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Vorticity–vector potential formulations of the Stokes equations in the half-space

Tahar Zamène Boulmezaoud[1,2,*,†] and Mohamed Medjden[3,‡]

[1] *Laboratoire des Mathématiques, Université de Versailles Saint-Quentin en Yvelines, 45, Avenue des Etats Unis, 75035 Versailles, France*
[2] *Laboratoire de Mathématiques Appliquées, Université de Pau et des Pays de l'Adour, IPRA- Avenue de l'université, 64000, Pau, France*
[3] *U.S.T.H.B., Faculté des mathématiques, BP32 El Alia Bab-Ezzouar, 16111, Algiers, Algeria*

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SUMMARY

The main objective of this paper is to propose two vorticity–vector potential formulations of the Stokes problem in the half space of $\mathbb{R}^3$. Weighted spaces are used for describing the behaviour at large distances.

RÉSUMÉ

On propose deux formulations de type vorticité–potentiel vecteur du problème de Stokes dans le demi-espace de $\mathbb{R}^3$. Les espaces de Sobolev avec poids sont utilisés pour décrire le comportement des fonctions à l'infini.

KEY WORDS: Stokes equations; weighted Sobolev spaces; half-space; unbounded domains

1. INTRODUCTION

Let n be an integer and let $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n,\ x_n > 0\}$ be the upper half-space of $\mathbb{R}^n$. In this paper, we consider the Stokes equations governing the motion of an incompressible viscous fluid in $\mathbb{R}^n_+$. They are given by

$$-\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \mathbb{R}^n_+, \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^n_+$$

*Correspondence to: Tahar Zamène Boulmezaoud, Laboratoire des Mathématiques, Université de Versailles Saint-Quentin en Yvelines, 45, Avenue des Etats Unis, 75035 Versailles, France.
†E-mail: boulmeza@math.uvsq.fr
‡E-mail: mohmedjden@yahoo.fr

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where $\boldsymbol{u}$ and p are the primitive variables, namely the velocity and the pressure of the fluid, and $\boldsymbol{f}$ is the external force. These equations must be completed by some conditions at the boundary $\{x_3 = 0\}$ and, since the half-space is unbounded, by a careful specification of what happens at infinity.

The Stokes system in unbounded regions of space has been studied by several authors, especially when the domain is the exterior of a bounded obstacle or the whole space (see References [1–5]). When the domain is the half-space, one can recall the works of Ukai [6], Borchers and Miyakawa [7], Borchers and Pileckas [8], Cattabriga [3], Galdi [2], Simader [9] and Boulmezaoud [10]. The results exposed in Reference [10], based on the use of weighted Sobolev spaces, cover a wide class of behaviours at infinity. One of these results states that for each integer $k \in \mathbb{Z}$, the Stokes system with an adhesion or a slip condition at $x_3 = 0$, admits at least one solution $(\boldsymbol{u}, p)$ satisfying

$$\int_{\mathbb{R}^n_+} (1 + |\boldsymbol{x}|^2)^{k-1} |\boldsymbol{u}|^2 \, \mathrm{d}x < +\infty, \quad \int_{\mathbb{R}^n_+} (1 + |\boldsymbol{x}|^2)^{k} |\nabla \boldsymbol{u}|^2 \, \mathrm{d}x < +\infty$$

$$\int_{\mathbb{R}^n_+} (1 + |\boldsymbol{x}|^2)^{k} |p|^2 \, \mathrm{d}x < +\infty$$

provided that f satisfies some natural conditions.

The main purpose of this paper is to prove that in the case $k = 0$, the 3D Stokes problem

$$(\mathrm{P}_1) \quad \begin{cases} -\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \mathbb{R}^3_+ \\ \operatorname{div} \boldsymbol{u} = h & \text{in } \mathbb{R}^3_+ \\ \boldsymbol{u} = \boldsymbol{g} & \text{at } x_3 = 0 \end{cases}$$

admits two well posed weak formulations in terms of *vorticity* and *vector potential*. Namely, we prove that an adequate use of weighted spaces for describing the decay of functions at large distances leads in an easy way to these formulations, without handling singular integrals. The results exposed in papers [10,11] concerning the Laplace equation, curl–div systems and the Stokes equations are extensively used here.

The remainder of this paper is divided into four sections. In Sections 2 and 3, we recall some properties of weighted Sobolev spaces, and some results concerning the Laplace operator and vector potentials in the half-space. The fourth section is devoted to a first formulation of the problem (P_1) in terms of vector potential and vorticity. A second formulation of the same problem is given in the fifth section.

In the sequel, $\Sigma = \{x_3 = 0\}$ will refer to the boundary of $\mathbb{R}^3$.

2. WEIGHTED SOBOLEV SPACES

In $\mathbb{R}^3$ we consider the basic weight

$$\rho(r) = \sqrt{1 + r^2}$$

with $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ the distance to the origin. Given two integers $m \geqslant 0$ and $k \in \mathbb{Z}$, $W^m_k(\mathbb{R}^3_+)$ denote the space of all the measurable functions u whose generalized derivatives for orders $|\mu| \leqslant m$ satisfy

$$\rho^{k+|\mu|-m} D^{\mu} u \in L^2(\mathbb{R}^3_+)$$

($D^{\mu} u = \partial_1^{\mu_1} \partial_2^{\mu_2} \partial_3^{\mu_3} u$, $\mu = (\mu_1, \mu_2, \mu_3)$). The space $W^m_k(\mathbb{R}^3_+)$ is a Hilbert space when it is equipped with the inner product:

$$(u, v)_{W^m_k(\mathbb{R}^3_+)} = \sum_{|\lambda| \leqslant m} \int_{\mathbb{R}^3_+} \rho(r)^{2(k-m+|\lambda|)} D^{\lambda} u . D^{\lambda} v \, \mathrm{d}x$$

In what follows $\mathscr{D}(\mathbb{R}^3_+)$ denotes the space of $\mathscr{C}^{\infty}$ functions with a compact support included in $\mathbb{R}^3_+$. The closure of $\mathscr{D}(\mathbb{R}^3_+)$ in $W^m_{\alpha}(\mathbb{R}^3_+)$ is denoted by $\overset{\circ}{W}{}^m_{\alpha}(\mathbb{R}^3_+)$. When $m<0$, $W^m_{\alpha}(\mathbb{R}^3_+)$ refers to the dual of $\overset{\circ}{W}{}^{-m}_{-\alpha}(\mathbb{R}^3_+)$.

The reader can consult, e.g. References [11–14] for a detailed study of these spaces. Note that all the *local* properties of $W^m_{\alpha}(\mathbb{R}^3_+)$ coincide with those of classical Sobolev spaces.

