for discretizing second- or fourth-order elliptic equations when set in standard Sobolev spaces. The aim of this paper is to extend this method to some problems formulated in spaces of square-integrable functions with square-integrable divergence. A discretization of the equations due to Darcy which model the flow in porous media is proposed. The numerical analysis of the discrete problem is performed and numerical experiments are presented, they turn out to be in good coherency with the theoretical results.

Résumé: La méthode d’éléments spectraux avec joints est une technique de décomposition de domaine permettant de discrétiser des équations elliptiques d’ordre 2 ou 4 posées dans des espaces de Sobolev usuels. Le but de cet article est d’étendre cette méthode à certains problèmes variationnels formulés dans des espaces de fonctions de carré intégrable à divergence de carré intégrable. On propose une discrétisation des équations de Darcy qui modélisent l’écoulement dans des milieux poreux, on en effectue l’analyse numérique et on présente des expériences numériques cohérentes avec les résultats de l’analyse.

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1. Introduction.

The mortar element method, due to Bernardi, Maday and Patera [16], is a domain decomposition technique which allows for working on general partitions of the domain, without conformity restrictions. It is specially important when combined with spectral type discretizations, since handling complex geometries from very simple subdomains can be performed with this method in a very efficient way. It can also be used to couple different kinds of variational discretizations on the subdomains, such as finite elements or spectral methods. So it leads to discrete problems which are most often non conforming in the Hodge sense, which means that the discrete space is not contained in the variational one. It was firstly analyzed in the case of the two-dimensional Laplace equation [16][17] which admits a natural variational formulation in the usual Sobolev space $H^1(\Omega)$ of functions with square-integrable first-order derivatives. We also refer to [9] for the first three-dimensional results. It was extended [6] to the bilaplacian equation where the variational space is the standard space $H^2(\Omega)$ of functions with square-integrable first-order and second-order derivatives and also to the Stokes problem which is of saddle-point type, however it still involves usual Sobolev spaces. We also quote [7] for an application of the mortar technique to weighted Sobolev spaces, in order to handle discontinuous boundary conditions for the Navier–Stokes equations.

However a number of interesting problems involve other types of Hilbert spaces which are often domains of operators issued from mechanics and physics. Let us quote among them the spaces $H(\text{div}, \Omega)$ of square-integrable vector fields with square-integrable divergence and the space $H(\text{curl}, \Omega)$ of square-integrable vector fields with square-integrable curl. Up to now, the mortar method has not yet been applied in this case, except when associated with finite element discretizations [8][19][31]. The main difficulty for handling this new type of spaces is that the associated space of traces is made of non local distributions.

In this paper, we are interested in the mortar spectral element approximation of functions in $H(\text{div}, \Omega)$. The problem that we consider is Darcy’s system which models the flow of a viscous incompressible fluid in a porous medium. It is also involved in Goda’s projection algorithm [26] for the Navier–Stokes equations, so that analyzing its mortar element discretization seems important. From its variational formulation in $H(\text{div}, \Omega)$, we construct a mortar discrete problem relying on spectral techniques (we refer to [4] for the analysis of its spectral discretization without domain decomposition and to [3] for the analysis on a conforming decomposition) and prove that it admits a unique solution.

In the next step, we investigate the approximation properties of divergence-free functions in the mortar discrete space. Indeed these properties are needed for the numerical analysis of the discrete problem. Relying on this, we prove a priori error estimates of spectral type: the order of convergence only depends on the regularity of the solution, more precisely on its local regularity in each subdomain.

The implementation of the mortar technique mainly relies on an appropriate treatment of the matching conditions on the interfaces that we briefly describe (we refer to [10] for another way of handling these conditions). We write the resulting linear system and we present the algorithm which is used to solve it. Numerical experiments are described, we check that they are in good agreement with the theoretical results.
Part II of this work is devoted to the mortar spectral element discretization of problems formulated in $H(\text{curl}, \Omega)$. The analysis of a vector potential problem is presently under consideration, and the discretization relies on very similar ideas.

An outline of the paper is as follows.
- In Section 2, we recall the main properties of the space $H(\text{div}, \Omega)$.
- In Section 3, we present Darcy’s equations and prove the well-posedness of the equivalent variational problem.
- Section 4 is devoted to the description of the mortar spectral element discretization of these equations, and the well-posedness of the discrete problem is also checked.
- In Section 5, we derive some approximation properties of the mortar space, concerning mainly the approximation of divergence-free smooth functions by divergence-free piecewise polynomial functions.
- Error estimates between the exact and discrete solutions are derived from these properties in Section 6.
- Finally, we present some numerical experiments in Section 7.
2. Some properties of the space $H(\text{div}, \Omega)$.

Let $\Omega$ denote a bounded domain in $\mathbb{R}^d$, $d = 2$ or $3$, with a Lipschitz-continuous boundary. We denote by $n$ the unit outward normal to $\Omega$ on $\partial\Omega$. The generic point in $\Omega$ is denoted by $x = (x, y)$ in the case $d = 2$, $x = (x, y, z)$ in the case $d = 3$, while the components of any vector field $v$ in $\mathbb{R}^d$, are denoted by $v_x$ and $v_y$ in the case $d = 2$, $v_x$, $v_y$ and $v_z$ in the case $d = 3$. However we most often state the notation and the proofs in the case $d = 3$, since the corresponding results in dimension $d = 2$ are easier.

Let us first introduce the divergence operator in the case of dimension $d = 3$:

$$\text{div } v = \partial_x v_x + \partial_y v_y + \partial_z v_z,$$

defined on all functions $v$ in $L^2(\Omega)^3$ in the distribution sense (as usual, $\mathcal{D}(\Omega)$ stands for the space of infinitely differentiable functions with a compact support in $\Omega$):

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \langle \text{div } v, \varphi \rangle = -\int_{\Omega} (v_x \partial_x \varphi + v_y \partial_y \varphi + v_z \partial_z \varphi)(x) \, dx.$$ 

This definition is the same in dimension $d = 2$, when taking $v_z$ equal to zero and forgetting the dependency with respect to $z$. With this operator, we can associate the space $H(\text{div}, \Omega) = \{ v \in L^2(\Omega)^d; \text{div } v \in L^2(\Omega) \}$, provided with the natural norm

$$\| v \|_{H(\text{div}, \Omega)} = \left( \| v \|^2_{L^2(\Omega)^d} + \| \text{div } v \|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

We note that $H(\text{div}, \Omega)$ is a Hilbert space and we recall from [25, Chap. I, Thm 2.4] or [32, Chap. 1, Thm 1.1] that the space $\mathcal{D}(\mathbb{R}^d)^d$ of restrictions of functions in $\mathcal{D}(\mathbb{R}^d)^d$ to $\Omega$ is dense in $H(\text{div}, \Omega)$. We also prove the trace theorem on $H(\text{div}, \Omega)$ in an obvious way.

**Proposition 2.1.** The trace operator: $v \mapsto v \cdot n$, defined from the formula

$$\forall \varphi \in H^1(\Omega), \quad \langle v \cdot n, \varphi \rangle = \int_{\Omega} (v \cdot \text{grad } \varphi + (\text{div } v) \varphi)(x) \, dx,$$

is continuous from $H(\text{div}, \Omega)$ onto the dual space $H^{-\frac{1}{2}}(\partial\Omega)$ of $H^{\frac{1}{2}}(\partial\Omega)$.

**Proof:** Thanks to the previous density result, we must only check that the trace mapping is onto. With any $\mu$ in $H^{-\frac{1}{2}}(\partial\Omega)$, we associate the solution $\psi$ of the Laplace equation with Neumann boundary condition

$$\begin{cases} -\Delta \psi = \lambda & \text{in } \Omega, \\ \partial_n \psi = \mu & \text{on } \partial\Omega, \end{cases}$$

where the constant $\lambda$ is chosen such that (here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$)

$$\lambda \, \text{meas (}\Omega) = -\langle \mu, 1 \rangle.$$
Clearly, the function \( v = \text{grad} \psi \) belongs to \( H(\text{div}, \Omega) \) and satisfies \( v \cdot n = \mu \), whence the result.

**Remark:** Let \( \Gamma \) be a connected part of \( \partial \Omega \) with a positive measure. Since the extension by zero is continuous from \( H^{1/2}_{00}(\Gamma) \) into \( H^{1/2}(\partial \Omega) \), the trace operator: \( v \mapsto v \cdot n \) is also continuous from \( H(\text{div}, \Omega) \) into the dual space of \( H^{1/2}_{00}(\Gamma) \), which we denote by \( H^{1/2}_{00}(\Gamma)^\prime \) (see [27, Chap. 1, Thm 11.7] for the definition of \( H^{1/2}_{00}(\Gamma) \)).

We can now define the subspace

\[
H_0(\text{div}, \Omega) = \{ v \in H(\text{div}, \Omega); \ v \cdot n = 0 \text{ on } \partial \Omega \}.
\]

Proposition 2.1 yields that it is also a Hilbert space. Moreover, it can be proven [25, Chap. I, Thm 2.6][32, Chap. 1, Thm 1.3] that \( \mathcal{D}(\Omega)^d \) is dense in \( H_0(\text{div}, \Omega) \).

