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ASYMPTOTIC STABILITY FOR INTERMITTENTLY CONTROLLED SECOND-ORDER EVOLUTION EQUATIONS*

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Abstract. Motivated by several works on ordinary differential equations, we are interested in the asymptotic stability of *intermittently controlled* partial differential equations. We give a condition of asymptotic stability for second-order evolution equations uniformly damped by an on/off feedback. This result extends to the case of partial differential equations a previous result of R. A. Smith concerning ordinary differential equations.

Key words. damped wave equation, second-order evolution equations, asymptotic behavior, on-off damping

AMS subject classifications. 35L05, 35L10, 35B35, 35B40

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1. Introduction.

Motivated by several works on ordinary differential equations, we are interested in the asymptotic stability of *intermittently controlled* partial differential equations. This question has been widely studied in the case of ordinary differential equations (see, for example, [1, 9, 10, 25, 27, 28]). The typical problem is the oscillator damped by an *on/off* damping:

$$u'' + u + a(t)u' = 0, \quad t > 0, \tag{1.1}$$

where $a : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous nonnegative. For each solution u of (1.1), we define its energy by

$$\forall t \geq 0, \quad E_u(t) = \frac{1}{2}u(t)^2 + \frac{1}{2}u'(t)^2.$$

The derivative of the energy is

$$E'(t) = u(t)u'(t) + u'(t)u''(t) = -a(t)u'(t)^2,$$

hence the energy is always nonincreasing, but remains constant on the time intervals for which $a = 0$, and the decay is "very small" if a is "very small." Denote $\ell := \lim_{t\to\infty} E(t)$. Many authors (see, in particular, [1, 9, 10, 25, 27, 28]) investigated the links between the distribution of sets where a is positive and the property $\ell = 0$.

Assume that there exists a sequence $(I_n)_{n\geq 0}$ of disjoint open intervals in $(0, +\infty)$, denoted by $I_n = (a_n, b_n)$, where $b_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, and such that

$$\forall t \in I_n, \quad 0 < m_n \leq a(t) \leq M_n < \infty.$$

Roughly speaking, the energy is strictly decreasing on the time intervals I_n and just nonincreasing elsewhere. It is natural to wonder whether the decay on the time

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intervals I_n is sufficient to drive the energy to zero. Obviously some condition on the length of the intervals I_n has to be imposed to ensure $\ell = 0$. Smith [27] proved the following *sufficient condition of asymptotic stability*:

THEOREM 1.1. [27]. *Assume that*

$$\sum_{n=0}^{\infty} m_n T_n \delta_n^2 = +\infty, \tag{1.2}$$

where m_n *and* M_n *are the minimum and the maximum values of* $a(t)$ *in* I_n, T_n *is the length of* I_n *and* $\delta_n = \min(T_n, (1 + M_n)^{-1})$. *Then* (1.1) *is asymptotically stable; i.e., every solution* u *of* (1.1) *satisfies* $E_u(t) \to 0$ *as* $t \to \infty$.

For example, in the case of a damping such that $0 < m \leq a(t) \leq M$ for all $t \in I_n$ for all $n \in \mathbb{N}$, the condition (1.2) reduces to

$$\sum_{n=0}^{\infty} T_n^3 = +\infty. \tag{1.3}$$

It is noteworthy that (1.3) is also necessary in the following sense: given $\varepsilon > 0$ as small as we want, Pucci and Serrin [25] constructed an example for which the sequence $(T_n)_n$ satisfies

$$\sum_{n=0}^{\infty} T_n^{3-\varepsilon} = +\infty, \quad \text{while} \quad \sum_{n=0}^{\infty} T_n^3 < +\infty,$$

and suitable initial conditions such that the energy decays to some $\ell > 0$.

Note also that, under condition (1.2), *the distribution of the intervals* I_n *has no importance*. Only their size is important.

Condition (1.2) also requires that the damping coefficient a is not "too small" or "too large," in order to prevent "underdamping" or "overdamping." These phenomena are also a source of lack of strong stability (see [20, 22, 26], where the stability is studied for the wave equation, but always under the condition that the function a remains positive).