Following Reference [14] one can extend the above definition of $W^m_{\alpha}(\mathbb{R}^3_+)$ to real values of m. This extension is skipped here by sake of simplicity. We retain only that for any integer $m \geqslant 1$ and any real α, there exists a linear, continuous, and onto trace mapping $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{m-1})$ from $W^m_{\alpha}(\mathbb{R}^3_+)$ into $\prod_{j=0}^{m-1} W^{m-j-1/2}_{\alpha}(\Sigma)$ such that $\gamma u = (u(x', 0), \partial_3 u(x', 0), \ldots, \partial_3^{m-1} u(x', 0))$ for each $u \in \mathscr{D}(\bar{\mathbb{R}}^3_+)$.

Now, for each real k, consider the spaces

$$H_k(\mathrm{div};\, \mathbb{R}^3_+) = \{\boldsymbol{u} \in \mathscr{D}'(\mathbb{R}^3_+)^3;\ \rho(r)^{k-1} \boldsymbol{u} \in L^2(\mathbb{R}^3_+)^3,\ \rho(r)^k \mathrm{div}\, \boldsymbol{u} \in L^2(\mathbb{R}^3_+)\}$$

$$H_k(\mathbf{curl};\, \mathbb{R}^3_+) = \{\boldsymbol{u} \in \mathscr{D}'(\mathbb{R}^3_+)^3; \rho(r)^{k-1} \boldsymbol{u} \in L^2(\mathbb{R}^3_+)^3,\ \rho(r)^k \mathbf{curl}\, \boldsymbol{u} \in L^2(\mathbb{R}^3_+)^3\}$$

equipped, respectively, with the norms:

$$\|\boldsymbol{u}\|_{H_k(\mathrm{div};\, \mathbb{R}^3_+)} = (\|\rho^{k-1} \boldsymbol{u}\|^2_{L^2(\mathbb{R}^3_+)} + \|\rho^k \mathrm{div}\, \boldsymbol{u}\|^2_{L^2(\mathbb{R}^3_+)})^{1/2}$$

$$\|\boldsymbol{u}\|_{H_k(\mathbf{curl};\, \mathbb{R}^3_+)} = (\|\rho^{k-1} \boldsymbol{u}\|^2_{L^2(\mathbb{R}^3_+)} + \|\rho^k \mathbf{curl}\, \boldsymbol{u}\|^2_{L^2(\mathbb{R}^3_+)})^{1/2}$$

Recall that the normal trace operator $\boldsymbol{u} \longrightarrow \boldsymbol{u}.\boldsymbol{e}_3 = u_3(x_3 = 0)$ is well defined and continuous from $H_k(\mathrm{div};\, \mathbb{R}^3_+)$ onto $W^{-1/2}_{k-1}(\Sigma)$, the dual of $W^{1/2}_{-k+1}(\Sigma)$. In particular, if u is a function in $W^1_{k-1}(\mathbb{R}^3_+)$ and if $\Delta u \in W^0_k(\mathbb{R}^3_+)$, then the normal derivative $\gamma_1 u = \partial_3 u_{|x_3=0}$ is meaningful and belongs to $W^{-1/2}_{k-1}(\mathbb{R}^3_+)$. Similarly, the tangential trace operator $\boldsymbol{u} \longrightarrow \boldsymbol{u} \wedge \boldsymbol{e}_3$ is also well defined from $H_k(\mathrm{curl}; \mathbb{R}^3_+)$ into $W^{-1/2}_{k-1}(\Sigma)^3$. We have the following Green's formula

$$\forall \boldsymbol{u} \in H_k(\mathrm{div};\, \mathbb{R}^3_+), \forall \varphi \in W^1_{-k+1}(\mathbb{R}^3_+);\ \ (\mathrm{div}\, \boldsymbol{u}, \varphi) + (\boldsymbol{u}, \nabla \varphi) = \langle \boldsymbol{u}.\boldsymbol{e}_3, \boldsymbol{\varphi} \rangle_{x_3=0} \tag{1}$$

$$\forall \boldsymbol{u} \in H_k(\mathbf{curl};\, \mathbb{R}^3_+), \forall \boldsymbol{\varphi} \in W^1_{-k+1}(\mathbb{R}^3_+)^3;\ \ (\mathbf{curl}\, \boldsymbol{u}, \boldsymbol{\varphi}) - (\boldsymbol{u}, \mathbf{curl}\, \boldsymbol{\varphi}) = \langle \boldsymbol{u} \wedge \boldsymbol{e}_3, \boldsymbol{\varphi} \rangle_{x_3=0} \tag{2}$$

Here $(.,.)$ refer to the L^2 inner product while $\langle .,. \rangle$ is the duality.

The closure of $\mathscr{D}(\mathbb{R}^3_+)^3$ in $H_k(\mathrm{div};\,\mathbb{R}^3_+)$ is denoted by $\mathring{H}_k(\mathrm{div};\mathbb{R}^3_+)$. Similarly $\mathring{H}_k(\mathbf{curl};\,\mathbb{R}^3_+)$ denotes the closure of $\mathscr{D}(\mathbb{R}^3_+)^3$ in $H_k(\mathbf{curl};\,\mathbb{R}^3_+)$. Recall that (see Reference [11])

$$\mathring{H}_k(\mathrm{div};\mathbb{R}^3_+) = \{\boldsymbol{u} \in H_k(\mathrm{div};\mathbb{R}^3_+);\ \ \boldsymbol{u}.\boldsymbol{e}_3 = 0 \quad \text{at } x_3 = 0\}$$

$$\mathring{H}_k(\mathbf{curl};\,\mathbb{R}^3_+) = \{\boldsymbol{u} \in H_k(\mathbf{curl};\,\mathbb{R}^3_+);\ \ \boldsymbol{u} \wedge \boldsymbol{e}_3 = 0 \quad \text{at} \quad x_3 = 0\}$$

Let $H^{-1}_{-k}(\mathrm{div};\,\mathbb{R}^3_+)$ (resp. $H^{-1}_{-k}(\mathbf{curl};\,\mathbb{R}^3_+)$) refer to the dual of $\mathring{H}_k(\mathrm{div};\mathbb{R}^3_+)$ (resp. $\mathring{H}_k(\mathbf{curl};\,\mathbb{R}^3_+)$). Both are spaces of distributions.

In what follows, $\mathscr{A}^\Delta_k$ (resp. $\mathscr{N}^\Delta_k$) denotes the space of harmonic polynomials with total degree $\leqslant k$ and which are odd (resp. even) with respect to the last variable x_3. When $k \leqslant -1$, we set $\mathscr{A}^\Delta_k = \mathscr{N}^\Delta_k = \{0\}$.

The following results concern the Laplace operator with Neumann or Dirichlet conditions at $x_3 = 0$. The reader can refer to Reference [11] for the proof.