Let us also introduce the curl operator, in the three-dimensional case for brevity of notation (the results are the same in dimension \( d = 2 \) but simpler): it is defined on smooth functions by

\[
\text{curl } v = \begin{pmatrix}
\partial_y v_z - \partial_z v_y \\
\partial_z v_x - \partial_x v_z \\
\partial_x v_y - \partial_y v_x
\end{pmatrix}.
\]

We consider its domain

\[
H(\text{curl}, \Omega) = \{ v \in L^2(\Omega)^3; \ \text{curl } v \in L^2(\Omega)^3 \}.
\]

Neither the space \( H(\text{div}, \Omega) \) nor the space \( H_0(\text{div}, \Omega) \) nor the intersection \( H(\text{div}, \Omega) \cap H(\text{curl}, \Omega) \) has further regularity properties. But the intersection \( H_0(\text{div}, \Omega) \cap H(\text{curl}, \Omega) \) is continuously imbedded in \( H^{1/2}(\Omega)^d \) [20] and, if the domain \( \Omega \) is convex, in \( H^1(\Omega)^d \) [1, Thm 2.17]. Further results are known [21][22][23] when \( \Omega \) is a polygon: a function \( u \) in \( H_0(\text{div}, \Omega) \cap H(\text{curl}, \Omega) \) can be written

\[
u = u_r + \text{grad } S,
\]

(2.4)

where \( u_r \) belongs to \( H^1(\Omega)^2 \) and \( S \) is a linear combination of the singularities of the Laplace equation provided with Neumann boundary conditions. We recall that each singularity in the neighbourhood of a corner of the polygon with aperture \( \omega \) has the form

\[
r^{\frac{\pi}{\omega}} (\varphi(\theta) + (\log r)^p \psi(\theta)),
\]

where \( r \) is the distance to the corner, \( \theta \) the corresponding angular variable, \( p \) is equal to 0 except when \( \frac{\pi}{\omega} \) is an integer where it is equal to 1. Finally, \( \varphi \) and \( \psi \) are combinations of cosine functions, in order that \( \text{grad } S \) satisfies the same nullity condition as \( u \) on the boundary edges \( \theta = 0 \) and \( \theta = \omega \). More generally, any such function \( u \) which has the further property

\[
\text{div } u \in H^s(\Omega), \quad \text{curl } u \in H^s(\Omega)^3,
\]

(2.5)

admits the expansion (2.4) with \( u_r \) in \( H^{s+1}(\Omega)^2 \) for all \( s, 0 < s < \frac{2\pi}{\omega} - 1 \).
3. The continuous Darcy’s equations.

As previously, \( \Omega \) denotes a bounded domain in \( \mathbb{R}^d \), \( d = 2 \) or \( 3 \), with a Lipschitz–continuous boundary. We consider Darcy’s equations

\[
\begin{aligned}
\begin{cases}
    \boldsymbol{u} + \text{grad } p &= \boldsymbol{f} & \text{in } \Omega, \\
    \text{div } \boldsymbol{u} &= 0 & \text{in } \Omega, \\
    \boldsymbol{u} \cdot \boldsymbol{n} &= 0 & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]

where the unknowns are the velocity \( \boldsymbol{u} \) and the pressure \( p \).

As noted in [4], system (3.1) admits several equivalent variational formulations, which lead to different discrete problems. However, in order that the discretization described below can be used in the projection part of Goda’s algorithm for the Stokes problem and without restriction for the application to porous media, we have rather work with the following one:

Find \((\boldsymbol{u}, p)\) in \( H_0(\text{div}, \Omega) \times L^2_0(\Omega) \) such that

\[
\begin{aligned}
\forall \boldsymbol{v} \in H_0(\text{div}, \Omega), & \quad \int_{\Omega} \boldsymbol{u} \cdot \boldsymbol{v} \, d\boldsymbol{x} - \int_{\Omega} p \,(\text{div } \boldsymbol{v}) \, d\boldsymbol{x} = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v} \, d\boldsymbol{x}, \\
\forall q \in L^2_0(\Omega), & \quad -\int_{\Omega} q \,(\text{div } \boldsymbol{u}) \, d\boldsymbol{x} = 0,
\end{aligned}
\]

where for simplicity we assume that the data \( \boldsymbol{f} \) belong to \( L^2(\Omega)^d \) and \( L^2_0(\Omega) \) is defined by

\[
L^2_0(\Omega) = \{ q \in L^2(\Omega); \int_{\Omega} q \, d\boldsymbol{x} = 0 \}.
\]

It follows from the density results quoted in Section 2 that problem (3.2) is equivalent to system (3.1).

Problem (3.2) is of saddle-point type. So, we introduce the kernel

\[
V = \left\{ \boldsymbol{v} \in H_0(\text{div}, \Omega); \forall q \in L^2_0(\Omega), \int_{\Omega} q \,(\text{div } \boldsymbol{v}) \, d\boldsymbol{x} = 0 \right\}.
\]

It is readily checked that

\[
V = \left\{ \boldsymbol{v} \in H_0(\text{div}, \Omega); \text{div } \boldsymbol{v} = 0 \text{ in } \Omega \right\},
\]

so that the norms \( \| \cdot \|_{H(\text{div}, \Omega)} \) and \( \| \cdot \|_{L^2(\Omega)^d} \) are equivalent on \( V \). This yields the ellipticity of the first bilinear form on \( V \). Moreover, the inf-sup condition of the second bilinear form is an obvious consequence of the standard one (where \( H_0(\text{div}, \Omega) \) is replaced by \( H^1_0(\Omega)^d \), see [25, Chap. I, Cor. 2.4]): There exists a positive constant \( \beta \) such that

\[
\forall q \in L^2_0(\Omega), \quad \sup_{\boldsymbol{v} \in H_0(\text{div}, \Omega)} \frac{-\int_{\Omega} q \,(\text{div } \boldsymbol{v}) \, d\boldsymbol{x}}{\| \boldsymbol{v} \|_{H(\text{div}, \Omega)}} \geq \beta \| q \|_{L^2(\Omega)}.
\]

Combining all this leads to the well-posedness result.
Proposition 3.1. For any data \( f \) in \( L^2(\Omega)^d \), problem (3.2) has a unique solution \((u, p)\) in \( H_0(\text{div}, \Omega) \times L^2_0(\Omega)\).

The regularity of the solution of problem (3.1) is derived by taking the curl of the first equation in (3.1), which yields

\[
\text{curl } u = \text{curl } f \quad \text{in } \Omega.
\]

Indeed, thanks to the results recalled in Section 2, for any data \( f \) in \( H(\text{curl}, \Omega) \), the solution \((u, p)\) of problem (3.1) belongs to \( H^s(\Omega)^d \times H^{s+1}(\Omega) \) with \( s = \frac{1}{2} \) in the general case and \( s = 1 \) if \( \Omega \) is convex. Moreover, when \( \Omega \) is a polygon, \( u \) admits the expansion (2.4), and, if \( \text{curl } f \) belongs to \( H^s(\Omega)^2 \), \( 0 \leq s < \frac{2\pi}{\omega} - 1 \), where \( \omega \) denotes the largest angle of \( \Omega \) (equal to either \( \frac{\pi}{2} \) or \( \frac{3\pi}{2} \) in what follows), the regular part \( u_r \) in this expansion belongs to \( H^{s+1}(\Omega)^2 \).
4. Discretization of Darcy’s equations.

We now assume that $\Omega$ admits a disjoint decomposition into a finite number of (open) rectangles in dimension $d = 2$, rectangular parallelepipeds in dimension $d = 3$, denoted by $\Omega_k$:

$$\Omega = \bigcup_{k=1}^{K} \Omega_k \quad \text{and} \quad \Omega_k \cap \Omega_{k'} = \emptyset, \quad 1 \leq k \neq k' \leq K. \quad (4.1)$$

Note that, as indicated in [28], the extension to more complex subdomains leads to similar results, however they involve very technical arguments that we prefer to avoid in this work.

We make the further assumption that the intersection of each $\partial \Omega_k$ with $\partial \Omega$, if not empty, is a corner, a whole edge or a whole face of $\Omega_k$. For $1 \leq k \leq K$, we denote by $\Gamma_{k,\ell}$, $1 \leq \ell \leq L(k)$, the (open) edges in dimension $d = 2$, faces in dimension $d = 3$, of $\Omega_k$ which are not contained in $\partial \Omega$. We denote by $n_k$ the unit outward normal vector to $\Omega_k$ on $\partial \Omega_k$.

Note that the decomposition is said to be conforming if the intersection of two different domains $\Omega_k$ is either empty or a corner or a whole edge or face of both of them, however we do not make this assumption since it is a priori not necessary for the mortar method.

Let us now introduce the skeleton $S$ of the decomposition, $S = \bigcup_{k=1}^{K} \partial \Omega_k \setminus \partial \Omega$. As suggested in [16][17], we choose a disjoint decomposition of this skeleton into mortars $\gamma_m$:

$$S = \bigcup_{m=1}^{M} \Gamma_m \quad \text{and} \quad \gamma_m \cap \gamma_{m'} = \emptyset, \quad 1 \leq m \neq m' \leq M, \quad (4.2)$$

where each $\gamma_m = \Gamma_{k(m),\ell(m)}$ is a whole edge in dimension $d = 2$, face in dimension $d = 3$, of a subdomain $\Omega_k$, denoted by $\Omega_k$.

To describe the discrete spaces, for each nonnegative integer $n$, we define on each $\Omega_k$, resp. $\Gamma_{k,\ell}$, the space $\mathbb{P}_n(\Omega_k)$, resp. $\mathbb{P}_n(\Gamma_{k,\ell})$, of restrictions to $\Omega_k$, resp. to $\Gamma_{k,\ell}$, of polynomials with $d$ variables, resp. $d - 1$ variables (the tangential coordinates on $\Gamma_{k,\ell}$), and degree $\leq n$ with respect to each variable. The discretization parameter $\delta$ is then a $K$–tuple of positive integers $(N_1, \ldots, N_K)$, with each $N_k \geq 2$.

