Derivatives and Control in the Presence of Shocks

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Abstract

Sensitivity of shocks to data is a key point for fluid-structure and flutter control, and even more for sonic boom reduction. The linearized equations of fluids have Dirac masses and so it is not clear that the standard tools of optimal control apply to these. We show here that indeed great care have to be applied to find the linearized equations but that there is no such difficulties for control problems which do not involve explicitly the position of the shock in the criteria.

Keywords: Partial differential equations, Burger equation, Euler flow, sensitivity, transonic equation.

Introduction

Sensitivity of the position of the shocks with respect to the parameters of the flow is the problem we would like to investigate here. There are many important applications such as the fluttering of wings and the sonic boom of supersonic airplanes.
In a land mark paper, Godlewski et al [6] have studied a similar situation for the shock tube flow problem and solved it completely for Burger's equation when the sensitivity is with respect to initial data; their proof is however somewhat dependent on the explicit form of the solution and we will give here what we believe to be a simpler proof based on the definition of derivatives of distributions. For control, Giles [3] showed that the adjoint equation of the time dependent Euler equation is well posed and continuous across the shock. Here too we confirm the result. Then we turn to stationary problems and apply the same method to a simple transonic nozzle flow.

1 Burger's Equation

Consider the one dimensional Burger's equation,

$$\partial_t u + \partial_x(\frac{u^2}{2}) = 0 \ \forall\{x,t\} \in Q := R \times (0,+\infty),$$

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$$u(x,0) = u^0(x,a) \;\; \forall x \in R \tag{1}$$

It may have discontinuous solutions (shocks) depending on the initial data u^0. Here we assume that the initial data is function of a parameter a and we wish to find an equation for the derivative of u with respect to a.
With compatible (entropy) discontinuous initial data at, say $x(0) = 0$, the solution has a discontinuity at, say $x(t)$, which propagates at velocity $\dot{x}_t = \bar{u} := (u^+ + u^-)/2$. Let us call S the shock, $S = \{(x(t), t \; : \; t \geq 0\}$ and $\vec{\nu}$ its normal; it has components $\nu_x = -\dot{x}_t$, $\nu_t = 1$.
In weak form Burger's equation is

$$\iint_Q (u\partial_t w + \frac{u^2}{2}\partial_x w)\mathrm{d}x\mathrm{d}t + \int_R u^0(a)w(x,0)\mathrm{d}x = 0 \;\; \forall w \in H^1(Q). \tag{2}$$

To find the derivative in a let us go back to the definition of derivatives in the sense of distributions and apply (2), integrated in a on $A = (-\infty, +\infty)$ and with $\partial_a w(x,t,a)$ instead $w(x,t)$ where w is $\mathcal{D}(\mathcal{Q})$ the space of C^∞ functions with compact support in $\mathcal{Q} = R \times [0, +\infty) \times A$:

$$\iiint_{\mathcal{Q}} (u\partial_t\partial_a w + \frac{u^2}{2}\partial_x\partial_a w)\mathrm{d}x\mathrm{d}t\mathrm{d}a + \iint_{\mathcal{R}} u^0(a)\partial_a w(x,0,a)\mathrm{d}x\mathrm{d}a = 0 \quad \forall w \in \mathcal{D}(\mathcal{Q}) \tag{3}$$

with $\mathcal{R} = R \times A$.

Equation (3) should define completely $\partial_a u$ and the difficulty lies in its interpretation.

1.1 Interpretation of (3)

We will assume that there is only one shock front and that u is regular on both side. This allows us to write

$$u(x,t,a) = u^-(x,t,a) + (u^+(x,t,a) - u^-(x,t,a))I_{\Omega^+} \tag{4}$$

where Ω^+ is the set of points on the right side of the shock at some instant t and for some value a of the parameter.