To our knowledge, stability properties for such "intermittently controlled" systems have not yet been studied in the case of *partial* differential equations.

In [21], we studied the effect of an on/off feedback on the wave equation. We considered the simplified case of a damping coefficient a that is $2T$-periodic and such that $a(t) = a_0 > 0$ on $(0, T)$ and $a(t) = 0$ on $(T, 2T)$. In particular, the condition (1.2) was always satisfied. And we studied the wave equation damped by a *boundary* on/off feedback or by a *locally distributed* on/off feedback. In both cases, we proved that the situation is *radically different* from the case of ordinary differential equations. Indeed, we proved that, except for a countable number of exceptional values of T, asymptotic stability occurs (and more precisely, exponential stability). But, for the exceptional values of T, asymptotic stability does not occur. This means that *the distribution of the intervals* I_n *is very important in the case of the locally damped wave equation.* This phenomenon is related to the optics rays propagation and the geometric condition of Bardos, Lebeau, and Rauch [2, 3]. See further comments in section 3.3.

In [21], the only case for which the situation was not different from the situation of the ordinary differential equations was the wave equation damped by an *uniformly distributed* on/off feedback. In that case, asymptotic stability occurs for any value of T. Thus the distribution of the intervals damping has no importance.

In the present work, we now study the wave equation *uniformly damped* by a *general on/off feedback* (in particular, not necessarily periodic). We prove that *"the uniformly damped wave equation behaves exactly like the oscillator"* in the sense that Theorem 1.1 is still true.

More generally, we prove this result in an abstract setting that includes both the oscillator and wavelike or platelike equations and that also includes *bounded or unbounded* and *linear and nonlinear* damping operators.

In particular, this gives for the result of Smith a new proof quite different from the original one, which was relying on monotonicity properties of the solutions of (1.1). Our method is based on a preliminary result which is interesting in itself: we provide an estimate of the energy decay on a *short* time interval (see Theorem 3.1). This estimate is true for both ordinary and partial differential equations.

The paper is organized as follows.

- In section 2, we introduce our abstract setting and we give the result of well-posedness (Theorem 2.1).
- In section 3, we provide an estimate of the energy decay on a *short* time interval (Theorem 3.1) and we deduce the asymptotic stability result (Theorem 3.2) extending the previous result of Smith. Then we make some further comments concerning the case of locally distributed dampings to explain the necessity of considering only uniformly distributed dampings.
- In section 4, we give some examples.
- In section 5, we present another application of the method to the case of a positive-negative damping (Theorem 5.1).

2. Abstract setting and well-posedness. Let H be a real Hilbert space endowed with the scalar product $(\cdot,\cdot)_H$ and the norm $|\cdot|_H$.

Assume that $A : D(A) \subset H \to H$ is a linear self-adjoint and coercive operator on H with dense domain. We define $V = D(A^{1/2})$ endowed with the scalar product $((\cdot,\cdot))_V$ and the norm $\|\cdot\|_V$ defined by

$$\forall v \in V, \quad \|v\|_V^2 = |A^{1/2}v|_H^2 = \langle \tilde{A}v, v\rangle_{V',V},$$

where $\tilde{A} \in \mathcal{L}(V, V')$ represents the extension of A.

Also let W be a Hilbert space endowed with the norm $\|\cdot\|_W$ and such that

$$V \hookrightarrow W \hookrightarrow H \equiv H' \hookrightarrow W' \hookrightarrow V'$$

with dense imbeddings. We also assume that A satisfies the following property:

$$\begin{aligned} &\exists\lambda_0,\, C_0 > 0, \ \text{ such that, } \forall\lambda \in [0, \lambda_0], \\ &\quad (I + \lambda A)^{-1} \in \mathcal{L}(W) \ \text{ and } \ \|(I + \lambda A)^{-1}\|_{\mathcal{L}(W)} \le C_0. \end{aligned} \tag{2.1}$$