From Proposition 2.1, $H_0(\text{div}, \Omega)$ coincides with the space of functions $v$ such that their restrictions to each $\Omega_k$, $1 \leq k \leq K$, belong to $H(\text{div}, \Omega_k)$ and their normal traces vanish on $\partial \Omega$ and are continuous through the skeleton $S$. In analogy with this definition, we define the corresponding discrete space $\mathbb{D}_\delta(\Omega)$ of functions $\mathbf{v}_\delta$ such that:

- their restrictions $\mathbf{v}_\delta|_{\Omega_k}$ to each $\Omega_k$, $1 \leq k \leq K$, belong to $\mathbb{P}_{N_k}(\Omega_k)^d$,
- their normal traces $\mathbf{v}_\delta \cdot n$ vanish on $\partial \Omega$,
- the mortar function $\varphi$ being defined on each $\gamma_m$, $1 \leq m \leq M$, by

$$\varphi|_{\gamma_m} = \mathbf{v}_\delta|_{\Omega_k} \cdot n_k(m),$$

the following matching condition holds on each edge $\Gamma_{k,\ell}$, $1 \leq k \leq K$, $1 \leq \ell \leq L(k)$, which is not a mortar:

$$\forall \chi \in \mathbb{P}_{N_k-2}(\Gamma_{k,\ell}), \quad \int_{\Gamma_{k,\ell}} (\mathbf{v}_\delta|_{\Omega_k} \cdot n_k + \varphi)(\tau) \chi(\tau) d\tau = 0. \quad (4.3)$$
We also introduce the space $\mathbb{M}_\delta(\Omega)$ of discrete pressures:

$$M_\delta(\Omega) = \left\{ q_\delta \in L^2_0(\Omega): q_\delta|_{\Omega_k} \in \mathbb{P}_{N_k-2}(\Omega_k), 1 \leq k \leq K \right\}. \quad (4.4)$$

Note that there are many different possible choices for this space, however this one is justified by two arguments: replacing $N_k - 2$ by $N_k$ would give rise to spurious modes on the pressure (see [4, Lemme 4.1]), so that this choice must be eliminated. If $M_\delta(\Omega)$ is defined by (4.4) but with $N_k - 2$ replaced by $N_k - 1$, it does not contain any spurious mode for the pressure of Darcy’s equations, so it can be kept when solving only this equation, however it contains spurious modes for the pressure of the Stokes problem [15, Thm 24.1]. So the previous choice seems reasonable, in order to be consistent with this problem when using Goda’s algorithm.

Starting from the standard Gauss–Lobatto formula on $]-1,1[\), we define on each $\Omega_k$ and in each direction:

- the nodes $x_i^k$ and $y_i^k$, and the weights $\rho_i^{x,k}$ and $\rho_i^{y,k}$, $0 \leq i \leq N_k$, in the case of dimension $d = 2$,
- the nodes $x_i^k$, $y_i^k$ and $z_i^k$, and the weights $\rho_i^{x,k}$, $\rho_i^{y,k}$ and $\rho_i^{z,k}$, $0 \leq i \leq N_k$, in the case of dimension $d = 3$.

A discrete product is then introduced on each $\Omega_k$ by

$$\langle u_\delta, v_\delta \rangle^k = \begin{cases} \sum_{i=0}^{N_k} \sum_{j=0}^{N_k} u_\delta(x_i^k, y_j^k) v_\delta(x_i^k, y_j^k) \rho_i^{x,k} \rho_j^{y,k} & \text{for } d = 2, \\ \sum_{i=0}^{N_k} \sum_{j=0}^{N_k} \sum_{p=0}^{N_k} u_\delta(x_i^k, y_j^k, z_p^k) v_\delta(x_i^k, y_j^k, z_p^k) \rho_i^{x,k} \rho_j^{y,k} \rho_p^{z,k} & \text{for } d = 3. \end{cases}$$

This leads to the global discrete product on $\Omega$:

$$\langle u_\delta, v_\delta \rangle_\delta = \sum_{k=1}^K \langle u_\delta, v_\delta \rangle^k, \quad (4.5)$$

which coincides with the scalar product of $L^2(\Omega)$ for all functions $u_\delta$ and $v_\delta$ such that each product $\langle u_\delta, v_\delta \rangle_{|\Omega_k}$, $1 \leq k \leq K$, belongs to $\mathbb{P}_{2N_k-1}(\Omega_k)$. We also define, for $1 \leq k \leq K$, $I_\delta^k$ as the Lagrange interpolation operator on all nodes $(x_i^k, y_j^k)$, $0 \leq i, j \leq N_k$, respectively $(x_i^k, y_j^k, z_p^k)$, $0 \leq i, j, p \leq N_k$, with values in $\mathbb{P}_{N_k}(\Omega_k)$, and finally the global operator $I_\delta$ by

$$\langle I_\delta v \rangle_{|\Omega_k} = I_\delta^k v_{|\Omega_k}, \quad 1 \leq k \leq K. \quad (4.6)$$

The discrete problem is now built from the variational formulation (3.2). For any continuous data $f$ on $\overline{\Omega}$, it reads:

\textbf{Find $(u_\delta, p_\delta)$ in $D_\delta(\Omega) \times M_\delta(\Omega)$ such that}

$$\forall v_\delta \in D_\delta(\Omega), \quad \langle u_\delta, v_\delta \rangle_\delta - (p_\delta, \text{div } v_\delta)_\delta = (f, v_\delta)_\delta, \quad (4.7)$$

\$$\forall q_\delta \in M_\delta(\Omega), \quad -\langle q_\delta, \text{div } u_\delta \rangle_\delta = 0.$$

Note however that, thanks to the exactness property of the quadrature formula, we have

$$\forall v_\delta \in D_\delta(\Omega), \forall q_\delta \in M_\delta(\Omega), \quad -\langle q_\delta, \text{div } v_\delta \rangle_\delta = - \int_\Omega q_\delta (\text{div } v_\delta) \, dx. \quad (4.8)$$
We first prove the inf-sup condition, that relies on the arguments given in [4, Lemme 4.2] and [12, Prop. 3.1] combined via the Boland and Nicolaides technique [18]. Since \( D_\delta(\Omega) \) is not contained in \( H(\text{div}, \Omega) \) in the general case, we introduce the “broken” norm:

\[
\|v\|_{H(\text{div}, \cup \Omega_k)} = \left( \sum_{k=1}^{K} \|v\|_{H(\text{div}, \Omega_k)}^2 \right)^{1/2}.
\]  

(4.9)

**Proposition 4.1.** There exists an integer \( N_D \) only depending on the decomposition of \( \Omega \) such that, if all the \( N_k \) are \( \geq N_D \), the following inf-sup condition

\[
\forall q_\delta \in M_\delta(\Omega), \quad \sup_{v_\delta \in D_\delta(\Omega)} \frac{- \int_{\Omega} q_\delta(\text{div } v_\delta) \, dx}{\|v_\delta\|_{H(\text{div}, \cup \Omega_k)}} \geq \beta_D \|q_\delta\|_{L^2(\Omega)},
\]

(4.10)

holds for a positive constant \( \beta_D \) depending on the decomposition of \( \Omega \) but not on \( \delta \).

**Proof:** Let \( q_\delta \) be any function in \( M_\delta(\Omega) \). The idea consists in writing \( q_\delta \) as

\[
q_\delta = \tilde{q}_\delta + \bar{q}_\delta, \quad \text{with } \tilde{q}_\delta|_{\Omega_k} = \frac{1}{\text{meas}(\Omega_k)} \int_{\Omega_k} q_\delta(x) \, dx, \quad 1 \leq k \leq K.
\]

This decomposition is orthogonal, in the sense that

\[
\|q_\delta\|_{L^2(\Omega)}^2 = \|\tilde{q}_\delta\|_{L^2(\Omega)}^2 + \|\bar{q}_\delta\|_{L^2(\Omega)}^2.
\]

(4.11)

Next, since \( \bar{q}_\delta|_{\Omega_k} \) has a null integral on \( \Omega_k \), it follows from [4, Lemme 4.2] that there exists a function \( v_k \) in \( P_{N_k}(\Omega_k) \cap H_0(\text{div}, \Omega_k) \) such that

\[
-\text{div } v_k = \bar{q}_\delta|_{\Omega_k} \quad \text{and} \quad \|v_k\|_{H(\text{div}, \Omega_k)} \leq \bar{c} \|\bar{q}_\delta|_{\Omega_k}\|_{L^2(\Omega_k)}.
\]

(4.12)

So, the function \( \tilde{v}_\delta \) defined by \( \tilde{v}_\delta|_{\Omega_k} = v_k, \quad 1 \leq k \leq K \), belongs to \( D_\delta(\Omega) \) (and even to \( H_0(\text{div}, \Omega) \)). Concerning the function \( \tilde{q}_\delta \) (note that its integral on \( \Omega \) is equal to the integral of \( q_\delta \), hence to zero), it follows from [12, Prop. 3.1] that there exists a function \( \bar{q}_\delta \) in \( H_0^1(\Omega)^d \), with \( \bar{q}_\delta|_{\Omega_k} \) in \( P_{N_D}(\Omega_k)^d \), such that

\[
-\text{div } \bar{q}_\delta = \tilde{q}_\delta \quad \text{and} \quad \|\bar{q}_\delta\|_{H(\text{div}, \Omega)} \leq \bar{c} \|\tilde{q}_\delta\|_{L^2(\Omega)}.
\]

(4.13)

This function \( \bar{q}_\delta \) belongs to \( D_\delta(\Omega) \) when all the \( N_k \) are \( \geq N_D \).