Proposition 2.1 (*The Dirichlet problem*)
Let k be an integer. Then, for each pair $(f,g) \in W^{-1}_k(\mathbb{R}^3_+) \times W^{1/2}_k(\Sigma)$, the problem

$$\Delta u = f \ \text{ in } \mathbb{R}^3_+, \quad u = g \ \text{ at } x_3 = 0 \tag{3}$$

has a solution in $W^1_k(\mathbb{R}^3_+)$ if and only if f and g satisfy

$$\forall q \in \mathscr{A}^\Delta_{k-1}, \quad \langle f, q\rangle - \int_\Sigma g(\boldsymbol{x}').\partial_3 q(\boldsymbol{x}')\,\mathrm{d}x' = 0$$

When it exists, the solution is unique up to an element of $\mathscr{A}^\Delta_{-k-1}$. Moreover, if $(f,g) \in W^{m-1}_{m+k}(\mathbb{R}^3_+) \times W^{m+1/2}_{m+k}(\Sigma)$ with $m \geqslant 0$ an integer, then $u \in W^{m+1}_{m+k}(\mathbb{R}^3_+)$ (and depends continuously on f and g with respect to the quotient norm).

Proposition 2.2 (*The Neumann problem*)
Let k be an integer. Then, for each pair $(f,g) \in W^0_{k+1}(\mathbb{R}^3_+) \times W^{-1/2}_k(\Sigma)$, the problem

$$\Delta u = f \ \text{ in } \mathbb{R}^3_+, \quad \partial_3 u = g \ \text{ at } x_3 = 0 \tag{4}$$

has a solution in $W^1_k(\mathbb{R}^3_+)$ if and only if f and g satisfy

$$\forall q \in \mathscr{N}^\Delta_{k-1}, \int_{\mathbb{R}^3_+} f(\boldsymbol{x}) q(\boldsymbol{x})\,\mathrm{d}x + \langle g, q\rangle_\Sigma = 0$$

When it exists, the solution is unique up to an element of $\mathscr{N}^\Delta_{-k-1}$. Moreover, if $(f,g) \in W^{m-1}_{m+k}(\mathbb{R}^3_+) \times W^{m-1/2}_{m+k}(\Sigma)$ with $m \geqslant 1$ an integer, then $u \in W^{m+1}_{m+k}(\mathbb{R}^3_+)$ (and depends continuously on f and g with respect to the quotient norm).

Proposition 2.3 (*Boulmezaoud* [10, 11])
Let $m \geqslant 1$ and k be two integers. For any h in $W^0_{k+1}(\mathbb{R}^3_+)$ and any $\boldsymbol{g} \in W^{1/2}_{k+1}(\Sigma)^3$, the problem

$$\begin{aligned} \operatorname{div} \boldsymbol{u} &= h \ \text{ in } \mathbb{R}^3_+ \\ \boldsymbol{u} &= \boldsymbol{g} \ \text{ at } x_3 = 0 \end{aligned} \tag{5}$$

admits at least one solution $\boldsymbol{u} \in W^1_{k+1}(\mathbb{R}^3_+)^3$ if and only if the following condition is fulfilled:

$$\text{if } k \geqslant 1 \quad \int_{\mathbb{R}^3_+} h(\boldsymbol{x})\,\mathrm{d}x + \int_{\Sigma} \boldsymbol{g}.\boldsymbol{e}_3\,\mathrm{d}x' = 0 \tag{6}$$

In that case, $\boldsymbol{u}$ can be selected such that

$$\|\boldsymbol{u}\|_{W^1_{1+k}(\mathbb{R}^3_+)^3} \leqslant C\{\|\boldsymbol{g}\|_{W^{1/2}_{k+1}(\Sigma)^3} + \|h\|_{W^0_{k+1}(\mathbb{R}^3_+)}\} \tag{7}$$

with C a constant depending only on k.

3. SOME BASIC RESULTS ON VECTOR POTENTIALS

Consider the space:

$$X_k(\mathbb{R}^3_+) = \{\boldsymbol{\varphi} \in W^0_{k-1}(\mathbb{R}^3_+)^3;\ \rho^k \,\mathbf{curl}\,\boldsymbol{\varphi} \in L^2(\mathbb{R}^3_+)^3,\ \rho^k \operatorname{div}\boldsymbol{\varphi} \in L^2(\mathbb{R}^3_+)\}$$

This is a Hilbert space for the inner product associated to the norm

$$\|\boldsymbol{\varphi}\|_{X_k(\mathbb{R}^3_+)} = (\|\rho^{k-1}\boldsymbol{\varphi}\|^2_{L^2(\mathbb{R}^3_+)} + \|\rho^k \operatorname{div}\boldsymbol{\varphi}\|^2_{L^2(\mathbb{R}^3_+)} + \|\rho^k\,\mathbf{curl}\,\boldsymbol{\varphi}\|^2_{L^2(\mathbb{R}^3_+)})^{1/2}$$

Consider also the subspaces of $X_k(\mathbb{R}^3_+)$

$$X^T_k(\mathbb{R}^3_+) = \{\boldsymbol{\varphi} \in X_k(\mathbb{R}^3_+);\ \boldsymbol{\varphi}.\boldsymbol{e}_3 = 0 \text{ at } x_3 = 0\}$$
$$X^N_k(\mathbb{R}^3_+) = \{\boldsymbol{\varphi} \in X_k(\mathbb{R}^3_+);\ \boldsymbol{\varphi} \times \boldsymbol{e}_3 = \mathbf{0} \text{ at } x_3 = 0\}$$

In Reference [11] (Corollaries 8 and 10), it is proved that

$$X^N_k(\mathbb{R}^3_+) \hookrightarrow W^1_k(\mathbb{R}^3_+)^3, \quad X^T_k(\mathbb{R}^3_+) \hookrightarrow W^1_k(\mathbb{R}^3_+)^3$$

It follows in particular that the norm $\|.\|_{W^1_k(\mathbb{R}^3_+)^3}$ is well defined on both the spaces $X^N_k(\mathbb{R}^3_+)$ and $X^T_k(\mathbb{R}^3_+)$ and is equivalent to the norm $\|.\|_{X_k(\mathbb{R}^3_+)}$.

Now, for each integer k, we denote by $\mathscr{G}^\Delta_k$ (resp. $\mathscr{H}^\Delta_k$) the space of functions $\boldsymbol{\varphi} \in X^N_{-k}(\mathbb{R}^3_+)$ (resp. $\boldsymbol{\varphi} \in X^T_{-k}(\mathbb{R}^3_+)$) satisfying

$$\mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{0} \text{ in } \mathbb{R}^3_+, \quad \operatorname{div}\boldsymbol{\varphi} = 0 \text{ in } \mathbb{R}^3_+$$

We know that (see Reference [11])

$$\mathscr{G}^\Delta_k = \{\nabla q;\ q \in \mathscr{A}^\Delta_k\}, \quad \mathscr{H}^\Delta_k = \{\nabla q;\ q \in \mathscr{N}^\Delta_k\}$$

In the sequel, for each $\boldsymbol{\varphi} \in X^N_{-k}(\mathbb{R}^3_+)$, $\boldsymbol{\Lambda}_k\boldsymbol{\varphi}$ denotes the orthogonal projection of $\boldsymbol{\varphi}$ on $\mathscr{G}^\Delta_k$ with respect to the inner product of $(.,.)_{X^N_{-k}(\mathbb{R}^3_+)}$. It is also its projection with respect to the weighted L^2 inner product $(.,.)_{W^0_{-k-1}(\mathbb{R}^3_+)^3}$.