Next we consider a *time-dependent* operator B such that

$$B \in L^\infty(J, \mathrm{Lip}\,(W, W')), \qquad B(t)0 = 0, \tag{2.2}$$

$$\forall t \in J,\ \forall w,\, z \in W, \quad \langle B(t)w - B(t)z, w - z\rangle_{W',W} \ \ge\ 0, \tag{2.3}$$

$$\forall t \in J,\ \forall w \in W, \quad \langle B(t)w, w\rangle_{W',W} \ \ge\ b^2(t)\|w\|_W^2, \tag{2.4}$$

$$\forall t \in J,\ \forall w,\, z \in W, \quad \|B(t)w - B(t)z\|_{W'} \ \leq\ Cb(t)^2\|w - z\|_W, \tag{2.5}$$

where $J = [0, T]$ with $T > 0$ and where $b(t) \geq 0$ with $b \in L^2(J)$. Note that $B(t)$ is *a priori unbounded and nonlinear*. (The choice $W = H$ corresponds to the particular case of a bounded operator.)

Now we consider the following second-order evolution equation

$$u'' + Au + B(t)u' = 0, \quad t > 0, \tag{2.6}$$

with the initial conditions

$$u(0) = u_0 \in V, \quad u'(0) = u_1 \in H \tag{2.7}$$

and we prove that this problem is well-posed.

THEOREM 2.1. *Under the previous assumptions, for any $(u_0, u_1) \in V \times H$, there exists a unique solution $u \in L^2(0, T; V) \cap W^{1,2}(0, T; H) \cap W^{2,2}(0, T; V')$ with $bu' \in L^2(0, T; W)$ and $B(t)u' = b(t)h(t)$, $h(t) \in L^2(0, T; W')$ of*

$$u'' + Au + B(t)u' = 0 \quad \text{in } L^2(0, T; V')$$

such that

$$u(0) = u_0 \in V, \quad u'(0) = u_1 \in H.$$

In addition $u \in C([0, T]; V) \cap C^1([0, T]; H)$ and the energy of the solution u defined by

$$\forall t \geq 0, \quad E_u(t) := \frac{1}{2}\|u(t)\|_V^2 + \frac{1}{2}|u'(t)|_H^2,$$

is absolutely continuous on $[0, T]$ with

$$E_u'(t) = -\langle B(t)u'(t), u'(t)\rangle_{W',W} \quad a.e \text{ on } \{t,\ b(t) > 0\},$$

and

$$E_u'(t) = 0 \quad a.e \text{ on } \{t,\ b(t) = 0\}.$$

For the proof of Theorem 2.1, we first need the following lemma.

LEMMA 2.1. *Let $b = b(t) \geq 0$ and consider*

$$u \in L^2(0, T; V) \cap W^{1,2}(0, T; H) \cap W^{2,2}(0, T; V')$$

with

$$bu' \in L^2(0, T; W),$$

where $V \subset W \subset H$ with continuous and dense imbeddings. Let

$$f = bg, \text{ with } g \in L^2(0, T; W'),$$

and assume

$$u'' + Au = f \quad \text{in } L^2(0, T; V').$$

Then, in fact, $u \in C([0,T];V) \cap C^1([0,T];H)$ *and the energy* $E_u(t)$ *is absolutely continuous on* $[0,T]$ *with*

$$E_u'(t) = \langle g(t), b(t)u'(t)\rangle_{W',W}.$$

Proof of Lemma 2.1. Let $J_\lambda = (I+\lambda A)^{-1} : H \to H$ for $\lambda > 0$. We have $J_\lambda \in \mathcal{L}(H)$, $J_{\lambda|V} \in \mathcal{L}(V)$, and $J_{\lambda|W} \in \mathcal{L}(W)$ for $\lambda \le \lambda_0$. And by (2.1), $\{J_\lambda\}_{0<\lambda\le\lambda_0}$ is uniformly equicontinuous on $W \to W$.

We claim that

$$\forall \varphi \in W, \quad J_\lambda\varphi \to \varphi \ \text{ in } W \quad \text{as } \lambda \to 0. \tag{2.8}$$

Indeed (2.8) is well-known if $\varphi \in V$, and since V is dense in W, the result follows by density.