We finally take: \( v_\delta = \tilde{v}_\delta + \mu \bar{q}_\delta \), for a positive parameter \( \mu \). Indeed, \( \text{div } \tilde{v}_\delta \) is orthogonal to \( \tilde{q}_\delta \) (this comes from the Stokes formula applied on each \( \Omega_k \)), so that, from (4.12) and (4.13),

\[
- \int_{\Omega} q_\delta(\text{div } v_\delta) \, dx \geq \|\tilde{q}_\delta\|_{L^2(\Omega)}^2 + \mu \|\tilde{q}_\delta\|_{L^2(\Omega)}^2 - \mu \|\text{div } \bar{q}_\delta\|_{L^2(\Omega)} \|\bar{q}_\delta\|_{L^2(\Omega)}.
\]

Using once more (4.13) yields

\[
- \int_{\Omega} q_\delta(\text{div } v_\delta) \, dx \geq \|\tilde{q}_\delta\|_{L^2(\Omega)}^2 + \mu \|\tilde{q}_\delta\|_{L^2(\Omega)}^2 - \mu \|\bar{q}_\delta\|_{L^2(\Omega)} \|\bar{q}_\delta\|_{L^2(\Omega)}
\]

\[
\geq \frac{1}{2} \|\tilde{q}_\delta\|_{L^2(\Omega)}^2 + \mu (1 - \frac{\mu \|\bar{q}_\delta\|_{L^2(\Omega)}^2}{2}) \|\bar{q}_\delta\|_{L^2(\Omega)}^2.
\]

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So choosing \( \mu = 1/c^2 \) and using (4.11) gives
\[
- \int_{\Omega} q_\delta (\text{div } v_\delta) \, dx \geq c \| q_\delta \|_{L^2(\Omega)}^2.
\]
Next, from (4.12) and (4.13) combined with (4.11), it is readily checked that
\[
\| v_\delta \|_{H(\text{div}, \cup; \Omega_k)} \leq \tilde{c} \| \tilde{q}_\delta \|_{L^2(\Omega)} + \mu c \| q_\delta \|_{L^2(\Omega)} \leq c \| q_\delta \|_{L^2(\Omega)},
\]
which ends the proof.

**Remark:** As explained in [12, form. (3.3) & (3.5)], the integer \( N_D \) can be computed explicitly as a function of the geometry of decomposition at least in dimension \( d = 2 \).

The well-posedness of problem (4.7) follows from Proposition 4.1.

**Proposition 4.2.** Assume that all the \( N_k, 1 \leq k \leq K \), are \( \geq N_D \). For any data \( f \) continuous on \( \overline{\Omega} \), problem (4.7) has a unique solution \( (u_\delta, p_\delta) \) in \( D_\delta(\Omega) \times M_\delta(\Omega) \).

**Proof:** Problem (4.7) results into a square linear system, hence the existence of a solution follows from its uniqueness. So we now take \( f \) equal to zero. Choosing \( v_\delta \) equal to \( u_\delta \) yields that
\[
(u_\delta, u_\delta)_\delta = 0,
\]
or equivalently, since the weights of the Gauss–Lobatto formula are positive, that each \( u_\delta \mid_{\Omega_k} \) vanishes in the \((N_k + 1)^d\) different points of a tensorized grid. Since it belongs to \( P_{N_k}(\Omega_k)^d \), it is zero. Next Proposition 4.1 yields that \( p_\delta \) is also equal to zero, which ends the proof.

To conclude, we introduce the discrete kernel
\[
V_\delta(\Omega) = \left\{ v_\delta \in D_\delta(\Omega); \forall q_\delta \in M_\delta(\Omega), \int_{\Omega} q_\delta (\text{div } v_\delta) \, dx = 0 \right\}. \tag{4.14}
\]
As usual, it plays a key role in the numerical analysis of problem (4.7). Moreover it is not contained in the space \( V \) introduced in (3.3) for the discretization that we have chosen.
5. Approximation of divergence-free functions.

We now intend to estimate the distance of a divergence-free function to the space \( \mathbb{V}_\delta(\Omega) \) introduced in (4.14). Here, we consider separately the cases of dimension \( d = 2 \) and dimension \( d = 3 \). For each \( \delta = (N_1, \ldots, N_K) \), we define the parameter \( \mu_\delta \)

- equal to 1 in the case of a conforming decomposition,
- equal to the maximum of the ratios \( N_k/N_{k'} \) for all adjacent subdomains \( \Omega_k \) and \( \Omega_{k'} \), \( 1 \leq k, k' \leq K \) (here, “adjacent” means that \( \Omega_k \) and \( \Omega_{k'} \) share a part of an edge in dimension \( d = 2 \), of a face in dimension \( d = 3 \)).

**Proposition 5.1.** Assume the function \( u \) in \( V \) such that each \( u|_{\Omega_k} \), \( 1 \leq k \leq K \), belongs to \( H^{s_k}(\Omega_k)^d \), \( s_k \geq \frac{1}{2} \). In the case of dimension \( d = 2 \), there exists a constant \( c \) independent of \( \delta \) such that
\[
\inf_{v_\delta \in \mathbb{V}_\delta(\Omega)} \| u - v_\delta \|_{L^2(\Omega)^d} \leq c \mu_\delta^{1/2} \sum_{k=1}^{K} N_k^{-s_k} \| u \|_{H^{s_k}(\Omega_k)^d}. \tag{5.1}
\]

**Proof:** Since \( u \) is divergence-free, there exists a stream–function \( \psi \) in \( H^1(\Omega) \) such that \( u = \text{curl } \psi \), with each \( \psi|_{\Omega_k} \) in \( H^{s_k+1}(\Omega_k) \) and \( \psi \) constant on each connected component of \( \partial \Omega \). It is also known [16, Appendix B] that there exists a function \( \psi_\delta \):

- which is equal to \( \psi \) on \( \partial \Omega \),
- which is continuous in all the corners of the \( \Omega_k \),
- such that each \( \psi_\delta|_{\Omega_k} \) belongs to \( \mathbb{P}_{N_k}(\Omega_k) \) and satisfies on each \( \Gamma_{k,\ell} \) (here, \( \bar{\varphi} \) stands for the mortar function associated with \( \psi_\delta|_{\Omega_k(m)} \) on each \( \gamma_m \))
\[
\forall \chi \in \mathbb{P}_{N_k-2}(\Gamma_{k,\ell}), \quad \int_{\Gamma_{k,\ell}} (\psi_\delta - \bar{\varphi}) \chi d\tau = 0,
\]
- and finally which satisfies
\[
\| \psi - \psi_\delta \|_{H^1(\bigcup \Omega_k)} \leq c \mu_\delta^{1/2} \sum_{k=1}^{K} N_k^{-s_k} \| \psi \|_{H^{s_k+1}(\Omega_k)^d}.
\]

Taking \( v_\delta = \text{curl } \psi_\delta \) gives estimate (5.1). So, we only must check that it belongs to \( \mathbb{D}_\delta(\Omega) \), more precisely that it satisfies the matching condition (4.3). We first observe that, if \( \tau \) denotes the unit vector directly orthogonal to \( n_{k(m)} \), the mortar function associated with \( v_\delta \) is defined by, for \( 1 \leq m \leq M \),
\[
\varphi = v_\delta|_{\Omega_{k(m)}} \cdot n_{k(m)} = \partial_\tau \psi_\delta|_{\Omega_{k(m)}} = \bar{\varphi}'.
\]

So, we have on each \( \Gamma_{k,\ell} \) which is not a mortar (recall that \( \psi_\delta|_{\Omega_k} - \bar{\varphi} \) vanishes at each endpoint of \( \Gamma_{k,\ell} \))
\[
\forall \chi \in \mathbb{P}_{N_k-2}(\Gamma_{k,\ell}), \quad \int_{\Gamma_{k,\ell}} (v_\delta|_{\Omega_k} \cdot n_k + \varphi) \chi d\tau = -\int_{\Gamma_{k,\ell}} (\partial_\tau \psi_\delta|_{\Omega_k} - \bar{\varphi}') \chi d\tau
\]
\[
= \int_{\Gamma_{k,\ell}} (\psi_\delta|_{\Omega_k} - \bar{\varphi}) \chi' d\tau = 0,
\]

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which is the desired matching condition. Moreover, in the case of a conforming decomposition, there exists [16, §3.1] a function $\psi_\delta$ in $H^1(\Omega)$ which satisfies all the previous properties with $\mu_\delta = 1$.

Now, we are interested in the case of dimension $d = 3$, which is much more complex. The proof mainly relies on special interpolation operators, which has been introduced by Nédélec [29] in the finite element framework and extended to spectral methods in [11]. We begin by proving approximation results involving the mortar space $\mathcal{C}_\delta(\Omega)$ in the finite element framework and extended to spectral methods in [11].

The proof mainly relies on special interpolation operators, which has been introduced by Nédélec [29] in the finite element framework and extended to spectral methods in [11]. The proof mainly relies on special interpolation operators, which has been introduced by Nédélec [29] in the finite element framework and extended to spectral methods in [11].

Proof: It follows from [11, Thm 4.9] that there exists, for $1 \leq k \leq K$, a polynomial $\xi_k^\delta$ in $\mathbb{P}_{N_k}(\Omega)^3$ such that $\xi_k^\delta \times n$ vanishes on $\partial \Omega \cap \partial \Omega_k$ and that

$$\|\text{curl } \xi - \text{curl } \xi_k^\delta\|_{L^2(\Omega)^3} \leq c N_k^{-s_k} \|u\|_{H^{s_k}(\Omega)^d}. \quad (5.4)$$

Moreover, this function $\xi_k^\delta$ satisfies, for $1 \leq \ell \leq L(k)$,

$$\forall q \in \mathbb{P}_{N_k-2}(\Gamma_k,\ell)^3, \quad \int_{\Gamma_k,\ell} (\psi - \xi_k^\delta) \times n \cdot q \, d\tau = 0. \quad (5.5)$$

In the case of a conforming decomposition and thanks to the assumption on the $N_k$, equation (5.5) implies that the function $\xi_\delta$ defined by $\xi_\delta|_{\Omega_k} = \xi_k^\delta$, $1 \leq k \leq K$, belongs to...
\(C_\delta(\Omega)\), whence the result.

In the case of a nonconforming decomposition, the previous approximation \(\xi_\delta\) does not belong to \(C_\delta(\Omega)\) but can be used in the first step of the construction of an approximation in this space. However proving this result seems to require the same restrictions on the decomposition as for more standard approximation results in \(H^1(\Omega)\), see [10, Chap. 3] or [13]. Moreover the final result does not take into account the local regularity of the function \(u\). So we have rather not present the result here.