Proposition 3.1 (*Boulmezaoud* [11])
The space $X_k^N(\mathbb{R}^3_+)$ is continuously imbedded in $W_k^1(R^3_+)^3$. In addition, there exists a constant C, which depends only on k such that:

$$\|\boldsymbol{\varphi}\|_{W_k^1(\mathbb{R}^3_+)} \leqslant (\|\rho^k \operatorname{div} \boldsymbol{\varphi}\|^2_{L^2(\mathbb{R}^3_+)} + \|\rho^k \mathbf{curl}\, \boldsymbol{\varphi}\|_{L^2(\mathbb{R}^3_+)^3} + \|\rho^{k-1} \boldsymbol{\Lambda}_{-k} \boldsymbol{\varphi}\|^2_{L^2(\mathbb{R}^3_+)^3})^{1/2}$$

for every function $\boldsymbol{\varphi} \in X_k^N(\mathbb{R}^3_+)$.

Note that when $k=-1$, one has $\mathscr{A}^{\Delta}_{-k} = \mathscr{A}^{\Delta}_1 = \{q(x) = cx_3,\ \ c \in \mathbb{R}\}$. Hence $\mathscr{G}^{\Delta}_{-k} = \mathscr{G}^{\Delta}_1 = \{v(x) = c\,\boldsymbol{e}_3,\ \ c \in \mathbb{R}\}$. It follows that for every $\boldsymbol{\varphi} \in X_{-1}^N(R^3_+)$,

$$\boldsymbol{\Lambda}_1 \boldsymbol{\varphi} = \frac{\int_{\mathbb{R}^3_+} \rho^{-4} \boldsymbol{\varphi}.\boldsymbol{e}_3 \, \mathrm{d}x}{\int_{\mathbb{R}^3_+} \rho^{-4} \, \mathrm{d}x} \boldsymbol{e}_3$$

Since

$$\int_{\mathbb{R}^3_+} \rho^{-4} \, \mathrm{d}x = \frac{\pi^2}{2}$$

one gets

$$\boldsymbol{\Lambda}_1 \boldsymbol{\varphi} = \frac{2}{\pi^2} \left(\int_{\mathbb{R}^3_+} \rho^{-4} \boldsymbol{\varphi}.\boldsymbol{e}_3 \, \mathrm{d}x \right) \boldsymbol{e}_3$$

Proposition 3.1 states that the semi norm

$$\boldsymbol{\varphi} \longrightarrow \left(\|\rho^{-1} \operatorname{div} \boldsymbol{\varphi}\|^2_{L^2(\mathbb{R}^3_+)} + \|\rho^{-1} \mathbf{curl}\, \boldsymbol{\varphi}\|^2_{L^2(\mathbb{R}^3_+)} + \left(\int_{\mathbb{R}^3_+} \rho^{-4} \boldsymbol{\varphi}.\boldsymbol{e}_3 \, \mathrm{d}x \right)^2 \right)^{1/2}$$

is a norm on $X_{-1}^N(\mathbb{R}^3_+)$ equivalent to the norm $\|.\|_{X_{-1}(\mathbb{R}^3_+)}$.

We need the two following propositions which assert the existence of vector potentials, with a vanishing normal or tangential component (see Reference [11] for a more general version).

Proposition 3.2
Let $k \leqslant 2$ and $m \geqslant 0$ be two integers. Let $\boldsymbol{v} \in W^m_{m+k}(\mathbb{R}^3_+)^3$ such that $\operatorname{div} \boldsymbol{v} = 0$. Then there exists a unique function $\boldsymbol{\varphi} \in W^{m+1}_{m+k}(\mathbb{R}^3_+)^3 / \mathscr{H}^{\Delta}_{-k}$ depending continuously on $\boldsymbol{u}$ such that

$$\mathbf{curl}\, \boldsymbol{\varphi} = \boldsymbol{v} \text{ in } \mathbb{R}^3_+, \quad \operatorname{div} \boldsymbol{\varphi} = 0 \text{ in } \mathbb{R}^3_+ \quad \text{and} \quad \boldsymbol{\varphi}.\boldsymbol{e}_3 = \mathbf{0} \text{ at } x_3 = 0$$

Proposition 3.3
Let $k \leqslant 2$ and $m \geqslant 0$ be two integers. Let $\boldsymbol{v} \in W^m_{k+m}(\mathbb{R}^3_+)^3$ such that $\operatorname{div} \boldsymbol{v} = 0$ in $\mathbb{R}^3_+$, $\boldsymbol{v}.\boldsymbol{e}_3 = 0$ at $x_3 = 0$. Then there exists a unique function $\boldsymbol{\varphi} \in W^{m+1}_{k+m}(\mathbb{R}^3_+)^3 / \mathscr{G}^{\Delta}_{-k}$ depending continuously on $\boldsymbol{v}$ such that

$$\mathbf{curl}\, \boldsymbol{\varphi} = \boldsymbol{v} \text{ in } \mathbb{R}^3_+, \quad \operatorname{div} \boldsymbol{\varphi} = 0 \text{ in } \mathbb{R}^3_+ \quad \text{and} \quad \boldsymbol{\varphi} \wedge \boldsymbol{e}_3 = \mathbf{0} \text{ at } x_3 = 0$$

4. A FIRST MIXED FORMULATION

Let k be an integer. Let us consider firstly the inhomogeneous Stokes system (P_1) with $\boldsymbol{f} \in W_k^{-1}(\mathbb{R}_+^3)^3$, $h \in W_k^0(\mathbb{R}_+^3)$ and $\boldsymbol{g} \in W_k^{1/2}(\Sigma)^3$. Then, according to Proposition 2.3, we can introduce a vector field $\boldsymbol{u}_0 \in W_k^1(\mathbb{R}_+^3)^3$ such that

$$\operatorname{div} \boldsymbol{u}_0 = h \text{ in } \mathbb{R}_+^3, \quad \boldsymbol{u}_0 = \boldsymbol{g} \text{ at } x_3 = 0$$

provided that h and $\boldsymbol{g}$ satisfy the compatibility condition (6) when $k \geqslant 1$. Consequently, we shall suppose without loss of generality that $h = 0$ and $\boldsymbol{g} = 0$. In the sequel, we deal with the homogeneous Stokes problem

Find $u \in W_k^1(\mathbb{R}_+^3)^3$ *and* $p \in W_k^0(\mathbb{R}_+^3)$ *solution of*

$$(\mathrm{P}_2) \quad \begin{cases} -\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \mathbb{R}_+^3 \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \mathbb{R}_+^3 \\ \boldsymbol{u} = \boldsymbol{0} & \text{at } x_3 = 0 \end{cases}$$

where $\boldsymbol{f} \in W_k^{-1}(\mathbb{R}_+^3)^3$.