Then we introduce

$$u_\lambda := J_\lambda u \ \text{ and } \ f_\lambda := J_\lambda(bg) = bJ_\lambda g.$$

We clearly have $u_\lambda \in L^2(0,T;D(A))$, $u_\lambda' \in L^2(0,T;D(A))$, and $u_\lambda'' \in L^2(0,T;V) \subset L^2(0,T;H)$.

Setting

$$E_\lambda(t) := \frac{1}{2}\|u_\lambda(t)\|_V^2 + \frac{1}{2}|u_\lambda'(t)|_H^2,$$

we have, a.e. on $(0,T)$,

$$E_\lambda' = ((u_\lambda, u_\lambda'))_V + (u_\lambda', u_\lambda'')_H = (Au_\lambda + u_\lambda'', u_\lambda')_H = b(J_\lambda g, u_\lambda')_H,$$

with $J_\lambda g \in L^2(0,T;V) \subset L^2(0,T;H)$.

Let α, β be two points of $[0,T]$ such that $\alpha < \beta$. Then we have

$$E_\lambda(\beta) - E_\lambda(\alpha) = \int_\alpha^\beta (bJ_\lambda g, u_\lambda')\,ds = \int_\alpha^\beta (J_\lambda g(s), b(s)u_\lambda'(s))\,ds. \tag{2.9}$$

On the other hand, we can prove

$$J_\lambda g \to g \ \text{ in } \quad L^2(0,T;W') \ \text{ as } \lambda \to 0 \tag{2.10}$$

and

$$bu_\lambda' = J_\lambda bu' \to bu' \ \text{ in } \quad L^2(0,T;W) \ \text{ as } \lambda \to 0. \tag{2.11}$$

Indeed, for the first property, we notice that $J_\lambda \in \mathcal{L}(W')$ for $0 < \lambda \le \lambda_0$ with a uniformly bounded norm (by duality, from (2.1)). Then $J_\lambda g(t) \to g(t)$ as $\lambda \to 0$, in W' a.e. on $(0,T)$, and

$$\|J_\lambda g(t) - g(t)\|_{W'} \le C\|g(t)\|_{W'}.$$

From Lebesgue's theorem, it follows that

$$\|J_\lambda g - g\|_{W'}^2 \to 0 \quad \text{in} \quad L^1(0,T) \ \text{ as } \lambda \to 0,$$

which gives (2.10).

Next, from (2.8), we also have $J_\lambda b(t)u' \to b(t)u'$ as $\lambda \to 0$ in W a.e. on $(0,T)$, and by (2.1), we have

$$\|J_\lambda b(t)u'(t) - b(t)u'(t)\|_W \le C\|b(t)u'(t)\|_W.$$

From Lebesgue's theorem, it follows that

$$\|J_\lambda bu' - bu'\|_W^2 \to 0 \quad \text{in} \quad L^1(0,T) \quad \text{as} \quad \lambda \to 0,$$

which gives (2.11).

Now assume for a moment that α and β are *both* such that

$$[u(\alpha), u'(\alpha)] \in V \times H \quad \text{and} \quad [u(\beta), u'(\beta)] \in V \times H.$$

Then as $\lambda \to 0$, we can pass to the limit in (2.9) to obtain

$$E_u(\beta) - E_u(\alpha) = \int_\alpha^\beta (g(s), b(s)u'(s))_{W',W}\, ds. \tag{2.12}$$

Now let α be *fixed* for a while and apply (2.12) with $\beta = \beta_n \to t \in [0,T]$ as $n \to +\infty$. We obtain that $E(\beta_n)$ is bounded and therefore (replacing if necessary $(\beta_n)_n$ by a subsequence) we have

$$(u(\beta_n), u'(\beta_n)) \rightharpoonup (\varphi, \psi) \quad \text{weakly in } V \times H \quad \text{as } n \to +\infty.$$

On the other hand, by the regularity assumptions on u, we have

$$(u(\beta_n), u'(\beta_n)) \to (u(t), u'(t)) \quad \text{strongly in } H \times V' \quad \text{as } n \to +\infty.$$

It follows that $(u(t), u'(t)) = (\varphi, \psi)$ and therefore $u(t) \in V$ and $u'(t) \in H$. Since this is valid for *any* t, (2.12) becomes true for any (α, β).