Let Σ be the shock surface in $\mathcal{Q}$,

$$\Sigma = \{(x,t,a) \; : \; [u](x,t,a) \neq 0\} := \{(x(t,a),t,a) \; : \; t \geq 0\} \tag{5}$$

Here we have assumed that Σ can be represented by a function $x(t,a)$, the position of the shock at time t when the parameter is equal to a.
Differentiating (4) with respect to a yields

$$\partial_a u = \partial_a u^- + (\partial_a u^+ - \partial_a u^-)I_{\Omega^+} - (u^+ - u^-)\dot{x}_a\delta_\Sigma \tag{6}$$

where δ_Σ is the 2-dimensional Dirac function on Σ.
So the singularity we can expect is that $\partial_a u$ will be the sum of a Dirac function on Σ plus a smooth function left and right of Σ with a possible jump across Σ.

let us integrate by parts each term of (3):

$$\begin{aligned} I_1 &:= \iiint_{\mathcal{Q}} (u\partial_t\partial_a w + \frac{u^2}{2}\partial_x\partial_a w)\mathrm{d}x\mathrm{d}t\mathrm{d}a \\ &= -\iiint_{\mathcal{Q}_\Sigma} (\partial_a u\partial_t w + \partial_a(\frac{u^2}{2})\partial_x w)\mathrm{d}x\mathrm{d}t\mathrm{d}a \\ &\quad + \iint_\Sigma \nu_a([u]\partial_t w + [\frac{u^2}{2}]\partial_x w) \equiv I_4 + I_3 \end{aligned} \tag{7}$$

where $\mathcal{Q}_\Sigma = \mathcal{Q}^- \cup \mathcal{Q}^+$, (before and after the shock) and where ν_a is the a-component of the normal of Σ; the local parametrization of Σ by t and a, as $x(t,a),t,a)$ gives for tangents and normal

$$\vec{s} = (\dot{x}_t, 1, 0)^T \quad \vec{\sigma} = (\dot{x}_a, 0, 1)^T, \quad \vec{\nu} = (1, -\dot{x}_t, -\dot{x}_a)^T$$

Here the argument to obtain (7) is that u and u^2 are smooth functions except across Σ, so integration by part must be done on each side of Σ and (7) is obtained when both parts are added.

Now notice that $[\frac{u^2}{2}] = [u]\bar{u}$ and that $\partial_t + \bar{u}\partial_x$ is the tangential derivative in the direction $\vec{s}$, so that

$$I_3 := \iint_\Sigma \nu_a([u]\partial_t w + [\frac{u^2}{2}]\partial_x w) = \iint_\Sigma \nu_a[u]\partial_s w. \tag{8}$$

Hence, if $\mathcal{S} = \{x, a(x)\}$ is the equation of the intersection of Σ with the plan $t = 0$ and if $[u]$ is a smooth function (8) becomes

$$I_3 = -\iint_\Sigma w(\partial_t + \bar{u}\partial_x)(\nu_a[u]) - \int_{\mathcal{S}} \nu_a[u]w\mathrm{d}\sigma \tag{9}$$

In (7) the triple integral I_4 must be integrated by part in time and space,

$$\begin{aligned} I_4 &:= -\iiint_{\mathcal{Q}_\Sigma} (\partial_a u\partial_t w + \partial_a(\frac{u^2}{2})\partial_x w)\mathrm{d}x\mathrm{d}t\mathrm{d}a \\ &= \iiint_{\mathcal{Q}_\Sigma} w(\partial_t\partial_a u + \partial_x\partial_a(\frac{u^2}{2}))\mathrm{d}x\mathrm{d}t\mathrm{d}a \\ &\quad + \iint_{\mathcal{R}} (\partial_a uw)|_{t=0}\mathrm{d}x\mathrm{d}a - \iint_\Sigma w(\nu_t[\partial_a u] + \nu_x[\partial_a(\frac{u^2}{2})]) \end{aligned} \tag{10}$$

Integrating by parts the second integral in (3) gives

$$I_2 := \iint_{\mathcal{R}} u^0(a)\partial_a w(x,0,a)\mathrm{d}x\mathrm{d}a = -\iint_{\mathcal{R}} \partial_a u^0 w(x,0,a)\mathrm{d}x\mathrm{d}a + \int_{\mathcal{S}} \nu_a[u^0]w\mathrm{d}\sigma$$