The well-posedness, the regularity and the kernel of problem (P_2) were treated in Reference [11] for several kinds of behaviours at infinity. Here, we focus our attention on a formulation of this problem in terms of *vorticity* and *vector potential*. Such a formulation seems to be well-posed in the sense of Babuska–Brezzi only in the intermediate case $k = 0$, for which there is in a way a duality between the behaviours at infinity of the vorticity and the vector potential. Note that the problem (P_2) admits also a direct mixed formulation in terms of the velocity and the pressure when $k = 0$ (see Reference [10]). In what follows, we consider the problem (P_2) when $k = 0$. Let us start with the following simple lemma.

Lemma 4.1

Let $\boldsymbol{f} \in H_0^{-1}(\mathrm{div};\ \mathbb{R}_+^3)$. Then, $\mathbf{curl}\,\boldsymbol{f} \in H_1^{-1}(\mathbf{curl};\ \mathbb{R}_+^3)$ and

$$\|\mathbf{curl}\,\boldsymbol{f}\|_{H_1^{-1}(\mathbf{curl};\ \mathbb{R}_+^3)} \leqslant \|\boldsymbol{f}\|_{H_0^{-1}(\mathrm{div};\ \mathbb{R}_+^3)} \tag{8}$$

Proof

Recall first that $\mathscr{D}(\mathbb{R}_+^3)^3$ is dense in $\mathring{H}_{-1}$ ($\mathbf{curl}$; $\mathbb{R}_+^3$). Let $\boldsymbol{\varphi} \in \mathscr{D}(\mathbb{R}_+^3)^3$. Then, $\mathbf{curl}\,\boldsymbol{\varphi} \in \mathring{H}_0$ (div; $\mathbb{R}_+^3$), and

$$\|\mathbf{curl}\,\boldsymbol{\varphi}\|_{H_0(\mathrm{div};\ \mathbb{R}_+^3)} = \|\mathbf{curl}\,\boldsymbol{\varphi}\|_{W_{-1}^0(\mathbb{R}_+^3)}$$

In the sense of distributions, we can write

$$\langle \mathbf{curl}\,\boldsymbol{f}, \boldsymbol{\varphi} \rangle_{\mathscr{D}'(\mathbb{R}_+^3), \mathscr{D}(\mathbb{R}_+^3)} = \langle \boldsymbol{f}, \mathbf{curl}\,\boldsymbol{\varphi} \rangle_{H_0^{-1}(\mathrm{div};\ \mathbb{R}_+^3), \mathring{H}_0(\mathrm{div};\ \mathbb{R}_+^3)}$$

It follows that

$$\begin{aligned} |\langle \mathbf{curl}\,\boldsymbol{f}, \boldsymbol{\varphi} \rangle| &\leqslant \|\boldsymbol{f}\|_{H_0^{-1}(\mathrm{div};\ \mathbb{R}_+^3)} \|\mathbf{curl}\,\boldsymbol{\varphi}\|_{H_0(\mathrm{div};\ \mathbb{R}_+^3)} \\ &\leqslant \|\boldsymbol{f}\|_{H_0^{-1}(\mathrm{div};\ \mathbb{R}_+^3)} \|\boldsymbol{\varphi}\|_{H_{-1}(\mathbf{curl};\ \mathbb{R}_+^3)} \end{aligned}$$

which ends the proof. □

Now, let $\boldsymbol{u} \in W_0^1(\mathbb{R}_+^3)$ be solution of (P_2). We associate to $\boldsymbol{u}$ two vector fields as follows:

- The *vorticity* of $\boldsymbol{u}$ is the vector field of $L^2(\mathbb{R}_+^3)^3$ given by

$$\boldsymbol{w} = \mathbf{curl}\, \boldsymbol{u} \tag{9}$$

- The *vector potential* of $\boldsymbol{u}$ is the unique vector field $\boldsymbol{\varphi} \in X_{-1}^N(\mathbb{R}_+^3)$ satisfying

$$\mathbf{curl}\, \boldsymbol{\varphi} = \boldsymbol{u} \text{ in } \mathbb{R}_+^3, \ \mathrm{div}\, \boldsymbol{\varphi} = 0 \text{ in } \mathbb{R}_+^3, \ \int_{\mathbb{R}_+^3} \frac{\boldsymbol{\varphi}.\boldsymbol{e}_3}{\rho^4}\, \mathrm{d}x = 0 \tag{10}$$

The existence and the uniqueness of $\boldsymbol{\varphi}$ is ensured by Proposition 3.3.

The vector functions $\boldsymbol{\varphi}$ and $\boldsymbol{w}$ are related by the identity

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{\varphi} = \boldsymbol{w} \text{ in } \mathbb{R}_+^3 \tag{11}$$

Consider now the spaces

$$X = \{\boldsymbol{\eta} \in \overset{\circ}{H}_1(\mathrm{div}; \mathbb{R}_+^3);\ \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{\eta} \in H_1^{-1}(\mathbf{curl}; \mathbb{R}_+^3)\}$$

$$M = \left\{ \boldsymbol{\varphi} \in X_{-1}^N(\mathbb{R}_+^3);\ \int_{\mathbb{R}_+^3} \rho^{-4} \boldsymbol{\varphi}.\boldsymbol{e}_3\, \mathrm{d}x = 0 \right\}$$

equipped with their natural norms. Notice that $M \hookrightarrow \overset{\circ}{H}_{-1}(\mathbf{curl}; \mathbb{R}_+^3)$.

We are now in position to state the following.