Now the vector $Y(t) = (u(t), u'(t))$ is weakly continuous on $[0,T]$ and its norm is continuous by (2.12). The remainder of the proof is obvious from (2.12). □

Proof of Theorem 2.1. (i) Uniqueness. Let u and $\tilde{u}$ be two solutions with the same initial data. We have

$$u'' + Au + B(t)u' = 0 \quad \text{and} \quad \tilde{u}'' + A\tilde{u} + B(t)\tilde{u}' = 0.$$

Then $z := \tilde{u} - u$ satisfies

$$z'' + Az = B(t)u' - B(t)\tilde{u}',$$

with

$$bz' \in L^2(0,T;W)$$

and

$$B(t)u' - B(t)\tilde{u}' = bg, \quad g \in L^2(0,T;W').$$

From Lemma 2.1, we deduce

$$E_z'(t) = \langle g(t), b(t)z' \rangle_{W',W} = \begin{cases} -\langle B(t)\tilde{u}' - B(t)u', \tilde{u}' - u' \rangle_{W',W} & \text{if } b(t) > 0, \\ 0 & \text{if } b(t) = 0. \end{cases}$$

Thus $E_z(t) = \frac{1}{2}[\|z\|_V^2 + \|z'\|_H^2]$ is nonincreasing by (2.3). Since $E_z(0) = 0$, we obtain $z \equiv 0$, i.e., $\tilde{u} \equiv u$. $\square$

(ii) Existence. We introduce, for $\psi \in W$,

$$C(t)\psi := \begin{cases} \frac{1}{b(t)} B(t) \left(\frac{\psi}{b(t)}\right) & \text{if } b(t) > 0, \\ 0 & \text{if } b(t) = 0. \end{cases}$$

It is clear that $C(t) \in \mathrm{Lip}\,(W, W')$ for all $t \in J$ and $\|C(t)\psi_1 - C(t)\psi_2\|_{W'} \le C\|\psi_1 - \psi_2\|_W$ for all $t \in J$ and ψ_1, $\psi_2 \in W$.

Next, for (u^0, u^1) given in $V \times H$ and for $0 < \lambda \le \lambda_0$, we solve

$$\begin{cases} u_\lambda'' + Au_\lambda + J_\lambda B(t) J_\lambda u_\lambda' = 0, \quad t \in J, \\ u_\lambda(0) = u_0, \quad u_\lambda'(0) = u_1. \end{cases} \tag{2.13}$$

We have

$$u_\lambda \in \mathcal{C}^0([0,T]; V) \cap \mathcal{C}^1([0,T]; H) \cap \mathcal{C}^2([0,T]; V'),$$

and

$$\begin{aligned} \int_0^T \langle BJ_\lambda u_\lambda', J_\lambda u_\lambda' \rangle_{W',W}\, ds + E_\lambda(T) &= E_\lambda(0) \\ &= \frac{1}{2}[\|u_0\|_V^2 + \|u_1\|_H^2] \;=\; E(0), \end{aligned} \tag{2.14}$$

which is fixed. Equation (2.13) can also be written as

$$u_\lambda'' + Au_\lambda + b(t) J_\lambda C(t)(b(t) J_\lambda u_\lambda') = 0. \tag{2.15}$$

From (2.14) we deduce that $bJ_\lambda u_\lambda'$ is bounded in $\mathcal{W} := L^2(J; W)$. Thus

$$B(t) J_\lambda u_\lambda' = b(t) C(t)(b(t) J_\lambda u_\lambda') = bh_\lambda,$$

where h_λ is bounded in $\mathcal{W}' := L^2(J; W')$.

