Mathematical Models and Methods in Applied Sciences
Vol. 16, No. 12 (2006) 1887–1918




# JUNCTION IN A THIN MULTIDOMAIN FOR A FOURTH ORDER PROBLEM

ANTONIO GAUDIELLO
*DAEIMI, Università degli Studi di Cassino, via G. Di Biasio 43,*
*03043 Cassino (FR), Italia*
*gaudiell@unina.it*

ELVIRA ZAPPALE
*DIIMA, Università degli Studi di Salerno, via Ponte Don Melillo,*
*84084 Fisciano (SA), Italia*
*zappale@diima.unisa.it*

Received 3 October 2005
Revised 30 November 2005
Communicated by F. Brezzi

We consider a thin multidomain of $\mathbb{R}^N$, $N \geq 2$, consisting (e.g. in a 3D setting) of a vertical rod upon a horizontal disk. In this thin multidomain we introduce a bulk energy density of the kind $W(D^2U)$, where $W$ is a convex function with growth $p \in ]1, +\infty[$, and $D^2U$ denotes the Hessian tensor of a scalar (or vector-valued) function $U$. By assuming that the two volumes tend to zero with the same rate, under suitable boundary conditions, we prove that the limit model is well-posed in the union of the limit domains, with dimensions, respectively, 1 and $N-1$. Moreover, we show that the limit problem is uncoupled if $1 < p \leq \frac{N-1}{2}$, "partially" coupled if $\frac{N-1}{2} < p \leq N-1$, and coupled if $N-1 < p$. The main result is applied in order to derive the equilibrium configuration of two joint beams, T-shaped, clamped at the three endpoints and subject to transverse loads. The main result is also applied in order to describe the equilibrium configuration of a wire upon a thin film with contact at the origin, when the thin structure is filled with a martensitic material.

*Keywords*: Fourth order operator; junction in thin multidomains; beam; wire, thin film; martensitic material; dimension reduction; Γ-convergence.

AMS Subject Classification: 74B20, 74K10, 74K20, 74K30, 74K35, 78M30, 78M35

## 1. Introduction

Recently, there has been a growing interest in the thin structures theory, since these structures arise in many applications (see, for instance, Ref. 6 and references therein). The usual approach consists of dimensional reduction, through asymptotic analysis, from high dimensions to lower ones, and in finding

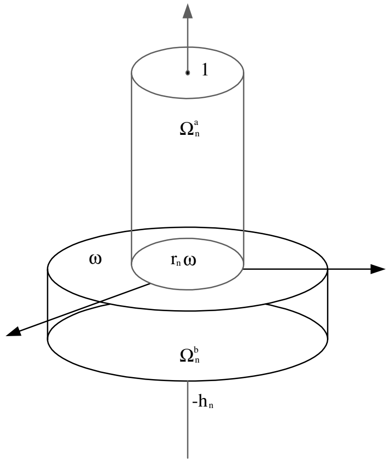


Fig. 1. The thin multidomain.

precise relations between the initial models and the limit ones (see for example Refs. 1, 9, 10, 12, 16, 18, 19, 31–35).

In the past few years, a new insight in dimensional reduction problems has been given in Refs. 22–25, where the authors analyze the junction of two thin cylinders (for a survey on junction problems see also Refs. 7, 8, 11, 15, 27, 29 and 30). The model, described in Ref. 22 through its integral energy and in Ref. 23 through the related constitutive equations, is a quasilinear Neumann second-order scalar problem in a thin multidomain of $\mathbb{R}^N$, $N \geq 2$. Precisely, the thin multidomain consists of two vertical cylinders, one placed upon the other: the first one with constant height and small cross-section, the second one with small thickness and constant cross-section (see Fig. 1). The authors derive the limit problem, by assuming that the volumes of the two cylinders tend to zero with the same rate. An analogous analysis is performed in Refs. 24 and 25, in order to derive the limit problem in the case of the linearized elasticity system in $\mathbb{R}^3$.

In the same spirit, we consider an analogous thin multidomain of $\mathbb{R}^N$ with a bulk energy density of the kind $W(D^2U)$, where $W$ is a convex function with growth $p \in ]1, +\infty[$, and $D^2U$ denotes the Hessian tensor of a scalar function $U$. By assuming that the volumes of the two cylinders tend to zero with the same rate, under suitable boundary conditions on the top of the vertical cylinder and on the lateral surface of the horizontal cylinder, we derive the limit energy, and the limit junction conditions in the origin. More precisely, we prove that the limit problem is well-posed in the union of the limit domains, with dimensions 1 and $N-1$, respectively, and involves 6 limit functions. Moreover, we show that the limit problem is uncoupled if $1 < p \leq \frac{N-1}{2}$, "partially" coupled if $\frac{N-1}{2} < p \leq N-1$, and coupled if $N-1 < p$ (see the limit space (3.8), Theorem 3.1 and Corollary 3.1). Furthermore, if $\frac{N-1}{2} < p$, the minimizers of the limit problem depend also on the limit of the ratio between the volumes of the two cylinders. They do not depend on it, if $1 < p \leq \frac{N-1}{2}$.

We do not take explicitly into account lower order terms, volume forces and surface forces which can be easily treated (see Remarks 3.3 and 3.4).

Since the considered energies are convex, we just study the problem in the scalar case. The main results can be extended, without effort, to the vectorial case with $\mathbb{R}^M$-valued functions $U$ $(M > 1)$, always for convex energies $W(D^2U)$ (see Remark 3.4).

By starting from the two-dimensional plate energy of Kirchhoff–Love (see Ref. 12), the main result is applied in order to obtain a rigorous derivation of the equilibrium configuration of two joint elastic beams, T-shaped, clamped at the three endpoints, and subject to transverse loads (see Sec. 4.1). In this case the limit problem is coupled. The flexion and the twist of the T-structure are explicitly calculated and discussed in the case of some particular loads.

Our result is also applied in order to describe the equilibrium configuration of a martensitic multistructure consisting of a wire upon a thin film with contact at the origin (see Sec. 4.2). The fact that the limit problem is "partially" coupled allows us to build up a model where there is a surprising sharp phase transition, without transition layers, from an austenite phase to a martensitic phase, with bulk interfacial energy not identically null. For a survey on martensitic materials see, for instance, Refs. 3–6, 28, 31, 32, 36 and 37 and the large bibliography quoted therein.

Our results could also be considered in order to describe "non-simple materials of grade 2" (see Refs. 14, 26, 38, 39 and references therein).

In the following section, after having introduced the problem in a thin multidomain, we reformulate it on a fixed domain through appropriate rescalings of the kind proposed by Ciarlet and Destuynder in Ref. 13. Section 3 is devoted to describe the main result. Some mechanical applications of this result are described in Sec. 4. The proof of the main result is developed in several steps and it is based on the $\Gamma$-convergence method introduced by E. De Giorgi (see Ref. 17). A density result for the limit space is given in Sec. 5, a recovery sequence for the $\Gamma$-limit is built in Sec. 6 and a compactness argument is presented in Sec. 7. Finally, the proof is completed in Sec. 8.

## 2. The Original Problem and the Rescalings

Let $N \geq 2$ be an integer number. In the sequel, $x = (x_1, \ldots, x_{N-1}, x_N) = (x', x_N)$ denotes the generic point of $\mathbb{R}^N$, $\mathbb{R}^{k\times k}_s$ (for $k = N, N-1$) the set of symmetric $k \times k$-matrices. Moreover, $D_{x'}$ and $D^2_{x'}$, $D_{x_N}$ and $D^2_{x_N}$ stand for the gradient and the Hessian tensor with respect to the first $N-1$ variables, for the first and the second derivative with respect to the last variable, respectively. Then, according to these notations, $D^2_{x',x_N}$ stands for $(D_{x_N})_{x'}$.

Let $\omega \subset \mathbb{R}^{N-1}$ be a bounded open connected set such that the origin in $\mathbb{R}^{N-1}$, denoted by $0'$, belongs to $\omega$, and let $\{r_n\}_{n\in\mathbb{N}}$, $\{h_n\}_{n\in\mathbb{N}} \subset ]0, 1[$ be two sequences such that

$$\lim_n h_n = 0 = \lim_n r_n. \tag{2.1}$$

For every $n \in \mathbb{N}$, consider the thin multidomain $\Omega_n = \Omega_n^a \cup \Omega_n^b$ ($a$ for "above", $b$ for "below") union of two vertical cylinders, one placed upon the other: $\Omega_n^a = r_n\omega \times [0,1[$ with small cross-section $r_n\omega$ and constant height, $\Omega_n^b = \omega \times ]-h_n, 0[$ with small thickness $h_n$ and constant cross-section (see Fig. 1). Moreover, set $\Omega = \omega \times ]-1,1[$.

In the thin multidomain introduce a convex bulk energy density of the kind $W(D^2U)$. Precisely, let

$$W : \mathcal{M} \in \mathbb{R}_s^{N\times N} \to W(\mathcal{M}) \in \mathbb{R} \tag{2.2}$$

be a function satisfying the following assumptions:

$$W \text{ is convex}; \tag{2.3}$$

$$\begin{aligned} &a + \alpha|\mathcal{M}|^p \le W(\mathcal{M}) \le b + \beta|\mathcal{M}|^p, \quad \forall\, \mathcal{M} \in \mathbb{R}_s^{N\times N};\\ &\text{for some } a, b \in \mathbb{R},\ \alpha, \beta \in ]0, +\infty[ \quad \text{and} \quad 1 < p < +\infty, \end{aligned} \tag{2.4}$$

where, if $\mathcal{M} = (m_{i,j})_{i,j=1,\dots,N}$, $|\mathcal{M}| = \left(\sum_{i,j=1,N}, m_{i,j}^2\right)^{\frac{1}{2}}$. Moreover, in the sequel, for a given $A \in \mathbb{R}_s^{(N-1)\times(N-1)}$, $B \in \mathbb{R}^{N-1}$ and $C \in \mathbb{R}$, $W\left(\left(\begin{smallmatrix} A & B^{\mathrm{T}} \\ B & C \end{smallmatrix}\right)\right)$ means $W(\mathcal{M})$, where $\mathcal{M} = (m_{i,j})_{i,j=1,\dots,N}$ and $(m_{i,j})_{i,j=1,\dots,N-1} = A$, $(m_{N,j})_{j=1,\dots,N-1} = B$, $(m_{i,N})_{i=1,\dots,N-1} = B^{\mathrm{T}}$ and $m_{N,N} = C$.

For every $n \in \mathbb{N}$, consider a function $\overline{U}_n$ which minimizes the energy

$$U_n \to \int_{\Omega_n} W(D^2U_n)dx = \int_{\Omega_n} W\left(\begin{pmatrix} D^2_{x'}U_n & (D^2_{x',x_N}U_n)^{\mathrm{T}} \\ D^2_{x',x_N}U_n & D^2_{x_N}U_n \end{pmatrix}\right)dx, \tag{2.5}$$

among all the functions $U_n \in W^{2,p}(\Omega_n)$ realizing the Dirichlet boundary condition $c^a + d^a \cdot x'$ on top of $\Omega_n^a$, and $f^b + g^b x_N$ on the lateral surface of $\Omega_n^b$, for some $c^a \in \mathbb{R}$, $d^a \in \mathbb{R}^{N-1}$ and $f^b, g^b \in W^{2,p}(\omega)$.

As usual, one tries to reformulate the problem on a fixed domain through appropriate rescalings which map $\Omega_n$ into $\Omega$ (see Fig. 2). Namely, by setting

$$\overline{u}_n(x) = \begin{cases} \overline{u}_n^a(x', x_N) = \overline{U}_n(r_n x', x_N), & (x', x_N) \text{ a.e. in } \Omega^a = \omega \times ]0,1[;\\ \overline{u}_n^b(x', x_N) = \overline{U}_n(x', h_n x_N), & (x', x_N) \text{ a.e. in } \Omega^b = \omega \times ]-1,0[; \end{cases} \tag{2.6}$$

it is easily seen that $\overline{u}_n^a \in W^{2,p}(\Omega^a)$ assumes the rescaled Dirichlet boundary condition $c^a + r_n d^a \cdot x'$ on top of $\Omega^a$, $\overline{u}_n^b \in W^{2,p}(\Omega^b)$ assumes the rescaled

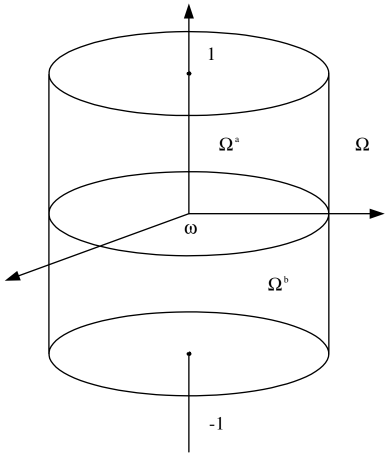


Fig. 2. The fixed domain.



Dirichlet boundary condition $f^b + h_n g^b x_N$ on the lateral boundary of $\Omega^b$. Moreover, $\overline{u}_n = (\overline{u}^a_n, \overline{u}^b_n)$ satisfies the following junction conditions:

$$\begin{cases} \overline{u}^a_n(x',0) = \overline{u}^b_n(r_n x',0), & x' \text{ a.e. in } \omega; \\ \dfrac{1}{r_n} D_{x'}\overline{u}^a_n(x',0) = (D_{x'}\overline{u}^b_n)(r_n x',0), & x' \text{ a.e. in } \omega; \\ D_{x_N}\overline{u}^a_n(x',0) = \dfrac{1}{h_n} D_{x_N}\overline{u}^b_n(r_n x',0), & x' \text{ a.e. in } \omega; \end{cases}$$

and minimizes the rescaled energy (divided through $r_n^{N-1}$):

$$u_n = (u^a_n, u^b_n) \to \int_{\Omega^a} W\left(\begin{pmatrix} \dfrac{1}{r_n^2} D^2_{x'} u^a_n & \left(\dfrac{1}{r_n} D^2_{x',x_N} u^a_n\right)^{\mathrm{T}} \\ \dfrac{1}{r_n} D^2_{x',x_N} u^a_n & D^2_{x_N} u^a_n \end{pmatrix}\right) dx$$

$$+ \frac{h_n}{r_n^{N-1}} \int_{\Omega^b} W\left(\begin{pmatrix} D^2_{x'} u^b_n & \left(\dfrac{1}{h_n} D^2_{x',x_N} u^b_n\right)^{\mathrm{T}} \\ \dfrac{1}{h_n} D^2_{x',x_N} u^b_n & \dfrac{1}{h_n^2} D^2_{x_N} u^b_n \end{pmatrix}\right) dx,$$

among all the functions $u_n$ subject to the same conditions of $\overline{u}_n$.