Proposition 4.2

Suppose that $\boldsymbol{f} \in H_0^{-1}(\mathrm{div};\ \mathbb{R}_+^3)$ $(\hookrightarrow W_0^{-1}(\mathbb{R}_+^3)^3)$, and let $(\boldsymbol{u}, p) \in W_0^1(\mathbb{R}_+^3)^3 \times L^2(\mathbb{R}_+^3)$ be the solution of (P_2). Then, the pair $(\boldsymbol{w}, \boldsymbol{\varphi})$ belongs to $X \times M$ and is solution of the problem

$$(\mathrm{P}_3) \quad \begin{cases} \forall \boldsymbol{\eta} \in X, & \displaystyle\int_{\mathbb{R}_+^3} \boldsymbol{w}.\boldsymbol{\eta}\, \mathrm{d}x - \langle \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{\eta}, \boldsymbol{\varphi} \rangle - \delta \int_{\mathbb{R}_+^3} \mathrm{div}\, \boldsymbol{\eta}.\mathrm{div}\, \boldsymbol{\varphi}\, \mathrm{d}x = 0 \\ \forall \boldsymbol{v} \in M, & \langle \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{w}, \boldsymbol{v} \rangle + \delta \displaystyle\int_{\mathbb{R}_+^3} \mathrm{div}\, \boldsymbol{w}.\mathrm{div}\, \boldsymbol{v}\, \mathrm{d}x = v^{-1} \langle \mathbf{curl}\, \boldsymbol{f}, \boldsymbol{v} \rangle \end{cases}$$

for any real number δ.

Proof

We have $\boldsymbol{\varphi} \in M$ by definition. On the other hand, observe that

$$\mathrm{div}\, \boldsymbol{w} = 0 \text{ in } \mathbb{R}_+^3, \ \boldsymbol{w}.\boldsymbol{e}_3 = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} (\boldsymbol{u} \times \boldsymbol{e}_3).\boldsymbol{e}_i = 0 \text{ at } x_3 = 0$$

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{w} = -\mathbf{curl}(\Delta \boldsymbol{u}) = v^{-1} \mathbf{curl}\, \boldsymbol{f}$$

Hence, $\boldsymbol{w} \in X$, thanks to Lemma 4.1. The second identity of (P_3) stems clearly from the last equality. The first identity stems from (11) and the inclusion $M \hookrightarrow \overset{\circ}{H}_{-1}(\mathbf{curl}; \mathbb{R}_+^3)$. □

The problem (P_3) can be put into the abstract framework:

$$(P_3')\quad \begin{cases} \text{find } \boldsymbol{w}\in X, \quad \boldsymbol{\varphi}\in M \text{ such that:} \\ \forall \boldsymbol{\eta}\in X,\ a(\boldsymbol{w},\boldsymbol{\eta}) + b_\delta(\boldsymbol{\eta},\boldsymbol{\varphi}) = 0 \\ \forall \boldsymbol{v}\in M, \qquad\qquad b_\delta(\boldsymbol{w},\boldsymbol{v}) = \ell(\boldsymbol{v}) \end{cases}$$

where

$$a(\boldsymbol{w},\boldsymbol{\eta}) = \int_{\mathbb{R}^3_+} \boldsymbol{w}.\boldsymbol{\eta}\,\mathrm{d}x$$

$$b_\delta(\boldsymbol{\eta},\boldsymbol{\varphi}) = -\langle \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\varphi},\boldsymbol{\eta}\rangle - \delta\int_{\mathbb{R}^3_+} \operatorname{div}\boldsymbol{\varphi}.\operatorname{div}\boldsymbol{\eta}\,\mathrm{d}x$$

$$\ell(\boldsymbol{v}) = \frac{1}{v}\int_{\mathbb{R}^3_+} \boldsymbol{f}.\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x$$

Theorem 4.3
Suppose that $\delta > 0$. Then, problem (P_3) admits one and only one solution $(\boldsymbol{w},\boldsymbol{\varphi})\in X\times M$. Furthermore,

$$\begin{aligned} \|\boldsymbol{w}\|_X &\leqslant C_1\beta_0^{-1}\|\boldsymbol{f}\|_{H_0^{-1}(\mathrm{div};\,\mathbb{R}^3_+)} \\ \|\boldsymbol{\varphi}\|_{X_{-1}(\mathbb{R}^3_+)} &\leqslant C_2\beta_0^{-1}\|\boldsymbol{f}\|_{H_0^{-1}(\mathrm{div};\,\mathbb{R}^3_+)} \end{aligned} \tag{12}$$

where $\beta_0 = v\min(1,\delta)$. C_1 and C_2 are two constants not depending on $\boldsymbol{f}$, v and δ.

Proof
The proof of Theorem 4.3 rests on the use well known Babuska–Brezzi theorem (see References [15,16] or [17]):

Theorem 4.4
Let X and M be two Hilbert spaces with norms $\|.\|_X$ and $\|.\|_M$ and let X' and M' be their duals. Let $a(.,.): X\times X\to\mathbb{R}$ and $b(.,.): X\times M\to\mathbb{R}$ be two bilinear and continuous forms. Assume that $a(.,.)$ is such that

$$a(v,v)\geqslant\alpha\|v\|_X^2,\ \forall v\in V = \{w\in X;\ b(w,\mu)=0\ \ \forall\mu\in M\}$$

for some constant $\alpha>0$. Then the problem: given $\ell\in X'$, $\chi\in M'$, find $(u,\lambda)\in X\times M$ such that

$$\begin{aligned} a(u,v) + b(v,\lambda) &= \langle \ell, v\rangle, \quad \forall v\in X \\ b(u,\mu) &= \langle \chi,\mu\rangle \quad \forall \mu\in M \end{aligned}$$

has a solution if and only if $b(.,.)$ satisfies

$$\inf_{\mu\in M}\sup_{v\in X}\frac{b(v,\mu)}{\|v\|_X\|\mu\|_M}\geqslant\beta$$

for some constant $\beta>0$. In that case, the solution is unique.

Now, let us consider the space

$$V_\delta=\{\boldsymbol{\eta}\in X;\ b_\delta(\boldsymbol{\eta},\boldsymbol{v})=0\}$$

Lemma 4.5
Suppose that $\delta\neq 0$. Then, a vector function $\boldsymbol{\eta}\in L^2(\mathbb{R}^3_+)^3$ belongs to V_δ if and only if

$$\mathbf{curl\,curl}\,\boldsymbol{\eta}=\mathbf{0}\ \text{ in }\mathbb{R}^3_+,\quad \operatorname{div}\boldsymbol{\eta}=0\ \text{ in }\mathbb{R}^3_+$$

Proof
Let $\boldsymbol{\eta}\in V_\delta$ and set $\boldsymbol{v}=\nabla h$ with $h\in W^2_{-1}(\mathbb{R}^3_+)$ solution of

$$\Delta h=\rho^2\operatorname{div}\boldsymbol{\eta}\quad\text{in }\mathbb{R}^3_+$$
$$h=0\quad\text{at }x_3=0$$

This problem admits one and only one solution. It follows that

$$b_\delta(\boldsymbol{\eta},\boldsymbol{v})=-\delta\int_{\mathbb{R}^3_+}\rho^2|\operatorname{div}\boldsymbol{\eta}|^2\,\mathrm{d}x=0$$

Thus $\operatorname{div}\boldsymbol{\eta}=0$. Moreover, for any $\boldsymbol{v}\in\mathscr{D}(\mathbb{R}^3_+)^3$ we have

$$b_\delta(\boldsymbol{\eta},\boldsymbol{v})=\langle\mathbf{curl\,curl}\,\boldsymbol{\eta},\boldsymbol{v}\rangle=0$$

Hence, $\mathbf{curl\,curl}\,\boldsymbol{\eta}=\mathbf{0}$ in $\mathscr{D}'(\mathbb{R}^3_+)$. This ends the proof of the lemma. □

We conclude from Lemma 4.5 that

$$a(\boldsymbol{\eta},\boldsymbol{\eta})=\|\boldsymbol{\eta}\|^2_{L^2(\mathbb{R}^3_+)}=\|\boldsymbol{\eta}\|^2_X$$

for each $\boldsymbol{\eta}\in V_\delta$. Hence, a is V_δ-elliptic. It remains to prove inf–sup condition on $b_\delta(.,.)$.