Finally, u_λ is bounded in $L^\infty(J; V) \cap W^{1,\infty}(J; H)$ and we may assume that there exists a subsequence such that

$$u_{\lambda_n} \rightharpoonup u \quad \text{weakly in } L^2(J; V) \cap H^1(J; H) \quad \text{as } n \to +\infty,$$

and

$$bJ_{\lambda_n} u_{\lambda_n}' \rightharpoonup z \quad \text{weakly in } \mathcal{W} \quad \text{as } n \to +\infty.$$

Since $u_{\lambda_n}' \rightharpoonup u'$ weakly in $\mathcal{H} := L^2(J; H)$ as $n \to +\infty$, we have (taking the inner product with some test function $\varphi \in H^1(J; D(A))$, for instance)

$$bJ_{\lambda_n} u_{\lambda_n}' \rightharpoonup bu' \quad \text{weakly in } \mathcal{H} \quad \text{as } n \to +\infty.$$

Therefore, $bu' = z \in \mathcal{W}$ and

$$bJ_{\lambda_n} u_{\lambda_n}' \rightharpoonup bu' \quad \text{weakly in } \mathcal{W} \quad \text{as } n \to +\infty.$$

On the other hand, we have (taking if necessary a subsequence)

$$C(t)(bJ_{\lambda_n} u_{\lambda_n}') \rightharpoonup \psi \quad \text{weakly in } \mathcal{W}' \quad \text{as } n \to +\infty.$$

It is not difficult, using a suitable test function, to check that

$$J_{\lambda_n} C(t)(bJ_{\lambda_n} u'_{\lambda_n}) \rightharpoonup \psi \quad \text{weakly in } \mathcal{W}' \quad \text{as } n \to +\infty,$$

so that, passing to the limit in (2.15),

$$u'' + Au + b\psi = 0 \quad \text{in } L^2(J; V'). \tag{2.16}$$

To obtain $b\psi = B(t)u'$, it remains to show that

$$\psi = C(t)bu'. \tag{2.17}$$

We introduce $\mathcal{C} : \mathcal{W} \to \mathcal{W}'$ defined by

$$\forall \varphi \in \mathcal{W}, \quad (\mathcal{C}\varphi)(t) := C(t)\varphi(t) \quad \text{a.e.} \quad \text{on } J.$$

We remark that

$$\begin{aligned} \langle \mathcal{C}(bJ_\lambda u'_\lambda), bJ_\lambda u'_\lambda \rangle_{\mathcal{W}',\mathcal{W}} &= \int_0^T \langle b\mathcal{C}(bJ_\lambda u'_\lambda), J_\lambda u'_\lambda \rangle_{W',W}\, dt \\ &= \int_0^T \langle B(t) J_\lambda u'_\lambda, J_\lambda u'_\lambda \rangle_{W',W}\, dt = E(0) - E_\lambda(T). \end{aligned}$$

Whereas, due to Lemma 2.1, we have

$$E(T) + \langle \Psi, bu' \rangle_{\mathcal{W}',\mathcal{W}} = E(0).$$

Since

$$E(T) \le \liminf_{n\to+\infty} E_{\lambda_n}(T),$$

we obtain

$$\begin{aligned} E(0) - \langle \Psi, bu' \rangle_{\mathcal{W}',\mathcal{W}} &\le \liminf_{n\to+\infty} E_{\lambda_n}(T) \\ &= \liminf_{n\to+\infty} (E(0) - \langle \mathcal{C}(bJ_{\lambda_n} u'_{\lambda_n}), bJ_{\lambda_n} u'_{\lambda_n} \rangle_{\mathcal{W}',\mathcal{W}}) \\ &= E(0) - \limsup_{n\to+\infty} \langle \mathcal{C}(bJ_{\lambda_n} u'_{\lambda_n}), bJ_{\lambda_n} u'_{\lambda_n} \rangle_{\mathcal{W}',\mathcal{W}}. \end{aligned}$$

Hence

$$\limsup_{n\to+\infty} \langle \mathcal{C}(bJ_{\lambda_n} u'_{\lambda_n}), bJ_{\lambda_n} u'_{\lambda_n} \rangle_{\mathcal{W}',\mathcal{W}} \le \langle \Psi, bu' \rangle_{\mathcal{W}',\mathcal{W}}.$$

Then we can apply the following lemma.