A further argument seems necessary to prove the desired approximation result in the case \(d = 3\).

**Proposition 5.3.** Assume the function \(u\) in \(V\) such that each \(u|_{\Omega_k}, 1 \leq k \leq K\), belongs to \(H^{s_k}(\Omega_k)^d, s_k \geq \frac{3}{2}\). In the case of dimension \(d = 3\), if the decomposition is conforming and if moreover for each mortar \(\gamma^m, 1 \leq m \leq M\), which is a face of both \(\Omega_{k(m)}\) and \(\Omega_k\), \(N_{k(m)}\) is \(\geq N_k\), there exists a constant \(c\) independent of \(\delta\) such that

\[
\inf_{v_\delta \in \nabla_\delta(\Omega)} \| u - v_\delta \|_{L^2(\Omega)^d} \leq c \sum_{k=1}^K N_k^{-s_k} \| u \|_{H^{s_k}(\Omega_k)^d}, \tag{5.6}
\]

**Proof:** It follows from [1, §3.b] that there exists a vector potential \(\xi\) such that \(u = \text{curl} \ \xi\) (it does not necessarily satisfy the boundary condition \(\xi \times n = 0\) on \(\partial \Omega\) when \(\Omega\) is multiply-connected). However the arguments of Lemma 5.2 still hold in this case, and estimate (5.3) yields that the function \(v_\delta = \text{curl} \ \xi_\delta\) satisfies the desired property (5.6). Moreover, it can be checked [11, §4] that, on each face \(\Gamma\) of \(\Omega_k\), \(\text{curl} \ \xi_\delta \cdot n\) is equal to the orthogonal projection (in \(L^2(\Gamma)\)) of \(u \cdot n\) onto \(P_{N-1}(\Gamma)\). So the jump of \(\text{curl} \ \xi_\delta \cdot n\) through \(\Gamma\) (or its trace if \(\Gamma\) is contained in \(\partial \Omega\)) is zero, and the function \(v_\delta\) belongs to \(\nabla_\delta(\Omega)\).

Note that, for a conforming decomposition, the condition \(N_{k(m)} \geq N_k\) is not at all restrictive since the choice of the mortar side is fully arbitrary. However enforcing the conformity of the decomposition is a little disappointing and the conditions \(s_k \geq \frac{3}{2}\) are not satisfied by all solutions (it is likely but not proven that estimate (5.6) holds for all \(s_k \geq 0\)).

We prove an error estimate, first for the velocity, second for the pressure. Let \( w_\delta \) be any function in the kernel \( V_\delta(\Omega) \). Multiplying the first line of (3.1) by \( w_\delta \) gives

\[
\int_{\Omega} u \cdot w_\delta \, dx - \int_{\Omega} p(\text{div} \, w_\delta) \, dx = \int_{\Omega} f \cdot w_\delta \, dx - \int_{S} [w_\delta \cdot n] \, p \, d\tau,
\]

where the notation \([\cdot]\) means the jump across \( S \). Using the definition of \( V_\delta(\Omega) \) thus implies, for any \( q_\delta \) in \( M_\delta(\Omega) \),

\[
\int_{\Omega} u \cdot w_\delta \, dx - \int_{\Omega} (p - q_\delta)(\text{div} \, w_\delta) \, dx = \int_{\Omega} f \cdot w_\delta \, dx - \int_{S} [w_\delta \cdot n] \, p \, d\tau. \tag{6.1}
\]

Next, we recall from the standard property of the Gauss–Lobatto formula [15, Rem. 13.3] that

\[
\forall z_\delta \in P_{N_k}(\Omega_k), \quad \|z_\delta\|_{L^2(\Omega_k)}^2 \leq (z_\delta, z_\delta)_\delta^k \leq 3^d \|z_\delta\|_{L^2(\Omega_k)}^2. \tag{6.2}
\]

So, we have for any \( v_\delta \) in \( V_\delta \)

\[
\|u_\delta - v_\delta\|_{L^2(\Omega)}^2 \leq (u_\delta - v_\delta, u_\delta - v_\delta)_\delta.
\]

Adding (6.1) with \( w_\delta = u_\delta - v_\delta \) and subtracting the first line of (4.7) leads to

\[
\|u_\delta - v_\delta\|_{L^2(\Omega)}^2 \leq \int_{\Omega} (u - v_\delta) \cdot (u_\delta - v_\delta) \, dx \\
+ \int_{\Omega} v_\delta \cdot (u_\delta - v_\delta) \, dx - (v_\delta, u_\delta - v_\delta)_\delta \\
- \int_{\Omega} (p - q_\delta)(\text{div} \, (u_\delta - v_\delta)) \, dx \\
+ (f, u_\delta - v_\delta)_\delta - \int_{\Omega} f \cdot (u_\delta - v_\delta) \, dx \\
+ \int_{S} [(u_\delta - v_\delta) \cdot n] \, p \, d\tau.
\]

The last idea consists in integrating once more by parts the term in the third line, which gives

\[
- \int_{\Omega} (p - q_\delta)(\text{div} \, (u_\delta - v_\delta)) \, dx = \int_{\Omega} (u_\delta - v_\delta) \cdot \text{grad} \, (p - q_\delta) \, dx \\
- \int_{S} [(u_\delta - v_\delta) \cdot n] \, (p - q_\delta) \, d\tau - \int_{S} (u_\delta - v_\delta) \cdot n \, (p - q_\delta) \, d\tau,
\]

where, in the decomposition of the jump in the last line, \( p - q_\delta \) is taken on the non-mortar side of \( S \). Inserting this in the previous line and noting that the jump \([u_\delta - v_\delta \cdot n]\) is
orthogonal to $q_\delta$ by definition of $\mathbb{D}_\delta(\Omega)$ yields

$$\|u_\delta - v_\delta\|_{L^2(\Omega)^d}^2 \leq \int_\Omega (u - v_\delta) \cdot (u_\delta - v_\delta) \, dx$$

$$+ \int_\Omega v_\delta \cdot (u_\delta - v_\delta) \, dx - (v_\delta, u_\delta - v_\delta)_{\delta}$$

$$+ \int_\Omega (u_\delta - v_\delta) \cdot \text{grad} (p - q_\delta) \, dx$$

$$+ (f, u_\delta - v_\delta)_{\delta} - \int_\Omega f \cdot (u_\delta - v_\delta) \, dx$$

$$- \int_S (u_\delta - v_\delta) \cdot n [p - q_\delta] \, d\tau. \quad (6.3)$$

Next, let $\Pi_{\delta-}$ be defined by $(\Pi_{\delta-} z)_{\Omega_k} = \Pi_{\delta-}^k z_{\Omega_k}$, where $\Pi_{\delta-}^k$ stands for the orthogonal projection operator from $L^2(\Omega_k)$ onto $P_{N_k-1}(\Omega_k)$. By adding and subtracting the function $\Pi_{\delta-} u$, we deduce from the exactness of the quadrature formula and (6.2) that

$$\int_\Omega v_\delta \cdot (u_\delta - v_\delta) \, dx - (v_\delta, u_\delta - v_\delta)_{\delta} \leq (1 + 3^d) \left( \|u - v_\delta\|_{L^2(\Omega)^d} + \|u - \Pi_{\delta-} u\|_{L^2(\Omega)^d} \right) \|u_\delta - v_\delta\|_{L^2(\Omega)^d}. \quad (6.4)$$

Similarly, we also have

$$(f, u_\delta - v_\delta)_{\delta} - \int_\Omega f \cdot (u_\delta - v_\delta) \, dx \leq (1 + 3^d) \left( \|f - I_\delta f\|_{L^2(\Omega)^d} + \|f - \Pi_{\delta-} f\|_{L^2(\Omega)^d} \right) \|u_\delta - v_\delta\|_{L^2(\Omega)^d}. \quad (6.5)$$

We refer to [15, Thm 7.1] and [15, Thm 14.2] for the approximation properties of the operators $\Pi_{\delta-}$ and $I_\delta$, respectively.

To estimate the last term in (6.3), we need a lemma. For a while, we work on the model domain $\Sigma = [-1,1]^d$ and we use the Legendre polynomials $L_n$, $n \geq 0$: each polynomial $L_n$ has degree $n$, is orthogonal to the other ones in $L^2(-1,1)$ and satisfies

$$L_n(\pm 1) = (\pm 1)^n, \quad \int_{-1}^1 L_n^2(\zeta) \, d\zeta = \frac{1}{n + \frac{1}{2}}. \quad (6.6)$$

**Lemma 6.1.** There exists a constant $c$ independent of $N$ such that, for any polynomial $w_N$ in $P_N(\Sigma)$ and for any $(d - 1)$–face $\Gamma$ of $\Sigma,$

$$\|w_N\|_{L^2(\Gamma)} \leq c N \|w_N\|_{L^2(\Sigma)}. \quad (6.7)$$

**Proof:** We check this inequality in the case $d = 3$, since the case $d = 2$ is simpler. First, we write any $w_N$ in $P_N(\Sigma)$ as

$$w_N(x, y, z) = \sum_{m=0}^N \sum_{n=0}^N \sum_{p=0}^N \alpha_{mnp} L_m(x)L_n(y)L_p(z),$$

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so that
\[ \|w_N\|_{L^2(\Sigma)}^2 = \sum_{m=0}^{N} \sum_{n=0}^{N} \sum_{p=0}^{N} \left( m + \frac{1}{2} \right)^2 (n + \frac{1}{2})^2 (p + \frac{1}{2})^2 \alpha_{mnp}^2 \] (6.8)

On the other hand, on the face \( \Gamma \) of equation \( x = \pm 1 \) for instance,
\[ w_N |_{\Gamma(y, z)} = \sum_{m=0}^{N} \sum_{n=0}^{N} \sum_{p=0}^{N} (\pm 1)^m \alpha_{mnp} L_n(y)L_p(z) \]
so that
\[ \|w_N\|_{L^2(\Gamma)}^2 = \sum_{n=0}^{N} \sum_{p=0}^{N} \left( \sum_{m=0}^{N} (\pm 1)^m \alpha_{mnp} \right)^2 \frac{1}{(n + \frac{1}{2})(p + \frac{1}{2})} \]

Using a Cauchy–Schwarz inequality then implies
\[ \|w_N\|_{L^2(\Gamma)}^2 = \left( \sum_{m=0}^{N} (m + \frac{1}{2}) \right) \sum_{n=0}^{N} \sum_{p=0}^{N} \left( \sum_{m=0}^{N} \frac{\alpha_{mnp}^2}{m + \frac{1}{2}} \right) \frac{1}{(n + \frac{1}{2})(p + \frac{1}{2})} \]
which, when compared with (6.8), yields the desired result.