(Notice the signs, due to the fact that when a increases Σ is crossed from + into - when $\dot{x}_t > 0$). Finally denoting $u' = \partial_a u$, we come to

$$0 = I_2 + I_3 + I_4 = \iiint_{Q_\Sigma} (\partial_t u' + \partial_x(uu'))w\mathrm{d}x\mathrm{d}t\mathrm{d}a$$
$$- \iint_\Sigma w(\nu_t[u'] + \nu_x[uu']) + \iint_{\mathcal{R}} (u'w)|_{t=0}\mathrm{d}x\mathrm{d}a - \iint_\Sigma w(\partial_t + \bar{u}\partial_x)(\nu_a[u])$$
$$- \iint_{\mathcal{R}} \partial_a(u^0)w(x,0,a)\mathrm{d}x\mathrm{d}a \tag{11}$$

This is interpreted as

$$\begin{aligned} &\partial_t u' + \partial_x(uu') = 0 \quad \forall\{x,t\} \in Q\backslash S, \qquad u'(x,0) = \partial_a u^0(x) \quad \forall x \\ &(\partial_t + \bar{u}\partial_x)(\nu_a[u]) + \nu_t[u'] + \nu_x[uu'] = 0 \;\text{ on } S \end{aligned} \tag{12}$$

As $\nu_t + \bar{u}\nu_x = 0$, notice that

$$\nu_t[u'] + \nu_x[uu'] = \bar{u'}[u]\nu_x$$

Notice also that $(\partial_t + \bar{u}\partial_x)g(x,t)|_S$ is the time derivative of $g(x(t,a),t)$; therefore, recalling that $\nu_a = -\dot{x}_a,\ \nu_x = 1$, the system found for u' and $\dot{x}_a$ is given by

Theorem 1 *The derivative of the solution of Berger's equation with respect to a parameter a in the data is*

$$\partial_a u(x,t) = u'(x,t) - [u]\dot{x}_a(t)\delta(x - x(t))$$

where $x(t)$ is the position of the shock at time t (which depends on a), u' is a discontinuous function at the shock, where $\delta(y)$ is the Dirac function at zero, $\dot{x}_a$ is the derivative of the shock position with respect to a. Furthermore $\{u', \dot{x}_a\}$ is completely determined by

$$\begin{aligned} &\partial_t u' + \partial_x(uu') = 0 \quad \forall\{x,t\} \in Q\backslash S, \qquad u'(x,0) = \partial_a u^0(x) \quad \forall x \\ &\frac{d}{dt}(\dot{x}_a[u]) = \bar{u'}[u] \;\text{ on } S \text{ with } \dot{x}_a(0) \text{ given by } \partial_a u^0. \end{aligned} \tag{13}$$

when the entropy condition is satisfied.

Indeed the entropy condition will insure that the linear equation for u' (13-a) has a unique solution right and left of the shock because the entropy condition ensures that characteristics left and right of the shock point in the right direction. No Rankine-Hugoniot condition are necessary. The unusual situation here is that the "natural" Rankine-Hugoniot condition implied by (13-a) if it was satisfied at the shock, is not verified.

1.2 Formal Derivation of Theorem 1

The same result can be obtained by differentiation of

1. Burger's equation, leading to

$$\partial_t u' + \partial_x(uu') = 0 \;\; \forall x, t \in Q_S \text{ i.e. } x \neq x(t)$$

2. and differentiation of the Rankine-Hugoniot condition

$$\frac{d}{da}(\dot{x}_t[u] - [\frac{u^2}{2}])|_{\{x(t,a),t,a\}} = 0 \tag{14}$$

Indeed, as $\frac{d}{da}b(x(t,a),t,a) = \dot{x}_a\partial_x b + \partial_a b$ and since $[u^2] = 2\bar{u}[u]$ (14) is

$$\begin{aligned} 0 &= \ddot{x}_{ta}[u] + \dot{x}_t([u'] + \dot{x}_a[\partial_x u]) - \bar{u}'[u] - \bar{u}[u'] - \dot{x}_a([u]\partial_x\bar{u} + \bar{u}[\partial_x u]) \\ &= \ddot{x}_{ta}[u] - \bar{u}'[u] - \dot{x}_a[u]\partial_x\bar{u} \end{aligned} \tag{15}$$

which is another form of (13-b).