Lemma 4.6 (*Inf–sup condition*)
Suppose that $\delta>0$. Then, there exists a constant $\beta>0$, not depending on δ, such that

$$\inf_{\boldsymbol{v}\in M}\sup_{\boldsymbol{\eta}\in X}\frac{b_\delta(\boldsymbol{\eta},\boldsymbol{v})}{\|\boldsymbol{\eta}\|_X\|\boldsymbol{v}\|_{X^N_{-1}(\mathbb{R}^3_+)}}\geqslant\min(1,\delta)\beta\tag{13}$$

Proof
Let $\boldsymbol{v}\in M$ and let $q\in W^2_2(\mathbb{R}^3_+)$ be solution of the Laplace equation

$$\begin{cases}\Delta q=-\operatorname{div}(\rho^{-2}\mathbf{curl}\,\boldsymbol{v})=-\nabla\rho^{-2}.\mathbf{curl}\,\boldsymbol{v}\in W^0_2(\mathbb{R}^3_+)\\ q=0\ \text{at}\ x_3=0\end{cases}$$

According to Proposition 2.1, q exists and satisfies

$$\|q\|_{W_2^2(\mathbb{R}^3_+)} \lesssim \|\rho^{-3}\mathbf{curl}\,\boldsymbol{v}\|_{W_2^0(\mathbb{R}^3_+)} \lesssim \|\boldsymbol{v}\|_{X^N_{-1}(\mathbb{R}^3_+)}$$

Consider also the function $\theta \in W_1^2(\mathbb{R}^3_+)$ solution of

$$\begin{cases} \Delta\theta = \rho^{-2}\mathrm{div}\,\boldsymbol{v} & \text{in } \mathbb{R}^3_+ \ \ (\in W_1^0(\mathbb{R}^3_+)) \\ \dfrac{\partial\theta}{\partial x_3} = 0 & \text{at } x_3 = 0 \end{cases}$$

By virtue of Theorem 2.1, θ exists, is unique and satisfies

$$\|\theta\|_{W_1^2(\mathbb{R}^3_+)} \lesssim \|\rho^{-2}\mathrm{div}\,\boldsymbol{v}\|_{W_1^0(\mathbb{R}^3_+)} \lesssim \|\boldsymbol{v}\|_{X^N_{-1}(\mathbb{R}^3_+)}$$

According to Proposition 3.2 there exists a unique vector field $\boldsymbol{w}_0 \in W_1^1(\mathbb{R}^3_+)^3$ solution of

$$\begin{cases} \mathbf{curl}\,\boldsymbol{w}_0 = \rho^{-2}\mathbf{curl}\,\boldsymbol{v} + \nabla q & \text{in } \mathbb{R}^3_+, \ \ (\in W_1^0(\mathbb{R}^3_+)) \\ \mathrm{div}\,\boldsymbol{w}_0 = 0 & \text{in } \mathbb{R}^3_+ \\ \boldsymbol{w}_0.\boldsymbol{e}_3 = 0 & \text{at } x_3 = 0 \end{cases}$$

We set $\boldsymbol{\eta} = -\boldsymbol{w}_0 - \nabla\theta$. Then, clearly $\boldsymbol{\eta}$ belongs to X and

$$b_\delta(\boldsymbol{\eta}, \boldsymbol{v}) = \int_{\mathbb{R}^3_+} |\rho^{-1}\mathbf{curl}\,\boldsymbol{v}|^2\,\mathrm{d}x + \delta\int_{\mathbb{R}^3_+} |\rho^{-1}\mathrm{div}\,\boldsymbol{v}|^2\,\mathrm{d}x$$

Here, we used the identity

$$\int_{\mathbb{R}^3_+} \mathbf{curl}\,\boldsymbol{v}.\nabla q\,\mathrm{d}x = 0$$

Inequality (13) stems from Proposition 3.1. This completes also the proof of Theorem 4.3 and Lemma 4.6. □

5. A SECOND FORMULATION

Our task here is to give another formulation of problem (P_2) when the right hand side $\boldsymbol{f}$ is slightly smoother. The first step of this formulation consists to introduce two vector potential operators P_T and P_N defined from the space

$$X^T_{1,0}(\mathbb{R}^3_+) = \{\boldsymbol{\eta} \in X^T_1(\mathbb{R}^3_+);\ \mathrm{div}\,\boldsymbol{\eta} = 0\}$$

into $W_1^2(\mathbb{R}^3_+)^3$ as follows:

- P_T assigns to each $\boldsymbol{\eta} \in X^T_{1,0}(\mathbb{R}^3_+)$ the unique vector field $\boldsymbol{\Phi} \in W_1^2(\mathbb{R}^3_+)^3$ satisfying

$$\mathbf{curl}\,\boldsymbol{\Phi} = \boldsymbol{\eta},\ \mathrm{div}\,\boldsymbol{\Phi} = 0,\ \text{and}\ \boldsymbol{\Phi}.\boldsymbol{e}_3 = 0 \text{ at } x_3 = 0$$

- P_N assigns to each $\boldsymbol{\eta} \in X_{1,0}^T(\mathbb{R}_+^3)$ the unique vector field $\boldsymbol{\Phi} \in W_1^2(\mathbb{R}_+^3)$ satisfying

$$\mathbf{curl}\,\boldsymbol{\Phi} = \boldsymbol{\eta}, \ \operatorname{div} \boldsymbol{\Phi} = 0, \ \text{and} \ \boldsymbol{\Phi} \times \boldsymbol{e}_3 = 0 \ \text{at} \ x_3 = 0$$

These linear operators are well defined and are continuous from $X_{1,0}^T(\mathbb{R}_+^3)$ into $W_0^1(\mathbb{R}_+^3)^3$, thanks to Propositions 3.2 and 3.3.