Applying Lemma 6.1 leads to, for the parameter \( \mu_{\delta} \) introduced at the beginning of Section 5,
\[ |\int_S (u_\delta - v_\delta) \cdot n [p-q_\delta] \, d\tau| \leq c \mu_{\delta} \left( \sum_{k=1}^{K} \sum_{\ell=1}^{L(k)} N_k \|p-q_\delta\|_{L^2(\Gamma_{k,\ell})} \|u_\delta - v_\delta\|_{L^2(\Omega_k)^d} \right) \]
whence
\[ |\int_S (u_\delta - v_\delta) \cdot n [p-q_\delta] \, d\tau| \leq c \mu_{\delta} \left( \sum_{k=1}^{K} \sum_{\ell=1}^{L(k)} N_k^2 \|p-q_\delta\|_{L^2(\Gamma_{k,\ell})}^2 \right)^{\frac{1}{2}} \|u_\delta - v_\delta\|_{L^2(\Omega)^d}. \]

Inserting (6.4), (6.5) and (6.9) into (6.3) and using a triangle inequality yields
\[ \|u - u_\delta\|_{L^2(\Omega)^d} \leq c \left( \|u - v_\delta\|_{L^2(\Omega)^d} + \|u - \Pi_{\delta} - u\|_{L^2(\Omega)^d} + \left( \sum_{k=1}^{K} \|p-q_\delta\|_{H^1(\Omega_k)}^2 + \mu_{\delta}^2 \sum_{\ell=1}^{L(k)} N_k^2 \|p-q_\delta\|_{L^2(\Gamma_{k,\ell})}^2 \right)^{\frac{1}{2}} \right) \]
\[ + \|f - \mathcal{I}_{\delta} f\|_{L^2(\Omega)^d} + \|f - \Pi_{\delta} - f\|_{L^2(\Omega)^d} \]
\[ + \int_{\Omega_k} \text{grad} (p - \Pi_{\delta}^1 p) \cdot \text{grad} q_\delta^k \, dx = 0 \quad \text{and} \quad \int_{\Omega_k} (p - \Pi_{\delta}^1 p) \, dx = 0. \]

Next, we choose \( q_\delta^k \) by: \( q_\delta^k \mid_{\Omega_k} = \Pi_{\delta}^1 p_\delta \), where \( \Pi_{\delta}^1 \) stands for the orthogonal projection operator from \( H^1(\Omega_k) \) onto \( \mathbb{P}_{N_k-2}(\Omega_k) \) defined by
\[ \forall q_\delta^k \in \mathbb{P}_{N_k-2}(\Omega_k), \quad \int_{\Omega_k} \text{grad} (p - \Pi_{\delta}^1 p) \cdot \text{grad} q_\delta^k \, dx = 0 \quad \text{and} \quad \int_{\Omega_k} (p - \Pi_{\delta}^1 p) \, dx = 0. \]
Indeed, the following approximation properties of this operator are well-known \[15, \text{Thm 7.3} \]: if the function \( p|_{\Omega_k} \) belongs to \( H^{s_k+1}(\Omega_k) \), \( s_k \geq 0 \),

\[
\| p - \Pi_{\delta}^{1,k} p \|_{H^1(\Omega_k)} + N_k \| p - \Pi_{\delta}^{1,k} p \|_{L^2(\Omega_k)} \leq c N_k^{-s_k} \| p \|_{H^{s_k+1}(\Omega_k)}, \tag{6.11}
\]

while estimating the norm of \( \| p - \Pi_{\delta}^{1,k} p \|_{L^2(\Gamma_{k,\ell})} \) results from a duality argument and is proven in the following lemma.

**Lemma 6.2.** For \( 1 \leq k \leq K \), there exists a constant \( c \) independent of \( N \) such that, if the function \( p|_{\Omega_k} \) belongs to \( H^{s_k+1}(\Omega_k) \), \( s_k \geq 0 \), and for \( 1 \leq \ell \leq L(k) \),

\[
\| p - \Pi_{\delta}^{1,k} p \|_{L^2(\Gamma_{k,\ell})} \leq c N_k^{-s_k-\frac{1}{2}} \| p \|_{H^{s_k+1}(\Omega_k)}. \tag{6.12}
\]

**Proof:** The duality argument reads

\[
\| p - \Pi_{\delta}^{1,k} p \|_{L^2(\Gamma_{k,\ell})} = \sup_{g \in L^2(\Gamma_{k,\ell})} \frac{\int_{\Gamma_{k,\ell}} (p - \Pi_{\delta}^{1,k} p) g d\tau}{\| g \|_{L^2(\Gamma_{k,\ell})}}.
\]

So, for any \( g \) in \( L^2(\Gamma_{k,\ell}) \), denoting by \( \overline{g} \) its extension by zero to \( \partial \Omega_k \), we consider the following Neumann problem

\[
\begin{cases}
-\Delta q = -\frac{1}{\text{meas}(\Omega_k)} \int_{\Gamma_{k,\ell}} g(\tau) d\tau & \text{in } \Omega_k, \\
\partial_n q = \overline{g} & \text{on } \partial \Omega_k.
\end{cases}
\]

This problem admits a solution \( q \) in \( H^1(\Omega_k) \cap L^2_0(\Omega_k) \). Moreover, it belongs to \( H^{\frac{s_k}{2}}(\Omega_k) \) \[23\] and satisfies

\[
\| q \|_{H^{\frac{s_k}{2}}(\Omega_k)} \leq c \| g \|_{L^2(\Gamma_{k,\ell})}. \tag{6.13}
\]

So, the desired estimate follows from the duality formula

\[
\int_{\Gamma_{k,\ell}} (p - \Pi_{\delta}^{1,k} p) g d\tau = \frac{1}{\text{meas}(\Omega_k)} \int_{\Gamma_{k,\ell}} g(\tau) d\tau \int_{\Omega_k} (p - \Pi_{\delta}^{1,k} p) dx + \int_{\Omega_k} \text{grad} (p - \Pi_{\delta}^{1,k} p) \cdot \text{grad} (q - \Pi_{\delta}^{1,k} q) dx,
\]

together with (6.11) and (6.13).

**Remark:** In the case of a conforming decomposition, a continuous approximation \( q_\delta \) of \( p \), i.e. in \( M_\delta(\Omega) \cap H^1(\Omega) \), can be built by standard arguments \[16, \S 3.1\], which satisfies

\[
\| p - q_\delta \|_{H^1(\Omega)} \leq c \sum_{k=1}^{K} N_k^{-s_k} \| p \|_{H^{s_k+1}(\Omega_k)}. \tag{6.14}
\]

Of course, with this choice, we have

\[
\int_S (u_\delta - v_\delta) \cdot n \ [p - q_\delta] d\tau = 0, \tag{6.15}
\]

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so that estimate (6.10) can be replaced by the simpler one
\[
\|u - u_\delta\|_{L^2(\Omega)^d} \leq c \left( \|u - v_\delta\|_{L^2(\Omega)^d} + \|u - \Pi_\delta u\|_{L^2(\Omega)^d} + \|p - q_\delta\|_{H^1(\Omega)} + \|f - I_\delta f\|_{L^2(\Omega)^d} + \|f - \Pi_\delta f\|_{L^2(\Omega)^d} \right)
\]

(6.16)

We are in a position to write the first error estimate, by inserting (6.11) and (6.12) into (6.16) and using the approximation properties stated in Propositions 5.1 and 5.3.

**Theorem 6.3.** Assume the data \( f \) such that each \( f_{\Omega_k} \), \( 1 \leq k \leq K \), belongs to \( H^{\sigma_k}(\Omega_k)^d \), \( \sigma_k > \frac{d}{2} \), and the solution \( (u, p) \) of problem (3.1) such that each \( (u_{\Omega_k}, p_{\Omega_k}) \), \( 1 \leq k \leq K \), belongs to \( H^{s_k}(\Omega_k)^d \times H^{s_k+1}(\Omega_k) \), \( s_k \geq d - \frac{d}{2} \). Then, in the two cases

(i) in dimension \( d = 2 \),

(ii) in dimension \( d = 3 \), if the decomposition is conforming and if moreover for each mortar \( \gamma^m \), \( 1 \leq m \leq M \), which is a face of both \( \Omega_{k(m)} \) and \( \Omega_k \), \( N_{k(m)} \) is \( \geq N_k \), the following error estimate holds between this velocity \( u \) and the velocity \( u_\delta \) of problem (4.7):
\[
\|u - u_\delta\|_{L^2(\Omega)^d} 
\leq c \sum_{k=1}^{K} (\lambda_k \mu_\delta N_k^{-s_k} (\|u\|_{H^{s_k}(\Omega_k)^d} + \|p\|_{H^{s_k+1}(\Omega_k)}) + N_k^{-\sigma_k} \|f\|_{H^{\sigma_k}(\Omega_k)^d}),
\]

(6.17)

where each \( \lambda_k \), \( 1 \leq k \leq K \), is equal to 1 if the decomposition is conforming, to \( N_k^{\frac{1}{2}} \) otherwise.