Exemple 1

Consider the case of a constant initial jump

$$u^0 = aI_{\{x<0\}} \quad \Rightarrow \quad u = aI_{\{x<at/2\}}$$

The shock position is $x(t,a) = at/2$. The derivative with respect to a is $\dot{x}_a = t/2$ and

$$\partial_a u = I_{\{x<at/2\}} + \frac{at}{2}\delta(x - at/2) := u' - [u]\dot{x}_a\delta(x - at/2)$$

Obviously it satisfies (13-a) and the second equation in (13-b) with $\dot{x}_a = t/2$ because $[u] = -a$. The first equation in (13-b) is also trivially satisfied.

Example 2

If we choose $u^0 = (a_1 + x)a_2^{-1}I_{\{x<0\}}$ then the solution to Burger's equation is

$$u = \frac{a_1 + x}{t + a_2}I_{\{x<x(t,a)\}} \text{ with } x(t,a) = a_1(\sqrt{\frac{t}{a_2} + 1} - 1)$$

The derivatives are ${u'}^+ = 0$ and

$${u'}_1^- = \frac{1}{t + a_2}, \quad \dot{x_1} = \sqrt{\frac{t}{a_2} + 1} - 1 \qquad {u'}_2^- = -\frac{a_1 + x}{(t + a_2)^2}, \quad \dot{x_2} = -\frac{ta_1}{2a_2^2\sqrt{\frac{t}{a_2} + 1}}$$

Let us verify (13-b) for the derivative with respect to a_1 first. It can be checked that

$$[u] = -\frac{(t + a_2)^{-1/2}}{\sqrt{a_2}} \qquad \bar{u}'[u] = -\frac{(t + a_2)^{-3/2}}{2\sqrt{a_2}} = \frac{d}{dt}(\frac{(t + a_2)^{-1/2}}{\sqrt{a_2}} + c)$$

because $a_1 + x = a_1\sqrt{(t + a_2)/a_2}$. So $\dot{x}_a$ is this last function above divided by $[u]$, giving

$$\dot{x}_a = -1 + c\sqrt{t + a_2}$$

Since $\dot{x}_a(0) = 0$ this gives the right value for c.

Let us now verify (13-b) for the derivative with respect to a_2.

$$\bar{u}' = -\frac{a_1 + x}{2(t + a_2)^2} = -\frac{a_1}{2\sqrt{a_2}(t + a_2)^{3/2}}$$

Hence

$$\bar{u}'[u] = \frac{a_1}{2a_2(t + a_2)^2} = -\frac{a_1}{2a_2}\frac{d}{dt}(\frac{1}{t + a_2} + c)$$

Henceforth

$$\dot{x}_2 = \frac{a_1}{2\sqrt{a_2}}\frac{1 + (t + a_2)c}{(t + a_2)^{1/2}}$$

but $\dot{x}_2 = 0$ implies $c = -1/a_2$ and that gives the right value for x_2.

2 Control

Applications of the previous result to the control of shocks is obvious and illustrated below on an academic problem.

Assume that we want to reach a desired state u_d on a sub time space interval $I = R \times (0, T)$. Optimal control would then set the following

$$\min_{a \in R^P} J(a) = \frac{1}{2}\int_I \|u - u_d\|^2 \quad \text{subject to (1)} \tag{16}$$

We assume here that a is multi-dimensional. Derivatives of J are obtained by

$$\begin{aligned}
&\partial_p J := \frac{\partial J}{\partial a_p} = \int_I (u - u_d)\partial_p u \text{ with } \partial_p u = u'_p + \dot{x}_p[u]\delta(x - x(t)) \\
&\partial_t u'_p + \partial_x(u u'_p) = 0 \quad \forall\{x, t\} \in Q \backslash S, \qquad u'_p(x, 0) = \partial_p u^0(x) \quad \forall x \\
&\frac{d}{dt}(\dot{x}_p[u]) = \bar{u'}_p[u] \;\text{ on } S
\end{aligned} \tag{17}$$