Now, we consider the closed subspace of $X_1^T(\mathbb{R}_+^3)$ defined as

$$\tilde{X} = \{\boldsymbol{\eta} \in X_{1,0}^T(\mathbb{R}_+^3);\ (P_T - P_N)\boldsymbol{\eta} = 0\}$$

It is quite clear that

$$\tilde{X} = \{\boldsymbol{\eta} = \mathbf{curl}\,\boldsymbol{\Phi};\ \boldsymbol{\Phi} \in \mathring{W}_0^1(\mathbb{R}_+^3)^3 \cap W_1^2(\mathbb{R}_+^3)^3\}$$

On the other hand, we set

$$\tilde{M} = \{\boldsymbol{\varphi} \in M;\ \operatorname{div} \boldsymbol{\varphi} = 0\}$$

We state the following

Theorem 5.1

Suppose that $\boldsymbol{f} \in W_1^0(\mathbb{R}_+^3)^3$ and let $(\boldsymbol{u}, p) \in W_0^1(\mathbb{R}_+^3)^3 \times L^2(\mathbb{R}_+^3)$ be solution of problem (P_2). Then, the corresponding pair $(\boldsymbol{w}, \boldsymbol{\varphi})$ belongs to $\tilde{X} \times \tilde{M}$ and is solution of the problem

$$(\mathrm{P}_4) \quad \begin{cases} \forall \boldsymbol{\eta} \in \tilde{X}, & \displaystyle\int_{\mathbb{R}_+^3} \boldsymbol{w}.\boldsymbol{\eta}\,\mathrm{d}x - \int_{\mathbb{R}_+^3} \mathbf{curl}\,\boldsymbol{\eta}.\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}x = 0 \\ \forall \boldsymbol{v} \in \tilde{M}, & \displaystyle\int_{\mathbb{R}_+^3} \mathbf{curl}\,\boldsymbol{w}.\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x = v^{-1} \int_{\mathbb{R}_+^3} \boldsymbol{f}.\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \end{cases}$$

Moreover, problem (P_4) has one and only one solution $(\boldsymbol{w}, \boldsymbol{\varphi}) \in \tilde{X} \times \tilde{M}$ and we have

$$\begin{aligned} \|\boldsymbol{w}\|_{W_1^1(\mathbb{R}_+^3)} &\leqslant c_1 v^{-1} \|\boldsymbol{f}\|_{W_1^0(\mathbb{R}_+^3)^3} \\ \|\boldsymbol{\varphi}\|_{W_{-1}^1(\mathbb{R}_+^3)} &\leqslant c_2 v^{-1} \|\boldsymbol{f}\|_{W_1^0(\mathbb{R}_+^3)^3} \end{aligned} \tag{14}$$

Proof

We know from Reference [10, Theorem 4.1] that $\boldsymbol{u}$ belongs to $W_1^2(\mathbb{R}_+^3)^3$. Hence, $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{u} \in \tilde{X}$ since $P_T \boldsymbol{w} = P_N \boldsymbol{w} = \boldsymbol{u}$. The pair $(\boldsymbol{w}, \boldsymbol{\varphi})$ is obviously solution of (P_4). Problem (P_4) is quite similar to (P_3') with X replaced by $\tilde{X}$, M by $\tilde{M}$ and $b_\delta(.,.)$ by

$$\tilde{b}(\boldsymbol{\eta}, \boldsymbol{v}) = -\int_{\mathbb{R}_+^3} \mathbf{curl}\,\boldsymbol{\eta}.\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x$$

We set $\tilde{V} = \{\boldsymbol{\eta} \in \tilde{X};\ \tilde{b}(\boldsymbol{\eta}, \boldsymbol{v}) = 0 \text{ for all } \boldsymbol{v} \in \tilde{M}\}$. Let $\boldsymbol{\eta} \in \tilde{V}$ and set $\boldsymbol{\Phi} = P_N \boldsymbol{\eta} = P_T \boldsymbol{\eta}$. Then, there exists a vector field $\boldsymbol{v}_0 \in M$ such that $\mathbf{curl}\,\boldsymbol{v}_0 = -\boldsymbol{\Phi}$, $\operatorname{div} \boldsymbol{v}_0 = 0$. Since $\boldsymbol{v}_0 \in \tilde{M}$ one gets

$$0 = \tilde{b}(\boldsymbol{\eta}, \boldsymbol{v}_0) = \int_{\mathbb{R}_+^3} \mathbf{curl}\,\boldsymbol{\eta}.\boldsymbol{\Phi}\,\mathrm{d}x = \int_{\mathbb{R}_+^3} |\boldsymbol{\eta}|^2\,\mathrm{d}x$$

Thus $\boldsymbol{\eta} = \mathbf{0}$, and $\tilde{V} = \{\mathbf{0}\}$. It follows that the $\tilde{V}$-ellipticity of $a(.,.)$ is trivial. Let us prove that $\tilde{b}(.,.)$ verifies the inf-sup condition.

Lemma 5.2
There exists a constant $\tilde{\beta} > 0$ such that

$$\inf_{\boldsymbol{v} \in \tilde{M}} \sup_{\boldsymbol{\eta} \in \tilde{X}} \frac{\tilde{b}(\boldsymbol{\eta}, \boldsymbol{v})}{\|\boldsymbol{\eta}\|_{X_1^T(\mathbb{R}^3_+)} \|\boldsymbol{v}\|_{X_{-1}^N(\mathbb{R}^3_+)}} \geqslant \tilde{\beta}$$

Proof
Let $\boldsymbol{v} \in \tilde{M}$ and consider $\boldsymbol{\Phi} \in W_1^2(\mathbb{R}^3_+)^3$ solution of the Laplace equation $-\Delta \boldsymbol{\Phi} = \rho^{-2} \, \mathbf{curl}\, \boldsymbol{v}$ in $\mathbb{R}^3_+$, $\boldsymbol{\Phi} = \mathbf{0}$ at $x_3 = 0$. Setting $\boldsymbol{\eta}_0 = -\,\mathbf{curl}\, \boldsymbol{\Phi} \in \tilde{X}$, gives

$$\tilde{b}(\boldsymbol{\eta}_0, \boldsymbol{v}) = \int_{\mathbb{R}^3_+} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\Phi}.\mathbf{curl}\,\boldsymbol{v}\, \mathrm{d}x = \int_{\mathbb{R}^3_+} [-\Delta \boldsymbol{\Phi} + \nabla(\operatorname{div} \boldsymbol{\Phi})].\mathbf{curl}\,\boldsymbol{v}\, \mathrm{d}x = \|\rho^{-1} \mathbf{curl}\, \boldsymbol{v}\|^2_{L^2(\mathbb{R}^3_+)^3}$$

The proof of the lemma is ended by observing that $\|\boldsymbol{\eta}_0\|_{X_1^T(\mathbb{R}^3_+)} \leqslant C \|\rho^{-1} \mathbf{curl}\, \boldsymbol{v}\|_{L^2(\mathbb{R}^3_+)^3}$. The proof of Theorem 5.1 is also ended. □

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