Estimate (6.17) is not fully optimal, except in the case of a conforming decomposition and with the further assumption that the mortars are chosen on the side of the smallest \( N_k \) in dimension 3. However, when \( \mu_\delta \) is bounded (this is most often the case in practical situations), the lack of optimality is only \( N_k^{\frac{1}{2}} \).

In the two-dimensional case of a polygon \( \Omega \), a more explicit estimate can be deduced from the previously quoted regularity results. We briefly explain the main arguments of the proof and refer to [14] for details.

**Corollary 6.4.** In the two-dimensional case of a polygon \( \Omega \), assume the data \( f \) such that each \( f_{\Omega_k} \), \( 1 \leq k \leq K \), belongs to \( H^{\sigma_k}(\Omega_k)^2 \), \( \sigma_k > 1 \). Then, the following error estimate holds between the velocity \( u \) of problem (3.1) and the velocity \( u_\delta \) of problem (4.7):
\[
\|u - u_\delta\|_{L^2(\Omega)^2} \leq c \mu_\delta \sum_{k=1}^{K} \lambda_k E_k \|f\|_{H^{\sigma_k}(\Omega_k)^2},
\]

(6.18)

for the \( \lambda_k \) introduced in Theorem 6.3 and where the \( E_k \), \( 1 \leq k \leq K \), are given by
\[
E_k = \begin{cases} 
N_k^{-\sigma_k} & \text{if } \overline{\Omega_k} \text{ contains no corner of } \Omega, \\
\max\{N_k^{-\sigma_k}, N_k^{-4}(\log N_k)^{\frac{3}{2}}\} & \text{if } \overline{\Omega_k} \text{ contains a corner of } \Omega \\
\max\{N_k^{-\sigma_k}, N_k^{-4}(\log N_k)^{\frac{1}{2}}\} & \text{if } \overline{\Omega_k} \text{ contains a nonconvex corner of } \Omega.
\end{cases}
\]

(6.19)
Proof: Going back to inequality (6.10), we keep the same estimate for the terms involving \( f \), however we further investigate the approximation of \( u \) and \( p \). Indeed,

1) when \( \Omega_k \) contains no corner of \( \Omega \), the regularity of \((u, p)\) on \( \Omega_k \) only depends on the data \( f \), namely it belongs to \( H^{\sigma_k}(\Omega_k)^2 \times H^{\sigma_k+1}(\Omega_k) \). So the desired approximation property follows from the same arguments as previously.

2) when \( \Omega_k \) contains a corner of \( \Omega \) with angle \( \omega \), we derive from (2.4) that

\[
\begin{aligned}
 u|_{\Omega_k} &= u_r + \text{grad} \, S, \\
p|_{\Omega_k} &= p_r - S,
\end{aligned}
\]

where \((u_r, p_r)\) belongs to \( H^{s}(\Omega_k)^2 \times H^{s+1}(\Omega_k) \) for \( s \leq \sigma_k \) and \( s < \frac{2\pi}{\omega} \). The idea now consists in looking for separate approximations of \( u_r \) and \( p_r \) and of \( S \). Indeed, the approximation properties of \( u_r \) and \( p_r \) are the same as stated previously, say for instance

\[
\| p_r - \Pi^{1,k}_{\delta} p_r \|_{H^1(\Omega_k)} \leq c N_k^{-\frac{s}{2}} \| p_r \|_{H^{\sigma_k}(\Omega_k)^2}.
\]

However when \( \sigma_k \) is \( \geq \frac{2\pi}{\omega} \), we note that the constant in the previous line is independent of \( s \), use the further estimate, for \( 0 < \varepsilon < \frac{1}{2} \),

\[
\| p_r \|_{H^{\frac{2\pi - \varepsilon}{\omega}}(\Omega_k)} \leq c \varepsilon^{-\frac{1}{2}} \| f \|_{H^{\sigma_k}(\Omega_k)^2},
\]

and take \( \varepsilon = (\log N_k)^{-1} \), which leads to

\[
\| p_r - \Pi^{1,k}_{\delta} p_r \|_{H^1(\Omega_k)} \leq c N_k^{-\frac{2\pi}{\omega}} (\log N_k)^{\frac{1}{2}} \| f \|_{H^{\sigma_k}(\Omega_k)^2}.
\]

Moreover, as explained in Section 2, the restriction of the function \( S \) to a small enough neighbourhood of the largest corner of \( \Omega \) contained in \( \Omega_k \) can be written

\[
r^{-\bar{p}} \left( \varphi(\theta) + (\log r)^p \psi(\theta) \right),
\]

with \( p \) equal to 1 or 0 according as \( \omega \) is equal to \( \frac{\pi}{2} \) or \( \frac{3\pi}{2} \). The local approximation properties of these functions are well-known \([14, \S 3]\)

\[
\| S - \Pi^{1,k}_{\delta} S \|_{H^1(\Omega_k)} \leq c N_k^{-\frac{2\pi}{\omega}} (\log N_k)^{p+\frac{1}{2}}.
\]

This, combined with the same duality argument as in Lemma 6.2, an analogous evaluation of \( u - \Pi_{\delta -} u \) and the construction of a global function \( v_{\delta} \) in \( V_{\delta}(\Omega) \) (which is rather simple in dimension \( d = 2 \), see Proposition 5.1), leads to the desired estimate.

Estimating the error on the pressure is now easy.

**Theorem 6.5.** If the assumptions of Theorem 6.3 are satisfied and in cases (i) and (ii) of this theorem, the following error estimate holds between the pressure \( p \) of problem (3.1) and the pressure \( p_{\delta} \) of problem (4.7):

\[
\begin{aligned}
\| p - p_\delta \|_{L^2(\Omega)} &\leq c \sum_{k=1}^{K} \left( \lambda_k \mu_\delta \, N_k^{-\sigma_k} \left( \log N_k \right)^{\frac{1}{2}} \left( \| u \|_{H^{\sigma_k}(\Omega_k)^d} + \| p \|_{H^{\sigma_k+1}(\Omega_k)} \right) \right) \\
&+ N_k^{-\sigma_k} \| f \|_{H^{\sigma_k}(\Omega_k)^d}\).
\end{aligned}
\]
for the $\lambda_k$ introduced in Theorem 6.3, with $\nu$ equal to zero in the case of a conforming decomposition and to 1 otherwise.

**Proof:** From the inf-sup condition (4.10), we derive that, for any $q_\delta$ in $M_\delta(\Omega),
\beta_D \|p_\delta - q_\delta\|_{L^2(\Omega)} \leq \sup_{v_\delta \in D_\delta(\Omega)} -\int_\Omega (p_\delta - q_\delta)(\text{div } v_\delta) \, dx
\|v_\delta\|_{H(\text{div},\Omega)}.

In order to evaluate $-\int_\Omega (p_\delta - q_\delta)(\text{div } v_\delta) \, dx$, we first use the discrete problem (4.7):

$-\int_\Omega (p_\delta - q_\delta)(\text{div } v_\delta) \, dx = (f, v_\delta) - (u_\delta, v_\delta) + \int_\Omega q_\delta(\text{div } v_\delta) \, dx.$

Next, we apply equation (3.1) to the function $v_\delta$, integrate by parts and add it to the previous line. This yields

$-\int_\Omega (p_\delta - q_\delta)(\text{div } v_\delta) \, dx = (f, v_\delta) - \int_\Omega f \cdot v_\delta \, dx + \int_\Omega (u - u_\delta) \cdot v_\delta \, dx + \int_\Omega u_\delta \cdot v_\delta \, dx - (u_\delta, v_\delta)
\quad - \int_\Omega (p - q_\delta)(\text{div } v_\delta) \, dx + \int_S [v_\delta \cdot n] p \, dx.$

Using the same arguments as in (6.4) and (6.5) together with a triangle inequality yields

$\|p - p_\delta\|_{L^2(\Omega)} \leq c (\|p - q_\delta\|_{L^2(\Omega)} + \|u - u_\delta\|_{L^2(\Omega)} + \|u - \Pi_{\delta} u\|_{L^2(\Omega)}^2
\quad + \|f - \mathcal{I}_{\delta} f\|_{L^2(\Omega)}^2 + \|f - \Pi_{\delta} f\|_{L^2(\Omega)}^2
\quad + \sup_{v_\delta \in D_\delta(\Omega)} \|v_\delta\|_{H(\text{div},\Omega)}^2) + \int_S [v_\delta \cdot n] p \, dx.$

(622)

All the terms in the right-hand side have been estimated previously, except the last one which represents the consistency error. To evaluate it, we use a partition of $S$ into non mortar edges or faces. Indeed, on each $\Gamma_{k,\ell}$ which is not a mortar and in the case of a non-conforming decomposition, we derive from the matching condition (4.3) that, for any $r_\delta$ in $P_{N_k - 2}(\Gamma_{k,\ell})$ and for any positive and small enough $\varepsilon$,

$\int_{\Gamma_{k,\ell}} [v_\delta \cdot n] p \, dx = \int_{\Gamma_{k,\ell}} [v_\delta \cdot n](p - r_\delta) \, dx
\leq c \varepsilon^{-\frac{1}{2}} \left( \sum_{k' \in C_{k,\ell}} \|v_\delta \cdot n\|_{H^{\frac{3}{2} + \varepsilon}(\partial \Omega_{k'})} \|p - r_\delta\|_{H^{\frac{3}{2} - \varepsilon}(\Gamma_{k,\ell})} \right)$

where $C_{k,\ell}$ is the set of indices $k'$, $1 \leq k' \leq K$, such that the intersection of $\partial \Omega_{k'}$ and $\Gamma_{k,\ell}$ has a positive measure. The term $c \varepsilon^{-\frac{1}{2}}$ represents the maximum norm of the extensions by zero from $H^{\frac{3}{2} - \varepsilon}(\Gamma_{k,\ell})$ into $H^{\frac{3}{2} - \varepsilon}(\partial \Omega_{k'})$ (this can be established with $c$ independent of $\varepsilon$ by using intrinsic norms on these spaces). Thanks to formula (2.3) combined with an inverse inequality, we have