So

$$\partial_p J = \int_I (u - u_d)u'_p + \int_{S \cap I}(u - u_d)\dot{x}_p[u]$$

Let v be solution of (T a large enough time)

$$\partial_t v + u\partial_x v = I_I(u - u_d) \qquad v(T) = 0 \tag{18}$$

The characterisics for this problem, starting from x_T at T:

$$\dot{x} = u(x(t), t) \qquad x(T) = x_T$$

never cross the shoc, due to the entropy condition again. Therefore v is continuous because it is the integral on the characteristics of $I_I(u-u_d)$. This continuity property was established for the Euler equation earlier by (Giles[3]), but by another method. Therefore

$$\int_I (u-u_d)u'_p = \int_{Q_S}(\partial_t v + u\partial_x v)u'_p = -\int_{Q_S} v(\partial_t u'_p + \partial_x(uu'_p)) - \int_S v(\nu_t[u'_p] + \nu_x[uu'_p])$$

where Q_S is the union of the the space-time domain before and after the shock. By (12) the first integral on the right hand side is given by (2) while second integral is on $-v\frac{d}{dt}\dot{x}_p[u]$. The integrals on $S \cap I$ cancel and

Theorem 2

$$\partial_p J = \int_R v(x,0)u^0_p(x)dx$$

with v *solution of (18)*

So have this rather surprising result that *with or without shocks the adjoint state and the derivative of* J *is the same*, calculus of variation applies without worrying about the Dirac masses. This could also have been derived from (7) where it is seen that if $\partial_t v + \bar{u}\partial_x v = 0$ then the integral on Σ disappears.

Notice however that the direct mode of automatic differentiation (see for exemple [5],[2]) produces numerical values which approximates Dirac masses.

3 Transonic Flow

The generalization to Euler's equations will be treated in a forthcoming paper. We consider here transonic irrotational stationary flows. After renormalization, the transonic equation in a domain Ω of boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, reads:

$$\nabla \cdot \big((1-|\nabla\phi|^2)^\beta \nabla\phi\big) = 0 \text{ in } \Omega, \quad \rho\frac{\partial\phi}{\partial n}|_{\Gamma_1} = g, \quad \phi|_{\Gamma_2} = \phi_\Gamma \tag{19}$$

with $\gamma = 1.4$, $\beta = 1/(\gamma-1) = 2.5$ in air. The flow velocity is $u = \nabla\phi$ and its density $\rho = (1-|\nabla\phi|^2)^\beta$.

Boundary conditions for nozzle flow on $\Gamma = \partial\Omega$ are of two kinds

$$\frac{\partial\phi}{\partial n}|_{\Gamma_n} = u_n \qquad \phi - <\phi> = \phi_d \text{ on } \Gamma_d = \Gamma\backslash\Gamma_n \tag{20}$$

for some averaging operator $<,>$ (this is because the problem must give a solution up to an undefined constant). Γ_n must contain the lateral walls of the nozzle Γ_w where the flow is tangent to the walls and so $u_n = 0$. It can also contain either Γ_i is the inflow boundary and/or Γ_o the outflow one. If $\Gamma_n = \Gamma$ then it is necessary to add a compatibility relation on the data for mass conservation. Finally an entropy inequality must be added for well posedness : $\Delta\phi > -\infty$.(see Glowinski [4] and Nečas [7]):

3.1 Differentiation

Assume now that ϕ is function of a scalar parameter a via the data of the partial differential equation. Denote by $S(a)$ the shock, i.e. the line of discontinuity of $u \cdot n, \quad u = \nabla\phi$. Denote $\dot{x}_a \cdot n_S \delta a$ the distance in the direction n_S, normal to S pointing inside Ω_+, between $S(a+\delta a)$ and $S(a)$, i.e.