LEMMA 2.2. *Let $\mathcal{W}$ be a Hilbert space and let $\mathcal{C} : \mathcal{W} \to \mathcal{W}'$ be monotone and Lipschitz continuous. Assume that $(z_n)_n$ is a sequence of $\mathcal{W}$ such that $z_n \rightharpoonup z$ weakly in $\mathcal{W}$ and $\mathcal{C}z_n \rightharpoonup \Psi$ weakly in $\mathcal{W}'$ as $n \to +\infty$.*

If

$$\limsup_{n\to+\infty} \langle \mathcal{C}z_n, z_n \rangle_{\mathcal{W}',\mathcal{W}} \le \langle \Psi, z \rangle_{\mathcal{W}',\mathcal{W}},$$

then $\Psi = \mathcal{C}z$

For the proof of this lemma, let $K : \mathcal{W}' \to \mathcal{W}$ be the duality map. Then $C := K\mathcal{C} : \mathcal{W} \to \mathcal{W}$ satisfies the assumptions of Proposition 2 of [8, p. 41]. (See also Brezis [5].)

This provides (2.17) and the proof of Theorem 2.1 is finished. ☐

3. Asymptotic stability.

3.1. An energy decay estimate on a short time interval. Assume that (2.1)–(2.5) hold. In order to study the asymptotic behavior of the energy, we first prove the following result, interesting in itself, concerning the *estimate of energy decay on a short interval of time.*

THEOREM 3.1. *Let $T > 0$ be fixed and assume that there exist $M, m > 0$ such that*

$$\forall t \in (0,T), \quad \forall v \in W, \quad \langle B(t)v, v\rangle_{W',W} \geq m\|v\|_W^2, \tag{3.1}$$

and

$$\forall t \in (0,T), \quad \forall v \in W, \quad \|B(t)v\|_{W'}^2 \leq M\langle B(t)v, v\rangle_{W',W}. \tag{3.2}$$

Then there exists $c > 0$ (independent of T) such that, for all $(u_0, u_1) \in V \times H$, the solution u of (2.6)–(2.7) *satisfies*

$$E(T) \;\leq\; \frac{1}{1 + c\,\frac{m}{T^{-3}+T^{-1}+MmT^{-1}}}\,E(0). \tag{3.3}$$

Theorem 3.1 is interesting in itself because it provides an estimate of the decay of the energy that is valid for t *small*. In particular, $E(t) < E(0)$. It has to be noted that, in general, estimates of the decay of the energy are provided for t *large enough*, even in the case of uniformly distributed damping terms. Of course if the damping is locally distributed in the domain, it is impossible to expect that $E(t) < E(0)$ for $t > 0$ small. (See, for example, [11, 12, 15, 16, 17, 18, 19, 23, 24] for classical estimates of the energy decay when t is large enough.)

Proof of Theorem 3.1. Following [7], we set

$$\theta(t) = t^2(T-t)^2.$$

Note that

$$\forall t \in [0,T], \quad |\theta'(t)| = |2t(T-t)(T-2t)| \leq 2T\theta^{1/2}(t), \tag{3.4}$$

$$\max_{t\in[0,T]} \theta(t) = \frac{T^4}{16}, \tag{3.5}$$

and

$$\int_0^T \theta(t)\,dt = \frac{T^5}{30}. \tag{3.6}$$

We also introduce $K_W, K_W' > 0$ such that

$$\forall v \in W, \quad K_W'|v|_H \leq \|v\|_W \leq K_W\|v\|_V = K_W|A^{1/2}v|_H. \tag{3.7}$$