The aim of this paper consists of describing the limit energy, as $n \to +\infty$, when the volumes of $\Omega^a_n$ and $\Omega^b_n$ tend to zero with same rate, that is

$$\lim_n \frac{h_n}{r_n^{N-1}} = q \in ]0, +\infty[. \tag{2.7}$$

## 3. Main Results

For every $n \in \mathbb{N}$, let $(\overline{u}_n^a, \overline{u}_n^b) \in \mathcal{U}_n$ be a solution of the following problem:

$$K_n^a(\overline{u}_n^a) + \frac{h_n}{r_n^{N-1}} K_n^b(\overline{u}_n^b) = \min_{(u_n^a, u_n^b) \in \mathcal{U}_n} \left\{ K_n^a(u_n^a) + \frac{h_n}{r_n^{N-1}} K_n^b(u_n^b) \right\}, \tag{3.1}$$

where

$$K_n^a : u^a \in W^{2,p}(\Omega^a) \to \int_{\Omega^a} W\left(\begin{pmatrix} \frac{1}{r_n^2} D_{x'}^2 u^a & \left(\frac{1}{r_n} D_{x',x_N}^2 u^a\right)^{\mathrm{T}} \\ \frac{1}{r_n} D_{x',x_N}^2 u^a & D_{x_N}^2 u^a \end{pmatrix}\right) dx, \tag{3.2}$$

$$K_n^b : u^b \in W^{2,p}(\Omega^b) \to \int_{\Omega^b} W\left(\begin{pmatrix} D_{x'}^2 u^b & \left(\frac{1}{h_n} D_{x',x_N}^2 u^b\right)^{\mathrm{T}} \\ \frac{1}{h_n} D_{x',x_N}^2 u^b & \frac{1}{h_n^2} D_{x_N}^2 u^b \end{pmatrix}\right) dx, \tag{3.3}$$

and

$$\begin{aligned} \mathcal{U}_n = \{&(u^a, u^b) \in (c^a + r_n d^a \cdot x' + W_a^{2,p}(\Omega^a)) \times (f^b + h_n g^b x_N + W_b^{2,p}(\Omega^b)) \\ &u^a(x', 0) = u^b(r_n x', 0), \qquad x' \text{ a.e. in } \omega; \\ &\frac{1}{r_n} D_{x'} u^a(x', 0) = (D_{x'} u^b)(r_n x', 0), \quad x' \text{ a.e. in } \omega; \\ &D_{x_N} u^a(x', 0) = \frac{1}{h_n} D_{x_N} u^b(r_n x', 0), \quad x' \text{ a.e. in } \omega\}, \end{aligned} \tag{3.4}$$

with $r_n$, $h_n$, $c^a$, $d^a$, $f^b$ and $g^b$ as defined in Sec. 2, $W_a^{2,p}(\Omega^a)$ the closure, with respect to $W^{2,p}$-norm, of $\{u^a \in C^\infty(\overline{\Omega^a}) : u^a = 0$ in a neighborhood of $\omega \times \{1\}\}$ and $W_b^{2,p}(\Omega^b)$ the closure, with respect to $W^{2,p}$-norm, of $\{u^b \in C^\infty(\overline{\Omega^b}) : u^b = 0$ in a neighborhood of $\partial\omega \times ]-1, 0[\}$. Without loss of generality, one can assume that

$$f^b = 0 = g^b \quad \text{a.e. in } B, \tag{3.5}$$

for some $(N-1)$-dimensional ball $B$ such that $0' \in B \subset\subset \omega$ .

To describe the limit energy of the sequence in (3.1), as $n \to +\infty$, when the volumes of $\Omega_n^a$ and $\Omega_n^b$ tend to zero with the same rate, introduce the limit functionals $K^a$, $K^b$, and the limit spaces $V^p$ (we point out the strong dependence on $p$ for the limit junction conditions):

$$\begin{aligned} K^a : (u^a, \xi^a, z^a) &\in W^{2,p}(\Omega^a) \times (W^{1,p}(\Omega^a))^{N-1} \times L^p(]0,1[; W^{2,p}(\omega)) \\ &\to \int_{\Omega^a} W\left(\begin{pmatrix} D_{x'}^2 z^a & (D_{x_N} \xi^a)^{\mathrm{T}} \\ D_{x_N} \xi^a & D_{x_N}^2 u^a \end{pmatrix}\right) dx, \end{aligned} \tag{3.6}$$

$$K^b : (u^b, \xi^b, z^b) \in W^{2,p}(\Omega^b) \times W^{1,p}(\Omega^b) \times L^p(\omega; W^{2,p}(]-1,0[))$$
$$\to \int_{\Omega^b} W\left(\begin{pmatrix} D^2_{x'}u^b & \left(D_{x'}\xi^b\right)^{\mathrm{T}} \\ D_{x'}\xi^b & D^2_{x_N}z^b \end{pmatrix}\right) dx, \tag{3.7}$$

and

$$V^p = \begin{cases} \mathcal{U}\times\Xi\times Z, & \text{if } p \le \dfrac{N-1}{2}; \\ \{((u^a,u^b),(\xi^a,\xi^b),(z^a,z^b)) \in \mathcal{U}\times\Xi\times Z : & \\ \quad u^a(0) = u^b(0')\}, & \text{if } \dfrac{N-1}{2} < p \le N-1; \\ \{((u^a,u^b),(\xi^a,\xi^b),(z^a,z^b)) \in \mathcal{U}\times\Xi\times Z : & \\ \quad u^a(0) = u^b(0'),\ \xi^a(0) = D_{x'}u^b(0'), & \\ \quad D_{x_N}u^a(0) = \xi^b(0')\} & \text{if } N-1 < p; \end{cases} \tag{3.8}$$

where

$$\begin{aligned} \mathcal{U} &= (c^a + W^{2,p}_a(]0,1[)) \times (f^b + W^{2,p}_0(\omega)), \\ \Xi &= (d^a + (W^{1,p}_a]0,1[)^{N-1}) \times (g^b + W^{1,p}_0(\omega)), \\ Z &= L^p(]0,1[; W^{2,p}_m(\omega)) \times L^p(\omega; W^{2,p}_m(]-1,0[)), \end{aligned} \tag{3.9}$$

($W^{1,p}_a(]0,1[) = \{u \in W^{1,p}(]0,1[) : u(1) = 0\}$, $W^{2,p}_a(]0,1[) = \{u \in W^{2,p}(]0,1[) : u(1) = 0 = Du(1)\}$, and for any subset $A \subset \mathbb{R}^k$, $W^{2,p}_m(A) = \{v \in W^{2,p}(A) : \int_A v d(x_1,\ldots,x_k) = 0,\ \int_A Dv d(x_1,\ldots,x_k) = 0\}$).

This paper is devoted to prove the following result

**Theorem 3.1.** *Let $W$ be a function satisfying* (2.2)–(2.4), *and let, for every $n \in \mathbb{N}$, $(\overline{u}^a_n, \overline{u}^b_n) \in \mathcal{U}_n$ be a solution of Problem* (3.1). *Let $K^a, K^b$ and $V^p$ be as in* (3.6)–(3.8), *respectively. Assume that* (2.1) *and* (2.7) *hold.*

*Then, there exist an increasing sequence of positive integers $\{n_i\}_{i\in\mathbb{N}}$ and $((\overline{u}^a,\overline{u}^b),(\overline{\xi}^a,\overline{\xi}^b),(\overline{z}^a,\overline{z}^b)) \in V^p$, depending possibly on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$, such that*

$$\begin{cases} \overline{u}^a_{n_i} \rightharpoonup \overline{u}^a \ \text{weakly in } W^{2,p}(\Omega^a), \\ \overline{u}^b_{n_i} \rightharpoonup \overline{u}^b \ \text{weakly in } W^{2,p}(\Omega^b), \end{cases} \tag{3.10}$$

$$\begin{cases} \dfrac{1}{r_{n_i}} D_{x'}\overline{u}^a_{n_i} \rightharpoonup \overline{\xi}^a \ \text{weakly in } (W^{1,p}(\Omega^a))^{N-1}, \\ \dfrac{1}{h_{n_i}} D_{x_N}\overline{u}^b_{n_i} \rightharpoonup \overline{\xi}^b \ \text{weakly in } W^{1,p}(\Omega^b), \end{cases} \tag{3.11}$$

$$\begin{cases} \dfrac{1}{r^2_{n_i}} D^2_{x'}\overline{u}^a_{n_i} \rightharpoonup D^2_{x'}\overline{z}^a \ \text{weakly in } (L^p(\Omega^a))^{(N-1)\times(N-1)}, \\ \dfrac{1}{h^2_{n_i}} D^2_{x_N}\overline{u}^b_{n_i} \rightharpoonup D^2_{x_N}\overline{z}^b \ \text{weakly in } L^p(\Omega^b), \end{cases} \tag{3.12}$$

*as* $i \to +\infty$, *and* $((\overline{u}^a, \overline{u}^b), (\overline{\xi}^a, \overline{\xi}^b), (\overline{z}^a, \overline{z}^b))$ *is a solution of the following problem*:

$$K^a(\overline{u}^a, \overline{\xi}^a, \overline{z}^a) + qK^b(\overline{u}^b, \overline{\xi}^b, \overline{z}^b) \\ = \min_{((u^a,u^b),(\xi^a,\xi^b),(z^a,z^b))\in V^p} \{K^a(u^a, \xi^a, z^a) + qK^b(u^b, \xi^b, z^b)\}. \tag{3.13}$$

*Moreover*, *the energies converge in the sense that*

$$\lim_n \left( K_n^a(\overline{u}_n^a) + \frac{h_n}{r_n^{N-1}} K_n^b(\overline{u}_n^b) \right) = K^a(\overline{u}^a, \overline{\xi}^a, \overline{z}^a) + qK^b(\overline{u}^b, \overline{\xi}^b, \overline{z}^b). \tag{3.14}$$

*Furthermore*, *if* $W$ *is strictly convex*, *Problem* (3.13) *admits a unique solution. Consequently convergences* (3.10)–(3.12), *hold true for the whole sequence.*

**Remark 3.1.** Let us point out that the limit problem is uncoupled if $1 < p \leq \frac{N-1}{2}$, partially coupled by the junction condition: $u^a(0) = u^b(0')$ if $\frac{N-1}{2} < p \leq N-1$, and coupled by the previous junction condition for $u$ and by the junction conditions: $\xi^a(0) = D_{x'}u^b(0')$, $D_{x_N}u^a(0) = \xi^b(0')$ if $N-1 < p$. Moveover, if $\frac{N-1}{2} < p$, the minimizers of the limit problem depend also on the limit of the ratio between the volumes of the beam and the plate. They do not depend on it, if $1 < p \leq \frac{N-1}{2}$.

**Remark 3.2.** If $W(D^2U) = |D^2U|^p$, then it is evident that $\overline{z}^a = 0 = \overline{z}^b$.

**Remark 3.3.** If one considers volume forces of the kind

$$\int_{\Omega_n} J_n U_n dx,$$

with $J_n \in L^{\frac{p}{p-1}}(\Omega_n)$, then in the rescaled problem (divided through $r_n^{N-1}$) one has terms of the kind:

$$\int_{\Omega^a} j_n^a u_n^a dx + \frac{h_n}{r_n^{N-1}} \int_{\Omega^b} j_n^b u_n^b dx,$$

where

$$j_n(x) = \begin{cases} j_n^a(x', x_N) = J_n(r_n x', x_N), & (x', x_N) \text{ a.e. in } \Omega^a; \\ j_n^b(x', x_N) = J_n(x', h_n x_N), & (x', x_N) \text{ a.e. in } \Omega^b. \end{cases}$$

Then, by assuming that

$$j_n^a \rightharpoonup j^a \text{ weakly in } L^{\frac{p}{p-1}}(\Omega^a) \text{ and } j_n^b \rightharpoonup j^b \text{ weakly in } L^{\frac{p}{p-1}}(\Omega^b),$$

the additional term

$$\int_{\Omega^a} j^a u^a dx + q \int_{\Omega^b} j^b u^b dx$$

will appear in the limit problem. Similarly, it is possible to add surface forces on the lateral surface of $\Omega_n^a$ and on the basis of $\Omega_n^b$.

As regards the original problem proposed in the previous section, from the rescaling (2.6) and Theorem 3.1, the result below follows:

**Corollary 3.1.** *Let $W$ be a function satisfying* (2.2)–(2.4), *and let, for every $n \in \mathbb{N}$, $\overline{U}_n$ be a minimizer of the original problem introduced in Sec.* 2. *Let $K^a$, $K^b$ and $V^p$ be as in* (3.6)–(3.8), *respectively. Assume that* (2.1) *and* (2.7) *hold.*

*Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and $((\overline{u}^a, \overline{u}^b), (\overline{\xi}^a, \overline{\xi}^b), (\overline{z}^a, \overline{z}^b)) \in V^p$, depending possibly on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$, such that*

$$\begin{cases} \displaystyle -\!\!\!\!\!\!\int_{r_{n_i}\omega} \overline{U}_{n_i}(x', x_N)dx' \rightharpoonup \overline{u}^a \ \textit{weakly in } W^{2,p}(]0,1[), \\ \displaystyle -\!\!\!\!\!\!\int_{-h_{n_i}}^0 \overline{U}_{n_i}(x', x_N)dx_N \rightharpoonup \overline{u}^b \ \textit{weakly in } W^{2,p}(\omega), \end{cases}$$

$$\begin{cases} \displaystyle -\!\!\!\!\!\!\int_{r_{n_i}\omega} D_{x'}\overline{U}_{n_i}(x', x_N)dx' \rightharpoonup \overline{\xi}^a \ \textit{weakly in } (W^{1,p}(]0,1[))^{N-1}, \\ \displaystyle -\!\!\!\!\!\!\int_{-h_{n_i}}^0 D_{x_N}\overline{U}_{n_i}(x', x_N)dx_N \rightharpoonup \overline{\xi}^b \ \textit{weakly in } W^{1,p}(\omega), \end{cases}$$

$$\begin{cases} \displaystyle -\!\!\!\!\!\!\int_{r_{n_i}\omega} D^2_{x'}\overline{U}_{n_i}(x', x_N)dx' \rightharpoonup -\!\!\!\!\!\!\int_{\omega} D^2_{x'}\overline{z}^a(x', x_N)dx' \quad \textit{weakly in } (L^p(]0,1[))^{(N-1)^2}, \\ \displaystyle -\!\!\!\!\!\!\int_{-h_{n_i}}^0 D^2_{x_N}\overline{U}_{n_i}(x', x_N)dx_N \rightharpoonup \int_{-1}^0 D^2_{x_N}\overline{z}^b(x', x_N)dx_N \ \textit{weakly in } L^p(\omega), \end{cases}$$

*as $i \to +\infty$, and $((\overline{u}^a, \overline{u}^b), (\overline{\xi}^a, \overline{\xi}^b), (\overline{z}^a, \overline{z}^b))$ is a solution of Problem* (3.13). *Moreover, the energies converge in the sense that*

$$\lim_n \left[ \frac{1}{r_n^{N-1}} \int_{\Omega_n} W \left( \begin{pmatrix} D^2_{x'}\overline{U}_n & \left(D^2_{x',x_N}\overline{U}_n\right)^{\mathrm{T}} \\ D^2_{x',x_N}\overline{U}_n & D^2_{x_N}\overline{U}_n \end{pmatrix} \right) dx \right]$$

$$= K^a(\overline{u}^a, \overline{\xi}^a, \overline{z}^a) + qK^b(\overline{u}^b, \overline{\xi}^b, \overline{z}^b).$$

*Furthermore, if $W$ is strictly convex, Problem* (3.13) *admits a unique solution. Consequently the previous convergences hold true for the whole sequence.*

**Remark 3.4.** Let $N \geq 2$ and $M \geq 1$ be two integers and

$$W : \mathcal{M} \in (\mathbb{R}_s^{N\times N})^M \to W(\mathcal{M}) \in \mathbb{R},$$

be a convex function satisfying (2.4) with $\mathcal{M} \in (\mathbb{R}_s^{N\times N})^M$. For every $n \in \mathbb{N}$, let $\overline{U}_n : \Omega_n \to \mathbb{R}^M$ be a minimizer of the energy (2.5) among all the admissible vectors

$U_n \in \left(W^{2,p}(\Omega_n)\right)^M$ realizing the Dirichlet boundary condition $c^a + d^a \cdot x'$ on top of $\Omega_n^a$, and $f^b + g^b x_3$ on the lateral surface of $\Omega_n^b$, for some $c^a \in (W^{2,p}(]0,1[))^M$, some $(M \times (N-1))$-matrix $d^a$ and $f^b, g^b \in \left(W^{2,p}(\omega)\right)^M$, where $\cdot$ reads as a row column product.