$$S(a+\delta a) = \{x + \delta a \dot{x}_a \cdot n_S \ : \ x \in S(a)\}.$$

Denote by $\Omega_\pm$ the region before the shock and after the shock. Then, I_D being the characteristic function of a set D,

$$u = u_- + (u_+ - u_-)I_{\Omega_+}$$

where $u_\pm$ are smooth functions. So

$$\delta u = \delta u_- + (\delta u_+ - \delta u_-)I_{\Omega_+} - \delta a(u_+ - u_-) \cdot n_S \dot{x}_a \cdot n_S \delta_S$$

where δ_S is the Dirac function on S. This is because the derivative of I_D is a Dirac mass on ∂D. So if ϕ', u' denotes the derivative of ϕ, u with respect to a, then we expect that

$$u' = u'_- + (u'_+ - u'_-)I_{\Omega_+} - [u.n_S]\dot{x}_a \cdot n_S \delta_S$$

and therefore if u' has a Dirac mass on S, ϕ' must be *discontinuous* across S, and u' being the x-derivative of ϕ':

Proposition 1 *An identity gives the displacement of the shock $\dot{x}_a$ once u' and ϕ' are known on the left and right side of the shock.*

$$[\phi']_S = -[u.n_S]_S \dot{x}_a \cdot n_S \tag{21}$$

Away from the shock the transonic equation can be differentiated, giving:

$$\nabla \cdot (\rho' \nabla\phi + \rho\nabla\phi') \equiv \nabla \cdot (\rho(1 - \frac{2\beta u \otimes u}{1 - |u|^2})\nabla\phi') = 0. \tag{22}$$

It contains the linearized Rankine-Hugoniot conditions

$$[\rho(1 - \frac{2\beta u^2}{1 - |u|^2})\frac{\partial\phi'}{\partial n_S}]_S = 0 \tag{23}$$

A formal proof of Proposition 1 is given in [1] for a linear problem arising with porous media. We reproduce below the main argument.

In weak form (19) is

$$\int_\Omega \rho u \cdot \nabla w = 0 \quad \forall w \in \mathcal{D}(\Omega)$$

So as for the Burger equation we integrate over the parameter space and replace w by $\partial_a w$.

$$\iint_{\mathcal{Q}} (\rho u \cdot \nabla \partial_a w = 0 \quad \forall w \in \mathcal{D}(\mathcal{Q})$$

where $\mathcal{Q} = \Omega \times A$. Now we integrate by parts in a:

$$-\iint_{\mathcal{Q}} (\partial_a(\rho u) \cdot \nabla w = 0 \quad \forall w \in \mathcal{D}(\mathcal{Q})$$

but here a new difficulty arise, differentiating ρu would give:

$$\partial_a(\rho u) = \rho(1 - \frac{2\beta u \otimes u}{1-|u|^2})\nabla\phi'$$

which is true in $\mathcal{Q}\backslash\mathcal{S}$ but makes no sense across S because of the Dirac function in $\nabla\phi'$ multiplied by a discontinuous function. The trick then is to define a new variable $v = \rho u$, make explicit use of the fact that v does not jump accross S and to write this definition in weak form. With $\rho = (1 - |\nabla\phi|^2)^\beta$ the system is

$$\iint_{\mathcal{Q}} (\frac{v}{\rho} \cdot \partial_a W + \phi\nabla \cdot \partial_a W + v \cdot \partial_a \nabla w) = 0 \quad \forall w, W_i \in \mathcal{D}(\mathcal{Q})$$

An integration by part and a differentiation of the quotient v/ρ gives

$$\iint_{\mathcal{Q}_S} ((\frac{v'}{\rho} - v\frac{\rho'}{\rho^2}) \cdot W + \phi'\nabla \cdot W + v'\nabla w) + \int_{\mathcal{S}} [\frac{1}{\rho}] v\nu_a \cdot W = 0$$

$$= \iint_{\mathcal{Q}_S} ((u' - \nabla\phi') \cdot W - w\nabla \cdot v') + \int_{\mathcal{S}} ([\frac{1}{\rho}] v\nu_a - [\phi'] n_S \nu_x) \cdot W = 0 \quad \forall w, W_i \in \mathcal{D}(\mathcal{Q}) \quad (24)$$

This gives $u' = \nabla\phi',\ \nabla \cdot v' = 0$ (which together with the differentiation of $v = \rho\nabla\phi$ gives (22)) and

$$\phi' + \dot{x}_a [\frac{1}{\rho}] \rho u \cdot n_S = 0$$

which is the same as (21)).