$\|v_\delta \cdot n\|_{H^{\frac{3}{2} + \varepsilon}(\partial \Omega_{k'})} \leq \|v_\delta\|_{H^\varepsilon(\Omega_{k'})^d} + \|\text{div } v_\delta\|_{L^2(\Omega_{k'})} \leq c N_{k'}^{2\varepsilon} \|v_\delta\|_{H(\text{div},\Omega_{k'})}.$
Next, we choose $r_\delta$ equal to the image of $p$ by the projection operator in $H^{\frac{1}{2}-\varepsilon}(\Gamma_{k,\ell})$, which gives
\[ \|p - r_\delta\|_{H^{\frac{1}{2}-\varepsilon}(\Gamma_{k,\ell})} \leq N_k^{-\varepsilon-s_k} \|p\|_{H^{s_k+1}(\Gamma_{k,\ell})}. \]
Combining all this and taking $\varepsilon$ equal to $\frac{1}{\log M_{k,\ell}}$, where $M_{k,\ell}$ denotes the maximum of the $N_{k'}$, $k' \in C_{k,\ell}$, yields the desired estimate. The same arguments hold in the case of a conforming decomposition with $\varepsilon = 0$ and $c \varepsilon^{-\frac{1}{2}}$ replaced by $c$.

We skip the proof of the corollary which is now obvious.

**Corollary 6.6.** In the two-dimensional case of a polygon $\Omega$, if the assumptions of Corollary 6.4 are satisfied, the following error estimate holds between the pressure $p$ problem (3.1) and the pressure $p_\delta$ of problem (4.7):
\[ \|p - p_\delta\|_{L^2(\Omega)} \leq c \mu_\delta \sum_{k=1}^{K} \lambda_k E_k \left( \log N_k \right)^{\frac{3}{2}} \|f\|_{H^{s_k}(\Omega_k)^2}, \]  
(6.23)
for the $\lambda_k$ and $\nu$ introduced in Theorems 6.3 and 6.5 and the $E_k$ defined in (6.19).

Estimates (6.17) and (6.21) are fully optimal in the case of a conforming decomposition, and the fact that each $u|_{\Omega_k}$ must belong to $H^{d-3}(\Omega_k)$ is not at all restrictive in dimension $d = 2$, see Section 2, however weakening this condition in dimension $d = 3$ is an interesting question (note however that the solution $(u, p)$ satisfies the desired regularity property at least for smooth data and for all $\Omega_k$ such that $\partial \Omega_k$ does not intersect an edge or a corner of $\Omega$). The results are a little more disappointing in the case of a non-conforming decomposition in dimension $d = 2$. However, if $\mu_\delta$ is bounded independently of $\delta$ (this is the case in most practical situations), the lack of optimality is only of order $(N_k \log N_k)^{\frac{1}{2}}$.

No convergence result is proven in the case of dimension 3 and with a non-conforming decomposition. However numerical experiments show this convergence in the case of second-order elliptic problems, see [24].
7. Numerical algorithms and experiments.

We briefly explain how to write the discrete problem described in Section 4 as a square linear system, and we propose an algorithm for solving it. Although the matching conditions through the skeleton are most often handled via the introduction of a Lagrange multiplier, see [10], we have chosen to construct a basis of the discrete velocity space made of functions that satisfy these conditions.

The whole unknowns of the discrete system are given by

- the vector $U$ of the values of $u_\delta$ at all nodes $(x_i^k, y_j^k)$, $0 \leq i, j \leq N_k$, $1 \leq k \leq K$, in dimension $d = 2$, $(x_i^k, y_j^k, z_p^k)$, $0 \leq i, j, p \leq N_k$, $1 \leq k \leq K$, in dimension $d = 3$,

- the vector $P$ of the values of $p_\delta$ at all nodes $(x_i^k, y_j^k)$, $1 \leq i, j \leq N_k - 1$, $1 \leq k \leq K$, in dimension $d = 2$, $(x_i^k, y_j^k, z_p^k)$, $1 \leq i, j, p \leq N_k - 1$, $1 \leq k \leq K$, in dimension $d = 3$, minus one node ($p_\delta$ is assumed to be zero in this node, in order to “fix” the constant and is enforced to belong to $L^2_0(\Omega)$ in a postprocessing step).

Next, the coefficients of $u$ in the basis of $D_\delta(\Omega)$ are now computed via the introduction of a rectangular matrix $Q$ in order to enforce the boundary and matching conditions, which leads to a new vector $QU$. Problem (4.7) is now equivalent to the following square linear system

$$
\begin{pmatrix}
Q^T AQ & Q^T B \\
B^T Q & 0
\end{pmatrix}
\begin{pmatrix}
U \\
P
\end{pmatrix}
= 
\begin{pmatrix}
Q^T AF \\
0
\end{pmatrix}.
$$

(7.1)

The matrix $A$ is fully diagonal, its diagonal terms are the $\rho_i^x \rho_j^y$ or the $\rho_i^x \rho_j^y \rho_p^z$ according to the dimension. The matrix $B$ is only block-diagonal, with $K$ blocks $B_k$ on the diagonal, one for each $\Omega_k$. Each $B_k$ represents the local discrete gradient, and $B_k^T$ the local discrete divergence.

System (7.1) is solved by using the Schwarz algorithm of Dirichlet-Neumann type for handling the domain decomposition, see [30, §1.3]. Each local problem is inverted via Uzawa algorithm, and a direct Choleski factorization is used to compute the discrete pressure (see [2]).

For the first numerical experiment, we consider the two-dimensional $L$-shaped domain $\Omega = [-1, 3]^2 \setminus [1, 3]^2$, partitioned into 3 subdomains illustrated in Figure 1 (left panel). For an integer $N \geq 6$, the $N_1$ associated with the middle square subdomain $\Omega_1 = [-1, 1]^2$ is equal to $N$ while $N_2$ and $N_3$ are equal to $N - 5$. We use our spectral method to compute an approximation of the analytical solution $(u, p)$ given by

$$
u(x, y) = \begin{pmatrix}
-\sin(\pi x) \cos(\pi y) \\
\cos(\pi x) \sin(\pi y)
\end{pmatrix}, \quad p(x, y) = \sin(\pi (x + y)).
$$

(7.2)

Figure 1 (right panel) depicts, in a semi-logarithmic scale, the curves of the errors $\|u - u_\delta\|_{L^2(\Omega)^2}$ and $\|p - p_\delta\|_{L^2(\Omega)}$ as a function of $N$, for $N$ varying from 6 to 22. As can be previewed from estimates (6.17) and (6.21), the convergence rate here is exponential despite the nonconformity of the discretization.
Next, we are interested in the computation of a nonsmooth solution in the same domain $\Omega$ with the same decomposition. We take all the $N_k$ equal to $N$, as illustrated in Figure 2 (left part). The data $f$ being given by

$$f(x, y) = \begin{pmatrix} y \\ 0 \end{pmatrix}. \quad (7.3)$$

We calculate the reference solution $(u_r, p_r)$ on the conforming partition with $N$ equal to 128, which is represented in Figures 3 and 4. Figure 2 (right part) presents the curves of the errors $\|u_r - u_\delta\|_{L^2(\Omega)}$ and $\|p_r - p_\delta\|_{L^2(\Omega)}$ as a function of $N$, in bilogarithmic scales and for $N$ varying from 8 to 64. The convergence order expected by the theory, see estimates (6.18) and (6.23), is $N^{-\frac{4}{3}}$. The slopes of the curves are $-2.1$ and $-4.5$, so they are better than the theoretical prediction (we refer to [5] for the first observation of
this superconvergence phenomenon). The same numerical test is performed in [3] using staggered grids and similar trends are observed.

We finally consider the square $\Omega = [0, 2]^2$, with a nonconforming decomposition into two squares $\Omega_1 = [0, 1]^2$ and $\Omega_2 = [1, 2] \times [0, 1]$ and a rectangle $\Omega_3 = [0, 2] \times [1, 2]$. For an integer $N \geq 8$, we take all the $N_k$ equal to $N$, see Figure 5.

We work successively with the solutions $(u, p)$ given by

\[ u(x, y) = \begin{pmatrix} -\sin(\pi x) \cos(\pi y) \\ \cos(\pi x) \sin(\pi y) \end{pmatrix}, \quad p(x, y) = \exp 2x(2 - x)(x + 1) \exp(-2y), \quad (7.4) \]

\[ u(x, y) = \begin{pmatrix} -\sin(\pi x) \cos(\pi y) \\ \cos(\pi x) \sin(\pi y) \end{pmatrix}, \quad p(x, y) = (x - 1)x^{\frac{1}{2}}(2 - x)^{\frac{1}{2}} \exp(-2y). \quad (7.5) \]

In Figure 6 are plotted, the curves of the errors $\|u - u_\delta\|_{L^2(\Omega)^2}$ and $\|p - p_\delta\|_{L^2(\Omega)}$ for both cases as a function of $N$. For the smooth solution, a linear/logarithmic scale is used and we observe that the exponential decaying of the error is preserved despite the nonconforming domain decomposition. For the nonsmooth solution rather a full logarithmic scale is
adopted, we observe the good convergence of the discretization. Due to the nonconformity of the discretization, the expected rate is $\frac{7}{2}$, up to some logarithmic terms, see estimates (6.18) and (6.23), while the experimental one is approximately 3.4. The results can then be considered in good agreement with the theoretical predictions.

**Figure 5:** The nonmatching grids for a nonconforming decomposition with $N=32$

**Figure 6:** The error curves for a smooth solution (left panel) and a nonsmooth solution (right panel).
References