Then, it is easily seen that the statements of Theorem 3.1 and Corollary 3.1 still hold true, with the limit functions in (3.10)–(3.12) belonging to the following spaces:

$$\begin{aligned}
&\overline{u}^a \in c^a + (W_a^{2,p}(]0,1[))^M, \quad \overline{u}^b \in f^b + \left(W_0^{2,p}(\omega)\right)^M,\\
&\overline{\xi}^a \in d^a + (W_a^{1,p}(]0,1[;\mathbb{R}^{N-1}))^M, \quad \overline{\xi}^b \in g^b + \left(W_0^{1,p}(\omega)\right)^M,\\
&\overline{z}^a \in (L^p(]0,1[;W_m^{2,p}(\omega)))^M, \quad \overline{z}^b \in (L^p(\omega;W_m^{2,p}(]-1,0[)))^M.
\end{aligned}$$

Moreover, if we add to (2.5) a term of the type $\int_{\Omega_n} \Phi(DU_n)dx$, where

$$\Phi : \mathcal{M} \in (\mathbb{R}^N)^M \to \Phi(\mathcal{M}) \in \mathbb{R}$$

is a continuous function satisfying $p$ - growth and coercivity assumptions, then it is evident that the additional term:

$$|\omega| \int_0^1 \Phi\left(\xi^a, D_{x_3}u^a\right) dx_3 + q\int_\omega \Phi\left(D_{x'}u^b, \xi^b\right) dx_1 dx_2$$

will appear in the limit problem. Let us point out that the presence of the leading energy depending on the Hessian tensor allows us to avoid of assuming $\Phi$ convex.

## 4. Applications

### 4.1. *Junction of two beams* (*coupled limit problem*)

In this section we derive rigorously the equilibrium configuration of two joint elastic beams, T-shaped and clamped at the three endpoints (see Fig. 3). We assume that the material of the beams is isotropic and homogeneous, and that the forces applied to the beams are transverse to the T-structure.

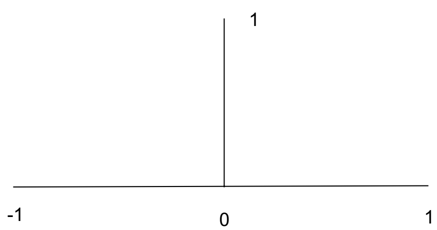


Fig. 3.

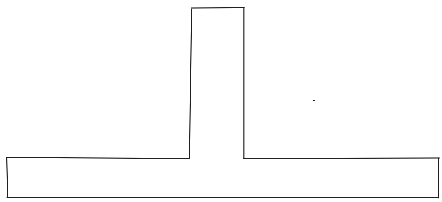


Fig. 4.



We derive our model via an asymptotic analysis based on a dimensional reduction of a Kirchhoff–Love[a] plate (see Ref. 12). Namely, for every $n \in \mathbb{N}$, we introduce an approximating T-shaped plate $\Omega_n = \Omega_n^a \bigcup \Omega_n^b = (]-r_n, r_n[\times[0,1[) \bigcup (]-1,1[\times ]-r_n, 0[)$ (see Fig. 4) with the following flexural energy:

$$c\int_{\Omega_n}(\Delta U_n)^2 + 2(1-\nu)((D^2_{x_1x_2}U_n)^2 - D^2_{x_1}U_nD^2_{x_2}U_n)dx_1dx_2 \\ - \int_{\Omega_n} J_nU_ndx_1dx_2, \tag{4.1}$$

where $\nu \in ]0, \frac{1}{2}[$ is the Poisson ratio, $2c > 0$ represents the flexural rigidity modulus of the plate (precisely, $c = \frac{Eh^3}{12(1-\nu^2)}$, with $E > 0$ Young modulus and $h$ denoting the small thickness of the plate), and $J_n$ represent the transverse forces to the plate.

For every $n \in \mathbb{N}$, let $\overline{U}_n : \Omega_n \to \mathbb{R}$ be the transverse displacement of the plate, which minimizes the energy (4.1) among all the kinematically admissible fields $U_n \in H^2(\Omega_n)$ such that $U_n$ and its normal derivative vanish on $]-r_n, r_n[\times\{1\}$ and on $\{-1,1\}\times]-r_n,0[$. Let $j^a$ and $j^b$ be the weak-$L^2$ limits of the rescaled forces (see Remark 3.3).

In order to derive our model, we pass to the limit, as $n \to +\infty$, in (4.1). Then, by applying Corollary 3.1, it is easily seen that

$$\begin{cases} \dfrac{1}{2r_n}\displaystyle\int_{-r_n}^{r_n} \overline{U}_n(x_1,x_2)dx_1 \rightharpoonup \overline{u}^a \text{ weakly in } H^2(]0,1[), \\ \dfrac{1}{h_n}\displaystyle\int_{-h_n}^{0} \overline{U}_n(x_1,x_2,)dx_2 \rightharpoonup \overline{u}^b \text{ weakly in } H^2(]-1,1[), \end{cases}$$

$$\begin{cases} \dfrac{1}{2r_n}\displaystyle\int_{-r_n}^{r_n} D_{x_1}\overline{U}_n(x_1,x_2)dx_1 \rightharpoonup \overline{\xi}^a \text{ weakly in } H^1\left(]0,1[\right), \\ \dfrac{1}{h_n}\displaystyle\int_{-h_n}^{0} D_{x_2}\overline{U}_n(x_1,x_2,)dx_2 \rightharpoonup \overline{\xi}^b \text{ weakly in } H^1(]-1,1[), \end{cases}$$

[a]In the Kirchhoff–Love theory, a plate, being a three-dimensional solid with a dimension (the thickness) very small with respect to the others, is described by a two-dimensional object.

and $(\overline{u}^a, \overline{u}^b, \overline{\xi}^a, \overline{\xi}^b)$ minimizes the following energy (in this case, an easy computation shows that the functions $D^2_{x_1} z^a$ and $D^2_{x_2} z^b$ which appear in (3.6) and (3.7), respectively, are expressed in terms of $u^a$ and $u^b$, respectively. Precisely, $D^2_{x_1} z^a = -\nu D^2_{x_2} u^a$, $D^2_{x_2} z^b = -\nu D^2_{x_1} u^b$):

$$\begin{aligned}2c\int_0^1 (1-\nu^2)(D^2_{x_2}u^a)^2 + 2(1-\nu)(D_{x_2}\xi^a)^2 dx_2 - \int_0^1 \left(\int_{-1}^1 j^a dx_1\right) u^a dx_2 \\ + c\int_{-1}^1 (1-\nu^2)(D^2_{x_1}u^b)^2 + 2(1-\nu)(D_{x_1}\xi^b)^2 dx_1 - \int_{-1}^1 \left(\int_{-1}^0 j^b dx_2\right) u^b dx_1,\end{aligned} \tag{4.2}$$

among all the fields $u^a \in H^2(]0,1[)$, $u^b \in H^2_0(]-1,1[)$, $\xi^a \in H^1(]0,1[)$, $\xi^b \in H^1_0(]-1,1[)$ with $u^a(1) = 0 = D_{x_2}u^a(1)$, and $\xi^a(1) = 0$, and satisfying the following junction conditions: $u^a(0) = u^b(0)$, $\xi^a(0) = D_{x_1}u^b(0)$, $\xi^b(0) = D_{x_2}u^a(0)$.

It is easily seen that $(\overline{u}^a, \overline{u}^b, \overline{\xi}^a, \overline{\xi}^b) \in H^2(]0,1[) \times H^2(]-1,1[) \times H^1(]0,1[) \times H^1(]-1,1[)$ is characterized as the unique weak solution of the following Euler–Lagrange system:

$$\begin{cases} 4c(1-\nu^2)D^4_{x_2}u^a = \displaystyle\int_{-1}^1 j^a dx_1 \quad \text{in } ]0,1[, \\ 2c(1-\nu^2)D^4_{x_1}u^b = \displaystyle\int_{-1}^0 j^b dx_2 \quad \text{in } ]-1,0[, \\ 2c(1-\nu^2)D^4_{x_1}u^b = \displaystyle\int_{-1}^0 j^b dx_2 \quad \text{in } ]0,1[, \\ D^2_{x_2}\xi^a = 0 \quad \text{in } ]0,1[, \\ D^2_{x_1}\xi^b = 0 \quad \text{in } ]-1,0[ \text{ and in } ]0,1[ \\ u^a(1) = D_{x_2}u^a(1) = 0, \\ u^b(-1) = D_{x_1}u^b(-1) = u^b(1) = D_{x_1}u^b(1) = 0, \\ \xi^a(1) = 0, \\ \xi^b(-1) = \xi^b(1) = 0, \\ u^a(0) = u^b(0), \\ D_{x_2}u^a(0) = \xi^b(0), \\ \xi^a(0) = D_{x_1}u^b(0), \\ 2D^3_{x_2}u^a(0) = D^3_{x_1-}u^b(0) - D^3_{x_1+}u^b(0), \\ (1+\nu)D^2_{x_2}u^a(0) = D_{x_1-}\xi^b(0) - D_{x_1+}\xi^b(0), \\ 4D_{x_2}\xi^a(0) = (1+\nu)\left(D^2_{x_1-}u^b(0) - D^2_{x_1+}u^b(0)\right), \end{cases} \tag{4.3}$$

where $D^m_{x_2-}$ and $D^m_{x_2+}$ denote the left and right derivative of order $m$, respectively.

From (4.3) it follows that

$$\begin{cases} \overline{\xi}^a = D_{x_1}\overline{u}^b(0)(1-x_2) \quad \text{in } ]0,1[, \\ \overline{\xi}^b = \begin{cases} D_{x_2}\overline{u}^a(0)(1+x_1) & \text{in } ]-1,0], \\ D_{x_2}\overline{u}^a(0)(1-x_1) & \text{in } [0,1[, \end{cases} \end{cases} \tag{4.4}$$

where $(\overline{u}^a, \overline{u}^b)$ is the unique weak solution of the following system:

$$\begin{cases} 4c(1-\nu^2)D^4_{x_2}u^a = \displaystyle\int_{-1}^{1} j^a dx_1 \quad \text{in } ]0,1[, \\ 2c(1-\nu^2)D^4_{x_1}u^b = \displaystyle\int_{-1}^{0} j^b dx_2 \quad \text{in } ]-1,0[, \\ 2c(1-\nu^2)D^4_{x_1}u^b = \displaystyle\int_{-1}^{0} j^b dx_2 \quad \text{in } ]0,1[, \\ u^a(1) = D_{x_2}u^a(1) = 0, \\ u^b(-1) = D_{x_1}u^b(-1) = u^b(1) = D_{x_1}u^b(1) = 0, \\ u^a(0) = u^b(0), \\ 2D^3_{x_2}u^a(0) = D^3_{x_1-}u^b(0) - D^3_{x_1+}u^b(0), \\ (1+\nu)D^2_{x_2}u^a(0) = 2D_{x_2}u^a(0), \\ 4D_{x_1}u^b(0) = (1+\nu)\big(D^2_{x_1-}u^b(0) - D^2_{x_1+}u^b(0)\big), \end{cases} \tag{4.5}$$

or equivalently, $(\overline{u}^a, \overline{u}^b)$ minimizes the following energy:

$$\begin{aligned} &2c(1-\nu^2)\int_0^1 (D^2_{x_2}u^a)^2 dx_2 + 4c(1-\nu)(D_{x_1}u^b(0))^2 \\ &\quad - \int_0^1 \left(\int_{-1}^1 j^a dx_1\right) u^a dx_2 + c(1-\nu^2)\int_{-1}^1 (D^2_{x_1}u^b)^2 dx_1 \\ &\quad + 4c(1-\nu)(D_{x_2}u^a(0))^2 - \int_{-1}^1 \left(\int_{-1}^0 j^b dx_2\right) u^b dx_1, \end{aligned} \tag{4.6}$$

among all the fields $u^a \in H^2(]0,1[)$, $u^b \in H^2_0(]-1,1[)$, with $u^a(1) = 0 = D_{x_2}u^a(1)$, and satisfying the following junction condition: $u^a(0) = u^b(0)$.

Note that the obtained limit problem is one-dimensional. The functions $u^a$ and $u^b$ represent the transverse displacement of the vertical beam and the transverse displacement of the horizontal beam, respectively. At the contact point, there is the junction of the transverse displacements. The boundary conditions mean that the T-structure is clamped at the three endpoints. The first and second terms in (4.6) represent the flexural energy and the torsional energy of the vertical beam, respectively. Analogously for the horizontal beam, the fourth and fifths terms in (4.6). The function $\xi^a$ measures, at each point, the tendency to twist of the vertical beam. Analogously, $\xi^b$ for the horizontal beam. The torsion of each beam is maximal at the contact point 0, while it vanishes at the three endpoints. Precisely, the maximal torsion of the vertical beam is equal to the slope of the horizontal beam in the origin, that is $\max|\xi^a| = |D_{x_1}\overline{u}^b(0)|$. Analogously, the maximal torsion of the horizontal beam is equal to the slope of the vertical beam in the origin, that is $\max|\xi^b| = |D_{x_2}\overline{u}^a(0)|$.

In some particular case, we have explicitly calculated the solution of (4.5), and consequently the twist (4.4) of our T-shaped beams. For instance, by choosing $j^a = x_2^2$ and $j^b = x_1^2$, it results that

$$u^a(x_2) = \frac{21 + 13\nu - 12(1+\nu)x_2 - 12x_2^2 - 6(1+\nu)x_2^3 + (9+5\nu)x_2^6}{720c(1-\nu^2)(9+5\nu)},$$

$$u^b(x_1) = \begin{cases} \dfrac{21 + 13\nu - 12(3+2\nu)x_1^2 - 6(1+\nu)x_1^3 + (9+5\nu)x_1^6}{720c(9+5\nu-9\nu^2-5\nu^3)} & \text{in } ]-1,0[, \\[2ex] \dfrac{21 + 13\nu - 12(3+2\nu)x_1^2 + 6(1+\nu)x_1^3 + (9+5\nu)x_1^6}{720c(9+5\nu-9\nu^2-5\nu^3)} & \text{in } ]0,1[, \end{cases}$$

$$\xi^a(x_2) = 0,$$

$$\xi^b(x_1) = \begin{cases} \dfrac{1+x_1}{60c(-9+4\nu+5\nu^2)} & \text{in } ]-1,0[, \\[2ex] \dfrac{1-x_1}{60c(-9+4\nu+5\nu^2)} & \text{in } ]0,1[. \end{cases}$$

Remark that, in this case, the vertical beam does not twist, since the force on the horizontal beam is symmetric (that is $j^b = x_1^2$ is an even function). Our model provides a precise description of this phenomenon. Indeed, we have $\xi^a(x_2) = 0$. For the same reason, $u^b$ is an even function.