4 Numerical Simulation

4.1 Orientation

To compute the displacement of the shock in a nozzle due to a change in the boundary conditions one should

1. Solve (22,23) with the same type of boundary condition as the nonlinear parent but homogeneous except the for the one which varies.
2. Compute the displacement of the shock by (21).

Consider the case of a divergent nozzle where we change only the outflow value of ϕ, keeping the inflow supersonic. Up to the shock the solution is completely determined by the inflow condition so it does not change. Hence $\phi' = 0$ before the shock.
Downstream of the shock we will use the Rankine-Hugoniot conditions to obtain an homogeneous Neumann condition on ϕ' which together with the downstream boundary condition allows us to integrate the PDE in this zone. Then we find the jump of ϕ' and deduce the change in the shock position from (21).

4.2 Numerical Algorithm

As the transonic equation is nonlinear we used a fixed point algorithm with a small under-relaxation parameter (for instance 0.01); convergence is obtained with 50 to 200 iterations:

$$\nabla \cdot (\rho^m \nabla \phi^{m+1}) = 0$$

To compute ρ^{m+1} we compute the two roots $u_\pm$ of

$$(1 - u^2)^\beta u = (1 - |\nabla \phi^{m+1}|^2)^\beta |\nabla \phi^{m+1}|$$

and set $\rho^{m+1} = (1 - u_-^2)^\beta$ if u decreases or if both u and ρu grows on the streamline and set $\rho^{m+1} = (1 - u_+^2)^\beta$ otherwise.

4.3 The Divergent Nozzle

Using `freefem+`(http://www.freefem.org) we computed the solution of the transonic equation in a symmetric nozzle of equation

$$\Gamma_w = \{(y(x), x) \ : \ y(x) = 1 + \frac{1}{8}(3x^2 - 2x^3), \ x \in (0, 1)\}$$

We performed two computations with $u_i = 0.4$, and a potential difference of 0.4 or 0.44. The level curves of $\frac{\partial \phi}{\partial x}$ for these are reported on figure 4.3. They show that only the region after the shock changes, as predicted by the theory. Finally we have solved numerically the PDE of ϕ' in the domain right to the shock with Neumann homogenous conditions except on the outflow boundary where we have a Dirichlet condition equal to 0.04. The level lines are shown on figure 4.3. It predicts a shock displacement parallel and of distance 0.04/(0.6-0.2)=0.1 which is compatible with the experiments of figure 4.3.

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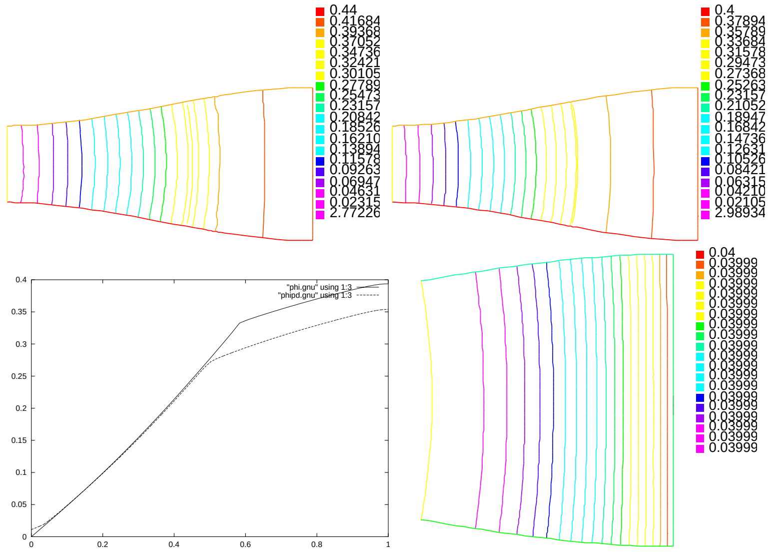

Figure 1: Level lines of ϕ with a changing potential at the outflow boundary going from a=0.40(left) to a=0.44 (right). Bottom left: Level lines of ϕ'. Bottom right: Plot of $x \to \phi(x)|_{\Gamma_w}$. for both values of a.

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