If one chooses $j^a = x_2$ and $j^b = x_1$, it results that

$$u^a(x_2) = \frac{-7 - 4\nu + (1+\nu)x_2 + x_2^2 + 2(7+4\nu)x_2^3 - (9+5\nu)x_2^5}{240c(-9-5\nu+9\nu^2+5\nu^3)},$$

$$u^b(x_1) = \begin{cases} \dfrac{(1+x_1)^2[21 + 26\nu + 8\nu^2 - 6(2+\nu)^2 x_1]}{240c(1-\nu)(1+\nu)(3+2\nu)(9+5\nu)} \\[2ex] \quad + \dfrac{(1+x_1)^2[-(54+66\nu+20\nu^2)x_1^2 + (27+33\nu+10\nu^2)x_1^3]}{240c(1-\nu)(1+\nu)(3+2\nu)(9+5\nu)} & \text{in } ]-1,0[, \\[2ex] \dfrac{(x_1-1)^2[21 + 26\nu + 8\nu^2 + (60+80\nu+26\nu^2)x_1]}{240c(1-\nu)(1+\nu)(3+2\nu)(9+5\nu)} \\[2ex] \quad + \dfrac{(x_1-1)^2[(54+66\nu+20\nu^2)x_1^2 + (27+33\nu+10\nu^2)x_1^3]}{240c(1-\nu)(1+\nu)(3+2\nu)(9+5\nu)} & \text{in } ]0,1[, \end{cases}$$

$$\xi^a(x_2) = \frac{1-x_2}{120c(3-\nu-2\nu^2)},$$

$$\xi^b(x_1) = \frac{1 + x_1}{240c(-9 + 4\nu + 5\nu^2)} \quad \text{in} \,]{-1}, 0[,$$
$$\xi^b(x_1) = \frac{1 - x_1}{240c(-9 + 4\nu + 5\nu^2)} \quad \text{in} \,]0, 1[.$$

In this case, since the force on the horizontal beam is odd, there is a torsional effect on the vertical beam. An observer, placed at the contact point of the beams, see the vertical beam twist to the left. Indeed, in our model, $\xi^a$ is positive (except $x_2 = 1$). Remark that $u^b$ is not odd, although the force on the horizontal beam is odd. This event occurs since the force $j^a = x_2$ shifts the origin: precisely $u^b(0) = u^a(0) = \frac{7+4\nu}{240c(9+5\nu-9\nu^2-5\nu^3)}$. For instance, if $j^a = 0$ and $j^b = x_1$, it results that $u^a = 0$, $\xi^a = \frac{1}{120c(\nu-1))(3+2\nu)}(x_2 - 1)$, $\xi^b = 0$ and $u^b$ is an odd polynomial of degree 5.

If one assumes $j^a = k^a$, $j^b = k_1$ in $]{-1}, 0[$ , $j^b = k_2$ in $]0, 1[$, with $k^a$, $k_1$ and $k_2$ constants, it results that the transverse displacements of the beams are described by polynomials of degree 4, which we do not detail here for the sake of clarity. Moreover,

$$\xi^a(x_2) = \frac{(-k_1 + k_2)(1 - x_2)}{96c(3 - \nu - 2\nu^2)},$$

$$\xi^b(x_1) = \begin{cases} \dfrac{(2k^a + 3k_1 + 3k_2)(1 + x_1)}{48c(-9 + 4\nu + 5\nu^2)} & \text{in} \,]{-1}, 0[, \\[2ex] \dfrac{(2k^a + 3k_1 + 3k_2)(1 - x_1)}{48c(-9 + 4\nu + 5\nu^2)} & \text{in} \,]0, 1[. \end{cases}$$

If $k_1 < k_2$, an observer placed at the contact point of the beams, see the vertical beam twisting to the left. Indeed, in our model, $\xi^a$ is positive (except $x_2 = 1$), if $k_1 < k_2$. On the contrary, if $k_2 < k_1$. To avoid torsional effects on the vertical beam, one has to impose symmetric forces on the horizontal beam, i.e. $k_1 = k_2$. In this case, to avoid torsional effects on the horizontal beam too, our model suggests to take $k^a = -3k_1 = -3k_2$.

### 4.2. *Martensitic materials (“partially” coupled limit problem)*

The result obtained in Sec. 3 can also be applied to martensitic materials. These materials are alloys that undergo a diffusionless phase trasition at a critical temperature due to a change in crystalline structure: austenite phase and martensitic phase. The martensitic materials often exhibit a shape-memory effect and they have been recently much studied because of their applications in microactuators and sensors. For a survey on martensitic materials see, for instance, Refs. 3–6, 21, 28, 31, 32, 36, 37 and the large bibliography quoted therein.

By making use of our result, it is possible to describe the equilibrium configuration of a 3D martensitic multistructure composed of a wire $\{(0, 0)\} \times [0, 1]$ orthogonal to a thin film $\omega \times \{0\}$ and with contact at the origin $(0, 0)$, where, for instance, $\omega$ is a circle of $\mathbb{R}^2$ with center $(0, 0)$ (see Fig. 5).

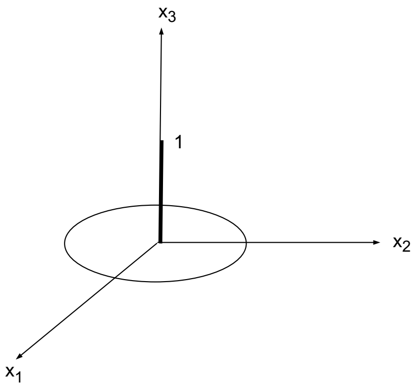


Fig. 5.

We deduce our model via an asymptotic analysis based on a dimensional reduction. Namely, for every $n \in \mathbb{N}$, we consider an approximating multistructure $\Omega_n = \Omega_n^a \bigcup \Omega_n^b$ consisting of two vertical cylinders, one placed upon the other: $\Omega_n^a$ with constant height 1 and small cross-section $r_n\omega$, $\Omega_n^b$ with small thickness $h_n$ and constant cross-section $\omega$ (see Fig. 1).

We assume that $\Omega_n$ is filled with a martensitic material. Hence, we consider free energies of the same type introduced by Bhattacharya and James in Ref. 6 for deriving martensitic thin films, and used by Le Dret and Meunier in Refs. 31 and 32 for obtaining martensitic wires. Namely, the considered energy is the sum of a term of bulk interfacial energy (see Ref. 6), and a term of internal elastic one coming from the geometrically nonlinear theory of martensite (see Ref. 4):

$$\int_{\Omega_n} [k \left|D^2U_n\right|^2 + \Phi(DU_n)]dx_1dx_2dx_3, \tag{4.7}$$

where $k$ is a positive constant, $U_n$ denotes the displacement, $DU_n$ the displacement gradient, $D^2U_n$ the gradient of the displacement gradient, and

$$\Phi : \mathcal{M} \in \mathbb{R}^{3\times 3} \to \Phi(\mathcal{M}) \in \mathbb{R}$$

is a continuous function satisfying the following growth and coercivity assumptions:

$$a + \alpha|\mathcal{M}|^2 \le \Phi(\mathcal{M}) \le b + \beta|\mathcal{M}|^q, \quad \forall\, \mathcal{M} \in \mathbb{R}^{3\times 3};$$
$$\text{for some } a, b \in \mathbb{R},\ \alpha, \beta \in ]0, +\infty[ \quad \text{and} \quad 2 \le q < 6.$$

Moreover, $\Phi$ is assumed to be frame-indifferent and to satisfy a condition of material symmetry (see (2.1) and (2.2) in Ref. 6).

For every $n \in \mathbb{N}$, let $\overline{U}_n : \Omega_n \to \mathbb{R}^3$ be a transformation field which minimizes energy (4.7) among all the kinematically admissible fields $U_n \in H^2(\Omega_n; \mathbb{R}^3)$ realizing the Dirichlet boundary condition $c^a+d^a\cdot(x_1,x_2)$ on top of the wire $\Omega_n^a$, and $f^b+g^bx_3$ on the lateral surface of the film $\Omega_n^b$, for some $c^a\in H^2(]0,1[,\mathbb{R}^3)$, for some $(3\times 2)$-matrix $d^a$ and some $f^b,g^b\in H^2(\omega;\mathbb{R}^3)$, where $\cdot$ reads as a row column product.

In order to deduce our model, we pass to the limit, as $n\to+\infty$, in the approximating problem, by assuming that the volumes of the two cylinders vanish with the same rate as $n\to+\infty$, i.e. $\lim_n \frac{h_n}{r_n^{N-1}}=q\in]0,+\infty[$. In this context, the vectorial version of Corollary 3.1 applies (see Remark 3.4), and hence it is easily seen that, up to a subsequence,

$$\begin{cases} \dfrac{1}{r_n^2|\omega|}\displaystyle\int_{r_n\omega}\overline{U}_n(x_1,x_2,x_3)dx_1dx_2 \rightharpoonup \overline{u}^a \text{ weakly in } H^2(]0,1[;\mathbb{R}^3),\\ \dfrac{1}{h_n}\displaystyle\int_{-h_n}^0 \overline{U}_n(x_1,x_2,x_3)dx_3 \rightharpoonup \overline{u}^b \text{ weakly in } H^2(\omega;\mathbb{R}^3),\end{cases}$$

$$\begin{cases} \dfrac{1}{r_n^2|\omega|}\displaystyle\int_{r_n\omega}D_{x'}\overline{U}_n(x_1,x_2,x_3)dx_1dx_2 \rightharpoonup \overline{\xi}^a \text{ weakly in } H^1\left(]0,1[;(\mathbb{R}^3)^2\right),\\ \dfrac{1}{h_n}\displaystyle\int_{-h_n}^0 D_{x_3}\overline{U}_n(x_1,x_2,x_3)dx_3 \rightharpoonup \overline{\xi}^b \text{ weakly in } H^1(\omega;\mathbb{R}^3),\end{cases}$$

and $(\overline{u}^a,\overline{u}^b,\overline{\xi}^a,\overline{\xi}^b)$ minimizes the following energy:

$$\begin{aligned}|\omega|\int_0^1&\left[k\left|\begin{pmatrix}0 & (D_{x_3}\xi^a)^{\mathrm{T}}\\ D_{x_3}\xi^a & D^2_{x_3}u^a\end{pmatrix}\right|^2+\Phi(\xi^a,D_{x_3}u^a)\right]dx_3\\ &+q\int_\omega\left[k\left|\begin{pmatrix}D^2_{x'}u^b & (D_{x'}\xi^b)^{\mathrm{T}}\\ D_{x'}\xi^b & 0\end{pmatrix}\right|^2+\Phi\left(D_{x'}u^b,\xi^b\right)\right]dx_1dx_2,\end{aligned}\tag{4.8}$$

among all the fields $u^a\in H^2(]0,1[;\mathbb{R}^3)$, $u^b\in H^2(\omega;\mathbb{R}^3)$, $\xi^a\in H^1\left(]0,1[;(\mathbb{R}^3)^2\right)$, $\xi^b\in H^1(\omega;\mathbb{R}^3)$ realizing the Dirichlet boundary condition $u^a(1)=c^a(1)$, $D_{x_3}u^a(1)=D_{x_3}c^a(1)$, $u^b=f^b$, $D_{x'}u^b=D_{x'}f^b$ on $\partial\omega$, $\xi^a(1)=d^a$ and $\xi^b=g^b$ on $\partial\omega$, and the following junction condition: $u^a(0)=u^b(0,0)$.

We have obtained a one-dimensional limit problem on the wire, a two-dimensional limit problem on the thin film and these problems are coupled by the junction condition on the displacement $u^a(0)=u^b(0,0)$.

Remark that the minimum of the limit energy is obtained as

$$\lim_n\frac{1}{r_n^2}\int_{\Omega_n}[k|D^2\overline{U}_n|^2+\Phi(D\overline{U}_n)]dx_1dx_2dx_3.$$

$\xi^a(x_3)\in(\mathbb{R}^3)^2$ represents the Cosserat vectors (see Ref. 2 for the Cosserat theory) which takes into account the limit deformation of the cross-section of the wire (we also emphasize that they are not describing bending and torsion effects), while $\xi^b(x_1,x_2)\in\mathbb{R}^3$ can be thought as a Cosserat director which keeps memory of the original normal to the middle surface of the thin film, and it is not a bending term. Both these vectors appear just as a contribution due to the presence of the interfacial energy. The triple $(\xi^a, D_{x_3}u^a)$ constitutes a Cosserat triple for the wire, while $(D_{x'}u^b, \xi^b)$ is the analogous one for the thin film.

We point out that there is no junction condition involving $(\xi^a, D_{x_3}u^a)$ and $(D_{x'}u^b, \xi^b)$ at the contact point between the wire and the thin film in the limit problem, but only the junction of the displacement $u^a(0) = u^b(0,0)$ (for this reason, we say that the limit problem is "partially" coupled). This fact allows us to build up a model where there is a surprising sharp phase transition, without transition layers, from an austenite phase to a martensitic phase, with bulk interfacial energy not identically null. Precisely, in a thin multistructure wire — thin film with contact at the origin (see Fig. 5), let us consider a free energy (4.8), where $\Phi$ has only two energy wells: SO(3) and SO(3)$U$, with

$$U = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_2 \end{pmatrix},$$

$0 < \eta_1 < 1 < \eta_2$, and SO(3)$U$ standing for the set of all rotation matrices post-multiplied by $U$. The set SO(3) describes the austenite phase, while SO(3)$U$ the martensitic one.

Let us minimize energy (4.8) among all the admissible fields realizing the boundary conditions: $u^a(1) = (0,0,\eta_2)$, $D_{x_3}u^a(1) = (0,0,\eta_2)$, $u^b = (x_1,x_2,0)$, $D_{x'}u^b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ on $\partial\omega$, $\xi^a(1) = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_1 & 0 \end{pmatrix}$ and $\xi^b = (0,0,1)$ on $\partial\omega$, and the following junction condition: $u^a(0) = u^b(0,0)$.

Then it is easily seen that the minimum value is reached at

$$\overline{u}^a = (0,0,\eta_2 x_3), \quad \overline{u}^b = (x_1,x_2,0), \quad \overline{\xi}^a = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_1 & 0 \end{pmatrix}, \quad \overline{\xi}^b = (0 \quad 0 \quad 1),$$

since the bulk interfacial energy is zero at $(\overline{u}^a, \overline{\xi}^a, \overline{u}^b, \overline{\xi}^a)$, and

$$\left(\overline{\xi}^a, D_{x_3}\overline{u}^a\right) = U, \quad \left(D_{x'}\overline{u}^b, \overline{\xi}^b\right) = I \in \mathrm{SO}(3).$$

This means that there is at least a solution which describes the film in the austenite phase, while the wire is in the martensitic one; so in the contact point a sharp transition phase happens. Let us underline that in (4.8) $k$ is positive, i.e. the bulk interfacial energy is not identically null. We emphasize that, if the bulk interfacial energy is not identically null and the structure is made by one and only wire, or by one and only thin film, or in general by a 3D medium without junction, then a sharp phase transition, without transition layers, cannot appear, since in these cases the displacement and the Cosserat vectors are $H^2$ and $H^1$ of the whole domain, respectively (see Refs. 6, 31 and 32).

## 5. Density

Introduce the auxiliary space:

$$\begin{aligned} V = \Big\{ &(u^a, u^b, \xi^a, \xi^b, z^a, z^b) \\ &\in (c^a + C_a^\infty([0,1])) \times \left(f^b + C_0^\infty(\omega)\right) \\ &\quad \times \left(d^a + (C_a^\infty([0,1]))^{N-1}\right) \times \left(g^b + C_0^\infty(\omega)\right) \\ &\quad \times C_0^\infty\left(]0,1[; C^\infty(\overline{\omega})\right) \times C_0^\infty\left(\omega; C^\infty([-1,0])\right) : \\ &\quad u^a(0) = u^b(0'), D_{x'}u^b(0') = \xi^a(0), D_{x_N}u^a(0) = \xi^b(0') \Big\}, \end{aligned} \tag{5.1}$$

where $C_a^\infty([0,1]) = \{u^a \in C^\infty([0,1]) : u^a(1) = 0,\ D^i u^a(1) = 0\,\forall\, i \in \mathbb{N}\}$. Point out that assumption (3.5) gives a meaning to the junction conditions in the definition (5.1).

This section is devoted to prove the following density result:

**Proposition 5.1.** *Let $V^p$ and $V$ be as in* (3.8) *and* (5.1), *respectively. Then $V$ is dense in $V^p$.*

**Proof.** Fix $(u^a, u^b, \xi^a, \xi^b, z^a, z^b) \in V^p$. Since $C_0^\infty(]0,1[; C^\infty(\overline{\omega})) \times C_0^\infty(\omega; C^\infty([-1,0])$ is dense in $Z$, defined in (3.9), it remains to approximate $(u^a, u^b, \xi^a, \xi^b)$.

Let $\{\hat{u}_n^a\}_{n\in\mathbb{N}} \subset C_a^\infty([0,1])$, $\{\hat{u}_n^b\}_{n\in\mathbb{N}} \subset C_0^\infty(\omega)$, $\{\xi_n^a\}_{n\in\mathbb{N}} \subset (C_a^\infty([0,1]))^{N-1}$ and $\{\xi_n^b\}_{n\in\mathbb{N}} \subset C_0^\infty(\omega)$ be such that

$$\hat{u}_n^a \to u^a - c^a \text{ strongly in } W^{2,p}(]0,1[) \text{ and in } C^1([0,1]), \tag{5.2}$$

$$\hat{u}_n^b \to u^b - f^b \text{ strongly in } W^{2,p}(\omega), \tag{5.3}$$

$$\xi_n^a \to \xi^a - d^a \text{ strongly in } \left(W^{1,p}(]0,1[)\right)^{N-1} \text{ and in } (C([0,1]))^{N-1}, \tag{5.4}$$

$$\xi_n^b \to \xi^b - g^b \text{ strongly in } W^{1,p}(\omega), \tag{5.5}$$

as $n \to +\infty$. Now, three cases will be considered.

(i) $N - 1 < p$. In this case, which is simpler to treat, convergences (5.3) and (5.5) also occur in $C^1(\overline{\omega})$ and in $C(\overline{\omega})$, respectively.

For every $n \in \mathbb{N}$, set

$$\begin{cases} \tilde{u}_n^a = c^a + \phi^a[\hat{u}_n^a + u^a(0) - \hat{u}_n^a(0) - c^a] + [1 - \phi^a]\hat{u}_n^a, \\ \tilde{u}_n^b = f^b + \phi^b[\hat{u}_n^b + u^b(0') - \hat{u}_n^b(0')] + [1 - \phi^b]\hat{u}_n^b, \end{cases} \tag{5.6}$$

$$\tilde{\xi}_n^a = d^a + \phi^a[\xi_n^a - \xi_n^a(0) + D_{x'}\hat{u}_n^b(0') - d^a] + [1 - \phi^a]\xi_n^a,$$

$$\tilde{\xi}_n^b = g^b + \phi^b[\xi_n^b - \xi_n^b(0') + D_{x_N}\hat{u}_n^a(0)] + [1 - \phi^b]\xi_n^b,$$

where $\phi^a \in C^\infty([0,1])$, $\phi^a = 1$ in $\left[0, \frac{1}{4}\right]$, $\phi^a = 0$ in $\left[\frac{1}{2}, 1\right]$; $\phi^b \in C_0^\infty(\omega)$, $\phi^b = 1$ in $\frac{1}{4}\omega$. Then, by taking into account (3.5), $(\tilde{u}_n^a, \tilde{u}_n^b, \tilde{\xi}_n^a, \tilde{\xi}_n^b) \in V$ and, as $n \to +\infty$,

$$(\tilde{u}_n^a, \tilde{u}_n^b, \tilde{\xi}_n^a, \tilde{\xi}_n^b) \to (u^a, u^b, \xi^a, \xi^b)$$
$$\text{strongly in } W^{2,p}(]0,1[) \times W^{2,p}(\omega) \times \left(W^{1,p}(]0,1[)\right)^{N-1} \times W^{1,p}(\omega).$$

(ii) $\frac{N-1}{2} < p \leq N-1$. In this case, convergence (5.3) occurs also in $C(\overline{\omega})$. Thus, by making use of the approximating sequences defined in (5.6), it is not restrictive to assume $u^a \in c^a + C_a^\infty([0,1])$ and $u^b \in f^b + C_0^\infty(\omega)$, with $u^a(0) = u^b(0')$. Moreover, by virtue of (5.4) and (5.5), one can also assume that $\xi^a \in d^a + (C_a^\infty([0,1]))^{N-1}$ and $\xi^b \in g^b + C_0^\infty(\omega)$. Now, fix $u^a$, $\xi^a$ and approximate $u^b$ and $\xi^b$ with sequences satisfying the junction conditions in (5.1).

Let $\varphi \in C^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in $]-\infty, 1]$, $\varphi = 0$ in $[2, +\infty[$, and, for every $n \in \mathbb{N}$, set $\varphi_n^b(x') = \varphi(n|x'|)$, $x' \in \mathbb{R}^{N-1}$. Point out that $\varphi_n^b \in C_0^\infty\left(\left\{x' \in \mathbb{R}^{N-1} : |x'| \leq \frac{2}{n}\right\}\right)$ and

$$\|\varphi_n^b\|_{(L^\infty(\mathbb{R}^{N-1}))} \leq 1, \quad \|D_{x'}\varphi_n^b\|_{(L^\infty(\mathbb{R}^{N-1}))^{N-1}} \leq cn, \tag{5.7}$$

$$\|D_{x'}^2\varphi_n^b\|_{(L^\infty(\mathbb{R}^{N-1}))^{(N-1)^2}} \leq cn^2, \tag{5.8}$$

for every $n \in \mathbb{N}$, where, from now on, $c$ denotes any positive constant independent of $n$.

For every $n \in \mathbb{N}$, define

$$\tilde{u}_n^b = f^b + (u^a(0) + \xi^a(0) \cdot x')\varphi_n^b + (1 - \varphi_n^b)(u^b - f^b). \tag{5.9}$$

Then, for $n \in \mathbb{N}$ large enough, $\tilde{u}_n^b \in f^b + C_0^\infty(\omega)$ and, by virtue of (3.5), $\tilde{u}_n^b(0') = u^a(0)$, $D_{x'}\tilde{u}_n^b(0') = \xi^a(0)$. Moreover, since $\tilde{u}_n^b - u^b = (u^a(0) + \xi^a(0) \cdot x' - u^b + f^b)\varphi_n^b$ and $u^a(0) + \xi^a(0) \cdot x' - u^b + f^b \in C^\infty(\overline{\omega})$, by making use of (5.7) it is easily seen that

$$\|\tilde{u}_n^b - u^b\|_{L^p(\omega)}^p \leq \frac{c}{n^{N-1}}, \quad \|D_{x'}\tilde{u}_n^b - D_{x'}u^b\|_{(L^p(\omega))^{N-1}}^p \leq \frac{c}{n^{N-1-p}}, \tag{5.10}$$

for every $n \in \mathbb{N}$. In what concerns the second-order derivatives, since $D_{x'}^2\tilde{u}_n^b - D_{x'}^2u^b = (-D_{x'}^2u^b + D_{x'}^2f^b)\varphi_n^b + \left[(\xi_i^a(0) - D_{x_i}u^b + D_{x_i}f^b)D_{x_j}\varphi_n^b + (\xi_j^a(0) - D_{x_j}u^b + D_{x_j}f^b)D_{x_i}\varphi_n^b\right]_{1 \leq i,j \leq N-1} + (u^a(0) + \xi^a(0) \cdot x' - u^b + f^b)D_{x'}^2\varphi_n^b$, from (5.7) and (5.8) it follows that

$$\|D_{x'}^2\tilde{u}_n^b - D_{x'}^2u^b\|_{(L^p(\omega))^{(N-1)^2}}^p \leq \frac{c}{n^{N-1-p}} + \frac{c}{n^{N-1-2p}}\|(u^a(0) + \xi^a(0) \cdot x' - u^b + f^b)\|_{L^\infty(\{x' \in \omega : |x'| < \frac{2}{n}\})}^p, \tag{5.11}$$

for every $n \in \mathbb{N}$. On the other hand, since $u^a(0) + \xi^a(0) \cdot x' - u^b + f^b \in C^\infty(\overline{\omega})$ vanishes in $0'$ (recall that $u^a(0) = u^b(0')$!) it results that

$$\|u^a(0) + \xi^a(0) \cdot x' - u^b + f^b\|^p_{L^\infty(\{x' \in \mathbb{R}^{N-1}: |x'| < \frac{2}{n}\})} \leq \frac{c}{n^p}, \tag{5.12}$$

for $n \in \mathbb{N}$ large enough. Then, by combining (5.11) with (5.12), it results that

$$\|D^2_{x'} \tilde{u}^b_n - D^2_{x'} u^b\|^p_{(L^p(\omega))^{(N-1)^2}} \leq \frac{c}{n^{N-1-p}}, \tag{5.13}$$

for $n \in \mathbb{N}$ large enough. Consequently, if $\frac{N-1}{2} < p < N-1$, (5.10) and (5.13) involve that

$$\tilde{u}^b_n \to u^b \text{ strongly in } W^{2,p}(\omega),$$

as $n \to +\infty$. If $p = N-1$, estimates (5.10), (5.13) and Mazur Theorem assure the existence of a sequence of convex combinations of elements of $\{\tilde{u}^b_n\}_{n \in \mathbb{N}}$ converging to $u^b$ strongly in $W^{2,p}(\omega)$.

Similarly, by setting, for every $n \in \mathbb{N}$,

$$\tilde{\xi}^b_n = g^b + \varphi^b_n D_{x_N} u^a(0) + (1 - \varphi^b_n)(\xi^b - g^b), \tag{5.14}$$

it results that $\tilde{\xi}^b_n \in g^b + C^\infty_0(\omega)$, for $n \in \mathbb{N}$ large enough, and $\tilde{\xi}^b_n(0') = D_{x_N} u^a(0)$. Moreover, by arguing as in (5.10), it is easily seen that, as $n \to +\infty$, $\{\tilde{\xi}^b_n\}_{n \in \mathbb{N}}$ converges to $\xi^b$ strongly in $W^{1,p}(\omega)$, if $\frac{N-1}{2} < p < N-1$; while a sequence of convex combinations of elements of $\{\tilde{\xi}^b_n\}_{n \in \mathbb{N}}$ converges to $\xi^b$ strongly in $W^{1,p}(\omega)$, if $p = N-1$.

(iii) $1 < p < \frac{N-1}{2}$. By virtue of (5.2)–(5.5), it is not restrictive to assume that $u^a \in c^a + C^\infty_a([0,1])$, $u^b \in f^b + C^\infty_0(\omega)$, $\xi^a \in d^a + (C^\infty_a([0,1]))^{N-1}$ and $\xi^b \in g^b + C^\infty_0(\omega)$. Fix $u^a$, $\xi^a$ and approximate $u^b$ and $\xi^b$ with the sequences defined in (5.9) and (5.14), respectively.

Remark only that (5.10) and (5.11) hold true, (point out that we are not allowed to use (5.12), since, in general, $u^a(0) + \xi^a(0) \cdot x' - u^b + f^b \in C^\infty(\overline{\omega})$ does not vanish in $0'$!). Consequently, $\{\tilde{u}^b_n\}_{n \in \mathbb{N}}$ converges to $u^b$ strongly in $W^{2,p}(\omega)$, if $1 < p < \frac{N-1}{2}$; while a sequence of convex combinations of elements of $\{\tilde{u}^b_n\}_{n \in \mathbb{N}}$ converges to $u^b$ strongly in $W^{2,p}(\omega)$, if $p = \frac{N-1}{2}$. $\square$

## 6. Steps to Γ-Convergence

This section is devoted to build a recovery sequence for the Γ-limit.

**Proposition 6.1.** *Let $W$ be a function satisfying* (2.2)–(2.4). *For every $n \in \mathbb{N}$, let $K^a_n$, $K^b_n$ and $\mathcal{U}_n$ be as in* (3.2)–(3.4), *respectively. Let $K^a$, $K^b$ and $V$ be as in* (3.6), (3.7) *and* (5.1), *respectively. Assume that* (2.1) *and* (2.7) *hold.*

*Then, for every* $(u^a, u^b, \xi^a, \xi^b, z^a, z^b) \in V$ *there exists a sequence* $\{(u_n^a, u_n^b)\}_{n\in\mathbb{N}}$, *with* $(u_n^a, u_n^b) \in \mathcal{U}_n$, *such that*

$$\lim_n K_n^a(u_n^a) = K^a(u^a, \xi^a, z^a), \quad \lim_n K_n^b(u_n^b) = K^b(u^b, \xi^b, z^b). \tag{6.1}$$

**Proof.** Fix $(u^a, u^b, \xi^a, \xi^b, z^a, z^b) \in V$ and, for every $n \in \mathbb{N}$, set

$$u_n^a(x', x_N) = \begin{cases} u^a(x_N) + r_n\xi^a(x_N)\cdot x' + r_n^2 z^a(x) \\ \quad \text{for a.e. } x = (x', x_N) \in \omega \times ]\varepsilon_n, 1[; \\ \Big[-\dfrac{2}{\varepsilon_n^3}u^a(\varepsilon_n) - 2\dfrac{r_n}{\varepsilon_n^3}\xi^a(\varepsilon_n)\cdot x' - 2\dfrac{r_n^2}{\varepsilon_n^3}z^a(x', \varepsilon_n) \\ \quad + \dfrac{2}{\varepsilon_n^3}u^b(r_n x') + 2\dfrac{h_n^2}{\varepsilon_n^3}z^b(r_n x', 0) \\ \quad + \dfrac{1}{\varepsilon_n^2}D_{x_N}u^a(\varepsilon_n) + \dfrac{r_n}{\varepsilon_n^2}D_{x_N}\xi^a(\varepsilon_n)\cdot x' + \dfrac{r_n^2}{\varepsilon_n^2}D_{x_N}z^a(x', \varepsilon_n) \\ \quad + \dfrac{1}{\varepsilon_n^2}\xi^b(r_n x') + \dfrac{h_n}{\varepsilon_n^2}D_{x_N}z^b(r_n x', 0)\Big]\, x_N^3 \\ \quad + \Big[\dfrac{3}{\varepsilon_n^2}u^a(\varepsilon_n) + 3\dfrac{r_n}{\varepsilon_n^2}\xi^a(\varepsilon_n)\cdot x' + 3\dfrac{r_n^2}{\varepsilon_n^2}z^a(x', \varepsilon_n) - \dfrac{3}{\varepsilon_n^2}u^b(r_n x') \\ \quad - 3\dfrac{h_n^2}{\varepsilon_n^2}z^b(r_n x', 0) - \dfrac{1}{\varepsilon_n}D_{x_N}u^a(\varepsilon_n) - \dfrac{r_n}{\varepsilon_n}D_{x_N}\xi^a(\varepsilon_n)\cdot x' \\ \quad - \dfrac{r_n^2}{\varepsilon_n}D_{x_N}z^a(x', \varepsilon_n) - \dfrac{2}{\varepsilon_n}\xi^b(r_n x') - 2\dfrac{h_n}{\varepsilon_n}D_{x_N}z^b(r_n x', 0)\Big]\, x_N^2 \\ \quad + \big[\xi^b(r_n x') + h_n D_{x_N}z^b(r_n x', 0)\big]\, x_N \\ \quad + u^b(r_n x') + h_n^2 z^b(r_n x', 0), \quad \text{for a.e. } x = (x', x_N) \in \omega \times ]0, \varepsilon_n[; \end{cases}$$

and

$$u_n^b(x', x_N) = u^b(x') + h_n\xi^b(x')x_N + h_n^2 z^b(x), \quad \text{for a.e. } x = (x', x_N) \in \Omega^b;$$

where $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset ]0, 1[$ will be determined later on.

It is easily seen that $(u_n^a, u_n^b) \in \mathcal{U}_n$, and the second convergence in (6.1) holds true.

In order to prove the first convergence in (6.1), observe that

$$\begin{aligned} &K_n^a(u_n^a) \\ &= \int_\omega\int_{\varepsilon_n}^1 W\left(\begin{pmatrix} D_{x'}^2 z^a & (D_{x_N}\xi^a + r_n D_{x',x_N}^2 z^a)^{\mathrm{T}} \\ D_{x_N}\xi^a + r_n D_{x',x_N}^2 z^a & D_{x_N}^2 u^a + r_n D_{x_N}^2\xi^a\cdot x' + r_n^2 D_{x_N}^2 z^a \end{pmatrix}\right) dx \\ &\quad + \int_\omega\int_0^{\varepsilon_n} W\left(\begin{pmatrix} \dfrac{1}{r_n^2}D_{x'}^2 u_n^a & \left(\dfrac{1}{r_n}D_{x',x_N}^2 u_n^a\right)^{\mathrm{T}} \\ \dfrac{1}{r_n}D_{x',x_N}^2 u_n^a & D_{x_N}^2 u_n^a \end{pmatrix}\right) dx \end{aligned}$$

$$= \int_{\Omega^a} W\left(\begin{pmatrix} D^2_{x'} z^a & (D_{x_N}\xi^a + r_n D^2_{x',x_N} z^a)^{\mathrm{T}} \\ D_{x_N}\xi^a + r_n D^2_{x',x_N} z^a & D^2_{x_N} u^a + r_n D^2_{x_N}\xi^a \cdot x' + r_n^2 D^2_{x_N} z^a \end{pmatrix}\right) dx$$

$$- \int_\omega \int_0^{\varepsilon_n} W\left(\begin{pmatrix} D^2_{x'} z^a & (D_{x_N}\xi^a + r_n D^2_{x',x_N} z^a)^{\mathrm{T}} \\ D_{x_N}\xi^a + r_n D^2_{x',x_N} z^a & D^2_{x_N} u^a + r_n D^2_{x_N}\xi^a \cdot x' + r_n^2 D^2_{x_N} z^a \end{pmatrix}\right) dx$$

$$+ \int_\omega \int_0^{\varepsilon_n} W\left(\begin{pmatrix} \frac{1}{r_n^2} D^2_{x'} u_n^a & \left(\frac{1}{r_n} D^2_{x',x_N} u_n^a\right)^{\mathrm{T}} \\ \frac{1}{r_n} D^2_{x',x_N} u_n^a & D^2_{x_N} u_n^a \end{pmatrix}\right) dx,$$

for every $n \in \mathbb{N}$. Consequently, since

$$\lim_n \int_{\Omega^a} W\left(\begin{pmatrix} D^2_{x'} z^a & (D_{x_N}\xi^a + r_n D^2_{x',x_N} z^a)^{\mathrm{T}} \\ D_{x_N}\xi^a + r_n D^2_{x',x_N} z^a & D^2_{x_N} u^a + r_n D^2_{x_N}\xi^a \cdot x' + r_n^2 D^2_{x_N} z^a \end{pmatrix}\right) dx$$

$$= \int_{\Omega^a} W\left(\begin{pmatrix} D^2_{x'} z^a & (D_{x_N}\xi^a)^{\mathrm{T}} \\ D_{x_N}\xi^a & D^2_{x_N} u^a \end{pmatrix}\right) dx = K^a(u^a, \xi^a, z^a)$$

and

$$\lim_n \int_\omega \int_0^{\varepsilon_n} W\left(\begin{pmatrix} D^2_{x'} z^a & (D_{x_N}\xi^a + r_n D^2_{x',x_N} z^a)^{\mathrm{T}} \\ D_{x_N}\xi^a + r_n D^2_{x',x_N} z^a & D^2_{x_N} u^a + r_n D^2_{x_N}\xi^a \cdot x' + r_n^2 D^2_{x_N} z^a \end{pmatrix}\right) dx = 0,$$

to obtain the first convergence in (6.1), it remains to show that

$$\lim_n \int_\omega \int_0^{\varepsilon_n} W\left(\begin{pmatrix} \frac{1}{r_n^2} D^2_{x'} u_n^a & \left(\frac{1}{r_n} D^2_{x',x_N} u_n^a\right)^{\mathrm{T}} \\ \frac{1}{r_n} D^2_{x',x_N} u_n^a & D^2_{x_N} u_n^a \end{pmatrix}\right) dx = 0,$$

i.e. by virtue of (2.4), it is enough to check that

$$\lim_n \int_\omega \int_0^{\varepsilon_n} \left|\frac{1}{r_n^2} D^2_{x'} u_n^a\right|^p dx = 0, \tag{6.2}$$

$$\lim_n \int_\omega \int_0^{\varepsilon_n} \left|\frac{1}{r_n} D^2_{x',x_N} u_n^a\right|^p dx = 0, \tag{6.3}$$

$$\lim_n \int_\omega \int_0^{\varepsilon_n} \left|D^2_{x_N} u_n^a\right|^p dx = 0. \tag{6.4}$$

To this aim, at first observe that assumption (3.5) ensures that

$$v^b, \xi^b \in W^{2,\infty}(B). \tag{6.5}$$

Let us prove convergence (6.2). Since

$$\begin{aligned}\frac{1}{r_n^2}D^2_{x'}u_n^a &= 2[-D^2_{x'}z^a(x',\varepsilon_n)+(D^2_{x'}u^b)(r_nx')+h_n^2(D^2_{x'}z^b)(r_nx',0)]\frac{x_N^3}{\varepsilon_n^3}\\&\quad+[D^3_{x',x',x_N}z^a(x',\varepsilon_n)+(D^2_{x'}\xi^b)(r_nx')+h_n(D^3_{x',x',x_N}z^b)(r_nx',0)]\frac{x_N^3}{\varepsilon_n^2}\\&\quad+3[D^2_{x'}z^a(x',\varepsilon_n)-(D^2_{x'}u^b)(r_nx')-h_n^2(D^2_{x'}z^b)(r_nx',0)]\frac{x_N^2}{\varepsilon_n^2}\\&\quad+[-D^3_{x',x',x_N}z^a(x',\varepsilon_n)-2(D^2_{x'}\xi^b)(r_nx')\\&\quad-2h_n(D^3_{x',x',x_N}z^b)(r_nx',0)]\frac{x_N^2}{\varepsilon_n}\\&\quad+[(D^2_{x'}\xi^b)(r_nx')+h_n(D^3_{x',x',x_N}z^b)(r_nx',0)]x_N\\&\quad+(D^2_{x'}u^b)(r_nx')+h_n^2(D^2_{x'}z^b)(r_nx',0),\quad \text{for a.e. } (x',x_N)\in\omega\times]0,\varepsilon_n[,\end{aligned}$$

for every $n\in\mathbb{N}$; by taking into account (6.5), it results that

$$\begin{aligned}&\int_\omega\int_0^{\varepsilon_n}\left|\frac{1}{r_n^2}D^2_{x'}u_n^a\right|^p dx\\&\quad\le c[\|D^2_{x'}z^a\|^p_{(L^\infty(\Omega^a))^{(n-1)^2}}+\|D^2_{x'}u^b\|^p_{(L^\infty(B))^{(n-1)^2}}+h_n^{2p}\|D^2_{x'}z^b\|^p_{(L^\infty(\Omega^b))^{(n-1)^2}}]\varepsilon_n\\&\qquad+c[\|D^3_{x',x',x_N}z^a\|^p_{(L^\infty(\Omega^a))^{(n-1)^2}}+\|D^2_{x'}\xi^b\|^p_{(L^\infty(B))^{(n-1)^2}}]\varepsilon_n^{p+1}\\&\qquad+ch_n^p\|D^3_{x',x',x_N}z^b\|^p_{(L^\infty(\Omega^b))^{(n-1)^2}}\varepsilon_n^{p+1}\\&\quad\le c\varepsilon_n,\end{aligned}$$

for $n\in\mathbb{N}$ large enough, where, from now on, $c$ denotes any positive constant independent of $n$. Consequently, to obtain convergence (6.2) it is enough to choose a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that

$$\lim_n \varepsilon_n = 0. \tag{6.6}$$

For what concerns convergence (6.3), since

$$\begin{aligned}\frac{1}{r_n}D^2_{x',x_N}u_n^a &= 6[(D_{x'}u^b)(r_nx')-\xi^a(\varepsilon_n)]\frac{x_N^2}{\varepsilon_n^3}+6[-r_nD_{x'}z^a(x',\varepsilon_n)\\&\quad+h_n^2(D_{x'}z^b)(r_nx',0)]\frac{x_N^2}{\varepsilon_n^3}+3[D_{x_N}\xi^a(\varepsilon_n)+r_nD^2_{x',x_N}z^a(x',\varepsilon_n)\\&\quad+(D_{x'}\xi^b)(r_nx')+h_n(D^2_{x',x_N}z^b)(r_nx',0)]\frac{x_N^2}{\varepsilon_n^2}+6[\xi^a(\varepsilon_n)\\&\quad-(D_{x'}u^b)(r_nx')]\frac{x_N}{\varepsilon_n^2}+6[r_nD_{x'}z^a(x',\varepsilon_n)-h_n^2(D_{x'}z^b)(r_nx',0)]\frac{x_N}{\varepsilon_n^2}\\&\quad+2[-D_{x_N}\xi^a(\varepsilon_n)-r_nD^2_{x',x_N}z^a(x',\varepsilon_n)-2(D_{x'}\xi^b)(r_nx')\\&\quad-2h_n(D^2_{x',x_N}z^b)(r_nx',0)]\frac{x_N}{\varepsilon_n}+(D_{x'}\xi^b)(r_nx')\\&\quad+h_n(D^2_{x',x_N}z^b)(r_nx',0),\quad\text{for a.e. }(x',x_N)\in\omega\times]0,\varepsilon_n[,\end{aligned}$$

for every $n \in \mathbb{N}$; by taking into account (6.5), it results that

$$\begin{aligned}
&\int_\omega \int_0^{\varepsilon_n} \left| \frac{1}{r_n} D^2_{x',x_N} u^a_n \right|^p dx \\
&\quad \leq c\|(D_{x'}u^b)(r_n\cdot) - \xi^a(\varepsilon_n)\|^p_{(L^\infty(\omega))^{n-1}} \frac{1}{\varepsilon_n^{p-1}} \\
&\qquad + c[r_n^p \|D_{x'}z^a\|^p_{(L^\infty(\Omega^a))^{n-1}} + h_n^{2p}\|D_{x'}z^b\|^p_{(L^\infty(\Omega^b))^{n-1}}] \frac{1}{\varepsilon_n^{p-1}} \\
&\qquad + c[\|D_{x_N}\xi^a\|^p_{(L^\infty(]0,1[))^{n-1}} + r_n^p\|D^2_{x',x_N} z^a\|^p_{(L^\infty(\Omega^a))^{(n-1)^2}} \\
&\qquad + \|D_{x'}\xi^b\|^p_{(L^\infty(B))^{n-1}} + h_n^p \|D^2_{x',x_N} z^b\|^p_{(L^\infty(\Omega^b))^{(n-1)^2}}]\varepsilon_n, \qquad (6.7)
\end{aligned}$$

for $n \in \mathbb{N}$ large enough. With regard to the first term on the right-hand side of (6.7), the junction condition $D_{x'}u^b(0') = \xi^a(0)$ and again (6.5) provide

$$\begin{aligned}
&\| \left(D_{x'}u^b\right)(r_n\cdot) - \xi^a(\varepsilon_n)\|^p_{(L^\infty(\omega))^{n-1}} \\
&\quad = \|(D_{x'}u^b)(r_n\cdot) - D_{x'}u^b(0') - \int_0^{\varepsilon_n} D_{x_N}\xi^a(t)dt\|^p_{(L^\infty(\omega))^{n-1}} \\
&\quad \leq c[r_n^p\|D^2_{x'}u^b\|^p_{(L^\infty(B))^{(n-1)^2}} + \varepsilon_n^p\|D_{x_N}\xi^a\|^p_{(L^\infty(]0,1[))^{n-1}}], \qquad (6.8)
\end{aligned}$$

for $n \in \mathbb{N}$ large enough. Then, from (6.7) and (6.8) it follows that

$$\int_\omega \int_0^{\varepsilon_n} \left| \frac{1}{r_n} D^2_{x',x_N} u^a_n \right|^p dx \leq c\left[\left(\frac{r_n}{\varepsilon_n}\right)^p \varepsilon_n + \left(\frac{h_n}{\varepsilon_n}\right)^{2p} \varepsilon_n^{p+1} + \varepsilon_n\right],$$

for $n \in \mathbb{N}$ large enough. Consequently, by virtue of (2.7), to obtain convergence (6.3) it is enough to choose a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ satisfying (6.6) and such that

$$\left\{\frac{r_n}{\varepsilon_n}\right\}_{n\in\mathbb{N}} \quad \text{is bounded.} \qquad (6.9)$$

For what concerns convergence (6.4), since

$$\begin{aligned}
D^2_{x_N} u^a_n &= 6\left[\frac{2}{\varepsilon_n^3}u^b(r_nx') - \frac{2}{\varepsilon_n^3}u^a(\varepsilon_n) + \frac{1}{\varepsilon_n^2}D_{x_N}u^a(\varepsilon_n) + \frac{1}{\varepsilon_n^2}\xi^b(r_nx')\right]x_N \\
&\quad + 12[-r_n\xi^a(\varepsilon_n)\cdot x' - r_n^2 z^a(x',\varepsilon_n) + h_n^2 z^b(r_nx',0)]\frac{x_N}{\varepsilon_n^3} \\
&\quad + 6[r_nD_{x_N}\xi^a(\varepsilon_n)\cdot x' + r_n^2 D_{x_N}z^a(x',\varepsilon_n) + h_nD_{x_N}z^b(r_nx',0)]\frac{x_N}{\varepsilon_n^2} \\
&\quad + \frac{6}{\varepsilon_n^2}u^a(\varepsilon_n) - \frac{6}{\varepsilon_n^2}u^b(r_nx') - \frac{2}{\varepsilon_n}D_{x_N}u^a(\varepsilon_n) - \frac{4}{\varepsilon_n}\xi^b(r_nx') \\
&\quad + 6[r_n\xi^a(\varepsilon_n)\cdot x' + r_n^2 z^a(x',\varepsilon_n) - h_n^2 z^b(r_nx',0)]\frac{1}{\varepsilon_n^2} \\
&\quad + [-2r_nD_{x_N}\xi^a(\varepsilon_n)\cdot x' - 2r_n^2 D_{x_N}z^a(x',\varepsilon_n) - 4h_nD_{x_N}z^b(r_nx',0)]\frac{1}{\varepsilon_n}, \\
&\qquad\quad \text{for a.e. } (x',x_N) \in \omega\times]0,\varepsilon_n[,
\end{aligned}$$

for every $n \in \mathbb{N}$; by taking into account (6.5), it results that

$$\begin{aligned}
&\int_\omega \int_0^{\varepsilon_n} |D^2_{x_N} u^a_n|^p dx \\
&\quad \le c \left\| \frac{2}{\varepsilon_n^3} u^b(r_n \cdot) - \frac{2}{\varepsilon_n^3} u^a(\varepsilon_n) + \frac{1}{\varepsilon_n^2} D_{x_N} u^a(\varepsilon_n) + \frac{1}{\varepsilon_n^2} \xi^b(r_n \cdot) \right\|^p_{L^\infty(\omega)} \varepsilon_n^{p+1} \\
&\qquad + c[r_n^p \|\xi^a\|^p_{(L^\infty(]0,1[))^{N-1}} + r_n^{2p} \|z^a\|^p_{L^\infty(\Omega^a)} + h_n^{2p} \|z^b\|^p_{L^\infty(\Omega^b)}] \frac{1}{\varepsilon_n^{2p-1}} \\
&\qquad + c[r_n^p \|D_{x_N} \xi^a|^p_{(L^\infty(]0,1[))^{N-1}} + r_n^{2p} \|D_{x_N} z^a\|^p_{L^\infty(\Omega^a)}] \frac{1}{\varepsilon_n^{p-1}} \\
&\qquad + c h_n^p \|D_{x_N} z^b\|^p_{L^\infty(\Omega^b)} \frac{1}{\varepsilon_n^{p-1}} \\
&\qquad + c \left\| \frac{6}{\varepsilon_n^2} u^a(\varepsilon_n) - \frac{6}{\varepsilon_n^2} u^b(r_n \cdot) - \frac{2}{\varepsilon_n} D_{x_N} u^a(\varepsilon_n) - \frac{4}{\varepsilon_n} \xi^b(r_n \cdot) \right\|^p_{L^\infty(\omega)} \varepsilon_n,
\end{aligned} \tag{6.10}$$

for every $n \in \mathbb{N}$. With regard to the first term on the right-hand side of (6.10), the mean value theorem allows us to write

$$u^a(\varepsilon_n) = \int_0^{\varepsilon_n} D_{x_N} u^a(t) dt + u^a(0) = \varepsilon_n D_{x_N} u^a(t_n) + u^a(0),$$

for a suitable $t_n \in ]0, \varepsilon_n[$ and for every $n \in \mathbb{N}$. Consequently, the junction conditions: $u^a(0) = u^b(0')$, $D_{x_N} u^a(0') = \xi^b(0')$ and again (6.5) give

$$\begin{aligned}
&\left\| \frac{2}{\varepsilon_n^3} u^b(r_n \cdot) - \frac{2}{\varepsilon_n^3} u^a(\varepsilon_n) + \frac{1}{\varepsilon_n^2} D_{x_N} u^a(\varepsilon_n) + \frac{1}{\varepsilon_n^2} \xi^b(r_n \cdot) \right\|_{L^\infty(\omega)} \\
&\quad = \left\| \frac{2}{\varepsilon_n^3} u^b(r_n \cdot) - \frac{2}{\varepsilon_n^2} D_{x_N} u^a(t_n) - \frac{2}{\varepsilon_n^3} u^a(0) + \frac{1}{\varepsilon_n^2} D_{x_N} u^a(\varepsilon_n) + \frac{1}{\varepsilon_n^2} \xi^b(r_n \cdot) \right\|_{L^\infty(\omega)} \\
&\quad = \left\| \frac{2}{\varepsilon_n^3} u^b(r_n \cdot) - \frac{2}{\varepsilon_n^3} u^b(0') + \frac{1}{\varepsilon_n^2} D_{x_N} u^a(\varepsilon_n) - \frac{1}{\varepsilon_n^2} D_{x_N} u^a(t_n) \right. \\
&\qquad \left. + \frac{1}{\varepsilon_n^2} D_{x_N} u^a(0) - \frac{1}{\varepsilon_n^2} D_{x_N} u^a(t_n) + \frac{1}{\varepsilon_n^2} \xi^b(r_n \cdot) - \frac{1}{\varepsilon_n^2} \xi^b(0') \right\|_{L^\infty(\omega)} \\
&\quad \le \frac{2}{\varepsilon_n^3} \|u^b(r_n \cdot) - u^b(0')\|_{L^\infty(\omega)} + \frac{1}{\varepsilon_n^2} |D_{x_N} u^a(\varepsilon_n) - D_{x_N} u^a(t_n)| \\
&\qquad + \frac{1}{\varepsilon_n^2} |D_{x_N} u^a(0) - D_{x_N} u^a(t_n)| + \frac{1}{\varepsilon_n^2} \|\xi^b(r_n \cdot) - \xi^b(0')\|_{L^\infty(\omega)} \\
&\quad \le c \left[ \frac{r_n}{\varepsilon_n^3} \|D_{x'} u^b\|_{(L^\infty(B))^{N-1}} + \frac{\varepsilon_n - t_n}{\varepsilon_n^2} \|D^2_{x_N} u^a\|_{L^\infty(]0,1[)} \right. \\
&\qquad \left. + \frac{t_n}{\varepsilon_n^2} \|D^2_{x_N} u^a\|_{L^\infty(]0,1[)} + \frac{r_n}{\varepsilon_n^2} \|D_{x'} \xi^b\|_{(L^\infty(B))^{N-1}} \right],
\end{aligned}$$

for a suitable $t_n \in ]0, \varepsilon_n[$ and for $n \in \mathbb{N}$ large enough. Analogously, with regard to the last term on the right-hand side of (6.10), it results

$$\begin{aligned}
&\left\| \frac{6}{\varepsilon_n^2} u^a(\varepsilon_n) - \frac{6}{\varepsilon_n^2} u^b(r_n \cdot) - \frac{2}{\varepsilon_n} D_{x_N} u^a(\varepsilon_n) - \frac{4}{\varepsilon_n} \xi^b(r_n \cdot) \right\|_{L^\infty(\omega)} \\
&\quad = \left\| \frac{6}{\varepsilon_n} D_{x_N} u^a(t_n) + \frac{6}{\varepsilon_n^2} u^a(0) - \frac{6}{\varepsilon_n^2} u^b(r_n \cdot) - \frac{2}{\varepsilon_n} D_{x_N} u^a(\varepsilon_n) - \frac{4}{\varepsilon_n} \xi^b(r_n \cdot) \right\|_{L^\infty(\omega)} \\
&\quad \le \frac{6}{\varepsilon_n^2} \| u^b(0') - u^b(r_n \cdot) \|_{L^\infty(\omega)} + \frac{2}{\varepsilon_n} |D_{x_N} u^a(t_n) - D_{x_N} u^a(\varepsilon_n)| \\
&\qquad + \frac{4}{\varepsilon_n} |D_{x_N} u^a(t_n) - D_{x_N} u^a(0)| + \frac{4}{\varepsilon_n} \| D_{x_N} u^a(0) - \xi^b(r_n \cdot) \|_{L^\infty(\omega)} \\
&\quad \le c \left[ \frac{r_n}{\varepsilon_n^2} \| D_{x'} u^b \|_{(L^\infty(B))^{N-1}} + \frac{\varepsilon_n - t_n}{\varepsilon_n} \| D^2_{x_N} u^a \|_{L^\infty(]0,1[)} \right. \\
&\qquad \left. + \frac{t_n}{\varepsilon_n} \| D^2_{x_N} u^a \|_{L^\infty(]0,1[)} + \frac{r_n}{\varepsilon_n} \| D_{x'} \xi^b \|_{(L^\infty(B))^{N-1}} \right],
\end{aligned}$$

for a suitable $t_n \in ]0, \varepsilon_n[$ and for $n \in \mathbb{N}$ large enough. Then, from (6.10) and the last two statements, it follows that

$$\int_\omega \int_0^{\varepsilon_n} |D^2_{x_N} u^a_n|^p \, dx \le c \left[ \left( \frac{r_n}{\varepsilon_n^2} \right)^p \varepsilon_n + \left( \frac{h_n}{\varepsilon_n} \right)^p \varepsilon_n + \left( \frac{h_n}{\varepsilon_n} \right)^{2p} \varepsilon_n + \varepsilon_n \right],$$

for $n \in \mathbb{N}$ large enough. Consequently, by virtue of (2.7), to obtain convergence (6.4) it is enough to choose a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ satisfying (6.6) and such that

$$\left\{ \frac{r_n}{\varepsilon_n^2} \right\}_{n\in\mathbb{N}} \quad \text{is bounded.} \tag{6.11}$$

Finally, by choosing $\varepsilon_n = r_n^{\frac{1}{2}}$, conditions (6.6), (6.9) and (6.11) are fulfilled and so the proof is completed. □

## 7. A Compactness Result

This section is devoted to prove convergences (3.10)–(3.12).

**Proposition 7.1.** *Let $W$ be a function satisfying* (2.2)–(2.4), *and let, for every $n \in \mathbb{N}$, $(\overline{u}^a_n, \overline{u}^b_n) \in \mathcal{U}_n$ be a solution of Problem* (3.1). *Let $V^p$ be as in* (3.8). *Assume that* (2.1) *and* (2.7) *hold.*

*Then, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and $((\overline{u}^a, \overline{u}^b), (\overline{\xi}^a, \overline{\xi}^b), (\overline{z}^a, \overline{z}^b)) \in V^p$, depending possibly on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$, such that convergences* (3.10)–(3.12), *hold true.*

**Proof.** At first, we prove (3.10)–(3.12), neglecting the junction properties.

In what concerns (3.10), we prove only the first convergence, the proof of the second one being similar.

Since the space $V$ defined in (5.1) is not empty, Proposition 6.1 ensures that

$$\sup_n \left\{ \int_{\Omega^a} \left( \frac{1}{r_n^{2p}} |D^2_{x'} \overline{u}^a_n|^p + \frac{1}{r_n^p} |D^2_{x',x_N} \overline{u}^a_n|^p + |D^2_{x_N} \overline{u}^a_n|^p \right) dx \right.$$
$$\left. + \frac{h_n}{r_n^{N-1}} \int_{\Omega^b} \left( |D^2_{x'} \overline{u}^b_n|^p + \frac{1}{h_n^p} |D^2_{x',x_N} \overline{u}^b_n|^p + \frac{1}{h_n^{2p}} |D^2_{x_N,x_N} \overline{u}^b_n|^p \right) dx \right\} < \infty. \quad (7.1)$$

Consequently, Poincaré inequality, (2.1) and a l.s.c. argument guarantee the existence of an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and a function $\overline{u}^a \in W^{2,p}(\Omega^a)$ such that the first convergence in (3.10) holds true, and

$$\overline{u}^a_{|_{\omega\times\{1\}}} = c^a, \quad (D\overline{u}^a)_{|_{\omega\times\{1\}}} = 0,$$
$$D^2_{x'}\overline{u}^a = 0, \quad D^2_{x',x_N}\overline{u}^a = 0, \quad \text{a.e. in } \Omega^a.$$

These properties provide that $\overline{u}^a$ is independent of $x'$.

Also for (3.11), we prove only the first convergence. The proof of the second one is similar. From (7.1), by using again Poincaré inequality, (2.1) and a l.s.c. argument, it follows the existence of an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and a function $\overline{\xi}^a \in (W^{1,p}(\Omega^a))^{N-1}$ such that first convergence in (3.11) holds true, and $\overline{\xi}^a{}_{|_{\omega\times\{1\}}} = d^a$, $D_{x'}\overline{\xi}^a = 0$ a.e. in $\Omega^a$. Then, $\overline{\xi}^a$ is independent of $x'$.

Let us prove the first convergence in (3.12). For every $n \in \mathbb{N}$ set

$$m_n^{(1)}(x_N) = \fint_\omega D_{x'}\overline{u}^a_n(x', x_N)dx', \quad x_N \text{ a.e. in } ]0,1[\,;$$

and

$$m_n^{(2)}(x_N) = \fint_\omega (\overline{u}^a_n(x', x_N) - m_n^{(1)}(x_N)\cdot x')dx', \quad x_N \text{ a.e. in } ]0,1[\,.$$

By virtue of the Poincaré–Wirtinger inequality (applied twice), there exists $c \in\, ]0,+\infty[$ (depending only on $\omega$ and not on $x_N$) such that

$$\left\| \frac{1}{r_n^2}(\overline{u}^a_n(\cdot, x_N) - m_n^{(1)}(x_N)\cdot x' - m_n^{(2)}(x_N)) \right\|_{W^{2,p}_m(\omega)}$$
$$\le \frac{c}{r_n^2} \|D^2_{x'}\overline{u}^a_n(\cdot, x_N)\|_{(L^p(\omega))^{(N-1)^2}}, \quad x_N \text{ a.e. in } (0,1), \quad \forall n \in \mathbb{N}.$$

Thus, by integrating this inequality over $x_N$, estimate (7.1) ensures the existence of an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and a function $\overline{z}^a \in L^p((0,1); W^{2,p}_m(\omega))$ such that

$$\frac{1}{r_{n_i}^2}(\overline{u}^a_{n_i} - m_{n_i}^{(1)}\cdot x' - m_{n_i}^{(2)}) \rightharpoonup \overline{z}^a \text{ weakly in } L^p((0,1); W^{2,p}_m(\omega)).$$

In what concerns the second convergence in (3.12), analogously it can be shown that

$$\frac{1}{h_{n_i}^2}\left(\overline{u}_{n_i}^b - \mu_{n_i}^{(1)} x_N - \mu_{n_i}^{(2)}\right) \rightharpoonup \overline{z}^b \text{ weakly in } L^p(\omega; W_m^{2,p}(]{-1}, 0[)),$$

where

$$\mu_n^{(1)}(x') = \int_{-1}^{0} D_{x_N} \overline{u}_n^b(x', x_N) dx_N, \qquad x' \text{ a.e. in } \omega,$$

$$\mu_n^{(2)}(x') = \int_{-1}^{0} (\overline{u}_n^b(x', x_N) - \mu_n^{(1)}(x') x_N) dx_N, \quad x' \text{ a.e. in } \omega.$$

It remains to prove that $\overline{u}^a$, $\overline{u}^b$, $\overline{\xi}^a$, $\overline{\xi}^b$ satisfy the junction properties given in (3.8). In order to prove that

$$\overline{u}^a(0) = \overline{u}^b(0'), \quad \text{if } \frac{N-1}{2} < p, \tag{7.2}$$

for every $i \in \mathbb{N}$ set

$$\rho_i(x_N) = \int_\omega (|\overline{u}_{n_i}^b(x', x_N)|^p + |D_{x'}\overline{u}_{n_i}^b(x', x_N)|^p + |D_{x'}^2 \overline{u}_{n_i}^b(x', x_N)|^p) dx', \qquad x_N \text{ a.e. in } ]{-1}, 0[\,.$$

By virtue of the second convergence in (3.10), it results that

$$\int_{-1}^{0} \liminf_i \rho_i(x_N) dx_N \le \liminf_i \int_{-1}^{0} \rho_i(x_N) dx_N \le \sup_i \|\overline{u}_{n_i}^b\|_{W^{2,p}(\Omega^b)} < +\infty.$$

Fix $\overline{x}_N \in ]{-1}, 0[$ such that $\liminf_i \rho_i(\overline{x}_N) < +\infty$. Then, up to a subsequence of $\{n_i\}$, $\left\{\overline{u}_{n_i}^b(\cdot, \overline{x}_N)\right\}_{i\in\mathbb{N}}$ is bounded in $W^{2,p}(\omega)$. Consequently, since $W^{2,p}(\omega)$ is compactly embedded into $C^0(\overline{\omega})$ for $\frac{N-1}{2} < p$, and since the second convergence in (3.10) provides that $\overline{u}_{n_i}^b(\cdot, \overline{x}_N) \to \overline{u}^b$ strongly in $W^{1,p}(\omega)$, as $i \to +\infty$, it follows that

$$\overline{u}_{n_i}^b(\cdot, \overline{x}_N) \to \overline{u}^b \text{ strongly in } C^0(\overline{\omega}), \quad \text{as } i \to +\infty, \quad \text{if } \frac{N-1}{2} < p. \tag{7.3}$$

By arguing as in the proof of Proposition 2.1 in Ref. 22, from (2.1), (2.7), the fact that the sequence $\{\|\frac{1}{h_n} D_{x_N} \overline{u}_n^b\|_{L^2(\Omega^b)}\}_{n\in\mathbb{N}}$ is bounded and (7.3) one can derive that

$$\lim_i \int_\omega \overline{u}_{n_i}^b(r_{n_i} x', 0) dx' = \overline{u}^b(0') \text{meas } \omega, \quad \text{if } \frac{N-1}{2} < p. \tag{7.4}$$

Since $\overline{u}_{n_i}^a(\cdot, 0) = \overline{u}_{n_i}^b(r_{n_i} \cdot, 0)$ a.e. in $\omega$, for every $i \in \mathbb{N}$, the first convergence in (3.10) and (7.4) imply (7.2).

By applying Proposition 2.1 in Ref. 22 to the sequence $\{D_{x_j} \overline{u}_{n_i}^b\}_{i\in\mathbb{N}}$, $j = 1, \ldots, N-1$, one obtains that

$$\lim_i \int_\omega (D_{x'} \overline{u}_{n_i}^b)(r_{n_i} x', 0) dx' = D_{x'} \overline{u}^b(0') \text{meas } \omega, \quad \text{if } N-1 < p;$$

from which, by taking into account the first convergence in (3.11), it follows that $\overline{\xi}^a(0) = D_{x'}\overline{u}^b(0')$, if $N-1<p$. Similarly, by applying the same proposition to the sequence $\{\frac{1}{h_{n_i}} D_{x_N}\overline{u}^b_{n_i}\}_{i\in\mathbb{N}}$, one deduces that $D_{x_N}\overline{u}^a(0) = \overline{\xi}^b(0')$, if $N-1<p$. $\square$

## 8. Proof of Theorem 3.1

**Proof.** Proposition 7.1 ensures the existence of an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and $((\overline{u}^a,\overline{u}^b),(\overline{\xi}^a,\overline{\xi}^b),(\overline{z}^a,\overline{z}^b)) \in V^p$, depending possibly on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$, satisfying convergences (3.10)–(3.12).

Now, let us prove that $((\overline{u}^a,\overline{u}^b),(\overline{\xi}^a,\overline{\xi}^b),(\overline{z}^a,\overline{z}^b))$ solves Problem (3.13). Assumptions (2.3) and (2.4) involve that the functionals

$$\mathcal{M} \in (L^p(\Omega^i))^{N\times N} \to \int_{\Omega^i} W(\mathcal{M})dx,$$

for $i=a,b$, are convex and strongly continuous, and hence weakly lower semicontinuous. Consequently, by virtue of (2.7), (3.10)–(3.12), it results that

$$K^a(\overline{u}^a,\overline{\xi}^a,\overline{z}^a) + qK^b(\overline{u}^b,\overline{\xi}^b,\overline{z}^b) \le \liminf_i \left( K^a_{n_i}(\overline{u}^a_{n_i}) + \frac{h_{n_i}}{r^{N-1}_{n_i}} K^b_{n_i}(\overline{u}^b_{n_i}) \right). \tag{8.1}$$

On the other hand, by virtue of Proposition 6.1, for every $(u^a,u^b,\xi^a,\xi^b,z^a,z^b)\in V$ there exists a sequence $\{(u^a_n,u^b_n)\}_{n\in\mathbb{N}}$, with $(u^a_n,u^b_n)\in\mathcal{U}_n$, such that

$$\lim_n K^a_n(u^a_n) = K^a(u^a,\xi^a,z^a), \quad \lim_n K^b_n(u^b_n) = K^b(u^b,\xi^b,z^b). \tag{8.2}$$

Then, by combining (2.7) with (8.1) and (8.2), one obtains that

$$\begin{aligned} &K^a(\overline{u}^a,\overline{\xi}^a,\overline{z}^a) + qK^b(\overline{u}^b,\overline{\xi}^b,\overline{z}^b)\\ &\quad\le \liminf_i \left( K^a_{n_i}(\overline{u}^a_{n_i}) + \frac{h_{n_i}}{r^{N-1}_{n_i}} K^b_{n_i}(\overline{u}^b_{n_i}) \right)\\ &\quad\le \limsup_i \left( K^a_{n_i}(\overline{u}^a_{n_i}) + \frac{h_{n_i}}{r^{N-1}_{n_i}} K^b_{n_i}(\overline{u}^b_{n_i}) \right)\\ &\quad\le \lim_i \left( K^a_{n_i}(u^a_{n_i}) + \frac{h_{n_i}}{r^{N-1}_{n_i}} K^b_{n_i}(u^b_{n_i}) \right)\\ &\quad= K^a(u^a,\xi^a,z^a) + qK^b(u^b,\xi^b,z^b), \quad \forall\,(u^a,u^b,\xi^a,\xi^b,z^a,z^b)\in V. \end{aligned} \tag{8.3}$$

Proposition 5.1 ensures that (8.3) is satisfied with $(u^a,u^b,\xi^a,\xi^b,z^a,z^b)\in V^p$, too. Consequently, $((\overline{u}^a,\overline{u}^b),(\overline{\xi}^a,\overline{\xi}^b),(\overline{z}^a,\overline{z}^b))$ solves Problem (3.13) and the convergence of the energies (3.14) comes true.

Moreover, if $W$ is strictly convex, Problem (3.13) admits a unique solution. Consequently, convergences (3.10)–(3.12) hold true for the whole sequence. $\square$

## Acknowledgments

The authors wish to thank Professor Hervé Le Dret and Professor Giovanni Romano for the helpful discussions they had on the subject.

## References

1. E. Acerbi, G. Buttazzo and D. Percivale, A variational definition of the strain energy for an elastic string, *J. Elasticity* **25** (1991) 137–148.
2. S. S. Antman, *Nonlinear Problems of Elasticity*, Applied Mathematical Sciences, Vol. 107 (Springer-Verlag, 1995).
3. J. M. Ball and R. D. James, Fine phase mixtures as minimizers of energy, *Arch. Rational Mech. Anal.* **100** (1987) 13–52.
4. J. M. Ball and R. D. James, Proposed experimental tests of a theory of fine microstructure and the two-well problem, *Phil. Trans. Roy. Soc. London A* **338** (1992) 389–450.
5. P. Belik and M. Luskin, A total-variation surface energy model for thin films of martensitic cristals, *Interfaces Free Bound.* **4** (2002) 71–88.
6. K. Bhattacharya and R. D. James, A theory of thin films of martensitic materials with applications to microactuators, *J. Mech. Phys. Solids* **47** (1999) 531–576.
7. D. Blanchard and A. Gaudiello, Homogenization of highly oscillating boundaries and reduction of dimension for a monotone problem, *ESAIM Control Optim. Calc. Var.* **9** (2003) 449–460.
8. D. Blanchard, A. Gaudiello and J. Mossino, Highly oscillating boundaries and reduction of dimension: The critical case, *Anal. Appl.*, to appear.
9. A. Braides, I. Fonseca and G. Francfort, 3D-2D asymptotic analysis for inhomogeneous thin films, *Indiana Univ. Math. J.* **49** (2000) 1367–1404.
10. D. Caillerie, Thin elastic and periodic plates, *Math. Meth. Appl. Sci.* **6** (1984) 159–191.
11. P. G. Ciarlet, *Plates and Junctions in Elastic Multistructures: An Asymptotic Analysis*, Research in Applied Mathematics, Vol. 14 (Masson/Springer-Verlag, 1990).
12. P. G. Ciarlet, *Mathematical Elasticity. Vol. II. Theory of plates*, Studies in Mathematics and its Applications, Vol. 27 (North-Holland, 1997).
13. P. G. Ciarlet and P. Destuynder, A justification of the two-dimensional linear plate model, *J. Mèc.* **18** (1979) 315–344.
14. M. Ciarletta and D. Ieşan, *Nonclassical Elastic Solids*, Pitman Research Notes in Mathematics Series, Vol. 293 (Longman/John Wiley and Sons, 1993).
15. D. Cioranescu and J. Saint Jean Paulin, *Homogenization of Reticulated Structures*, Applied Mathematical Sciences, Vol. 136 (Springer-Verlag, 1999).
16. A. Damlamian and M. Vogelius, Homogenization limits of the equations of elasticity in thin domains, *SIAM J. Math. Anal.* **18** (1987) 435–451.
17. E. De Giorgi and T. Franzoni, Su un tipo di convergenza variazionale, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **58** (1975) 842–850.
18. I. Fonseca and G. Francfort, 3D–2D asymptotic analysis of an optimal design problem for thin films, *J. Reine Angew. Math.* **505** (1998) 173–202.
19. D. D. Fox, A. Raoult and J. C. Simo, A justification of nonlinear properly invariant plate theories, *Arch. Rational Mech. Anal.* **124** (1993) 157–199.
20. G. Friesecke, R. D. James and S. Muller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity, *Comm. Pure Appl. Math.* **55** (2002) 1461–1506.
21. G. Gargiulo and E. Zappale, The energy density of non simple materials grade two thin films via a young measure approach, *Boll. UMI, Sec. A*, to appear.
22. A. Gaudiello, B. Gustafsson, C. Lefter and J. Mossino, Asymptotic analysis of a class of minimization problems in a thin multidomain, *Calc. Var. Partial Diff. Eqns.* **15** (2002) 181–201.
23. A. Gaudiello, B. Gustafsson, C. Lefter and J. Mossino, Asymptotic analysis for monotone quasilinear problems in thin multidomains, *Diff. Integral Eqns.* **15** (2002) 623–640.

24. A. Gaudiello, R. Monneau, J. Mossino, F. Murat and A. Sili, On the junction of elastic plates and beams, *C. R. Acad. Sci. Paris Sér. I Math.* **335** (2002) 717–722.
25. A. Gaudiello, R. Monneau, J. Mossino, F. Murat and A. Sili, Junction of elastic plates and beams, *ESAIM Control Optim. Calc. Var.*, to appear.
26. A. E. Green and R. S. Rivlin, Multipolar continuum mechanics, *Arch. Rational Mech. Anal.* **17** (1964) 113–147.
27. I. Gruais, Modélisation de la jonction entre une plaque et une poutre en élasticité linéarisée, *RAIRO Mod. Math. Anal. Numer.* **27** (1993) 77–105.
28. R. D. James and R. Rizzoni, Pressurized shape memory thin films, *J. Elasticity* **59** (2000) 399–436.
29. V. A. Kozlov, V. G. Ma'zya and A. B. Movchan, Asymptotic representation of elastic fields in a multi-structure, *Asympt. Anal.* **11** (1995) 343–415.
30. H. Le Dret, *Problèmes Variationnels dans le Multi-domaines: Modélisation des Jonctions et Applications*, Research in Applied Mathematics, Vol. 19 (Masson, 1991).
31. H. Le Dret and N. Meunier, Modeling heterogeneous wires made of martensitic materials, *C. R. Acad. Sci. Paris Sér. I Math.* **337** (2003) 143–147.
32. H. Le Dret and N. Meunier, Modeling heterogeneous martensitic wires, *Math. Mod. Meth. Appl. Sci.* **15** (2005) 375–406.
33. H. Le Dret and A. Raoult, The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity, *J. Math. Pures Appl.* **74** (1995) 549–578.
34. H. Le Dret and A. Raoult, Variational convergence for nonlinear shell models with directors and related semicontinuity and relaxation results, *Arch. Rational Mech. Anal.* **154** (2000) 101–134.
35. F. Murat and A. Sili, Problèmes monotones dans des cylindres de faibles diamètre formés de matériaux hétérogènes, *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995) 1199–1204.
36. P. M. Santos and E. Zappale, Second order analysis for thin structures, *Nonlinear Anal.* **56** (2004) 679–713.
37. Y. C. Shu, Heterogeneous thin films of martensitic materials, *Arch. Rational Mech. Anal.* **153** (2000) 39–90.
38. R. A. Toupin, Elastic materials with couple-stresses, *Arch. Rational Mech. Anal.* **11** (1962) 385–414.
39. R. A. Toupin, Theories of elasticity with couple-stress, *Arch. Rational Mech. Anal.* **17** (1964) 85–112.