First note that the energy of u is nonincreasing and satisfies

$$E(0) - E(T) = \int_0^T \langle Bu', u'\rangle_{W',W} \geq 0. \tag{3.8}$$

Multiplying (2.6) by θu, we obtain

$$\begin{aligned}\int_0^T \theta|A^{1/2}u|_H^2 &= -\int_0^T \theta\langle u''+Bu',u\rangle_{V',V}\\ &= \int_0^T ((\theta u)',u'))_H - \int_0^T \theta\langle Bu',u\rangle_{W',W}\\ &= \int_0^T \theta'(u,u')_H + \int_0^T \theta|u'|_H^2 - \int_0^T \theta\langle Bu',u\rangle_{W',W}\\ &\le \varepsilon\int_0^T {\theta'}^2(t)|u|_H^2 + \frac{1}{4\varepsilon}\int_0^T |u'|_H^2 + \int_0^T \theta(t)|u'|_H^2\\ &\quad + \eta\int_0^T \theta\|u\|_W^2 + \frac{1}{4\eta}\int_0^T \theta\|Bu'\|_{W'}^2,\end{aligned}$$

for all $\varepsilon,\eta>0$. Using (3.7), (3.4), (3.5) and (3.2) we deduce

$$\begin{aligned}\int_0^T \theta|A^{1/2}u|_H^2 \le{}& 4\frac{K_W^2}{{K_W'}^2}T^2\varepsilon\int_0^T \theta|A^{1/2}u|_H^2 + \frac{1}{4\varepsilon}\int_0^T |u'|_H^2\\ &+ \frac{T^4}{16}\int_0^T |u'|_H^2 + K_W^2\eta\int_0^T \theta|A^{1/2}u|_H^2 + \frac{T^4}{16}\frac{M}{4\eta}\int_0^T \langle Bu',u'\rangle_{W',W}.\end{aligned}$$

We choose ε and η such that

$$4\frac{K_W^2}{{K_W'}^2}T^2\varepsilon = K_W^2\eta = 1/4,$$

hence

$$\frac{1}{4\varepsilon} = 4\frac{K_W^2}{{K_W'}^2}T^2; \quad \frac{1}{4\eta} = K_W^2.$$

Thus we obtain

$$\begin{aligned}\frac12\int_0^T \theta|A^{1/2}u|_H^2 &\le \frac{1}{4\varepsilon}\int_0^T |u'|_H^2 + \frac{T^4}{16}\int_0^T |u'|_H^2 + \frac{T^4}{16}\frac{M}{4\eta}\int_0^T \langle Bu',u'\rangle_{W',W}\\ &= \left(4\frac{K_W^2}{{K_W'}^2}T^2 + \frac{T^4}{16}\right)\int_0^T |u'|_H^2 + \frac{K_W^2T^4M}{16}\int_0^T \langle Bu',u'\rangle_{W',W}.\end{aligned}$$

Hence,

$$\int_0^T \theta|A^{1/2}u|_H^2 \le F(T)\int_0^T |u'|_H^2 + \frac{K_W^2MT^4}{8}\int_0^T \langle Bu',u'\rangle_{W',W},$$

where

$$F(T) := 8\frac{K_W^2}{{K_W'}^2}T^2 + \frac{T^4}{8}.$$

Using

$$\forall t\in[0,T],\quad 2E(t) = |A^{1/2}u(t)|_H^2 + |u'(t)|_H^2 \ge 2E(T),$$

we deduce

$$\int_0^T \theta(2E(T) - |u'|_H^2) \le F(T)\int_0^T |u'|_H^2 + \frac{K_W^2 MT^4}{8}\int_0^T \langle Bu', u'\rangle_{W',W}.$$

Hence, using (3.5) and (3.7),

$$\begin{aligned} E(T)\int_0^T \theta &\le \frac{1}{2}\left(F(T) + \frac{T^4}{16}\right)\int_0^T |u'|_H^2 + \frac{K_W^2 MT^4}{16}\int_0^T \langle Bu', u'\rangle_{W',W} \\ &\le \frac{1}{{K'}_W^2}\left(4\frac{K_W^2}{{K'_W}^2}T^2 + \frac{3T^4}{32}\right)\int_0^T \|u'\|_W^2 + \frac{K_W^2 MT^4}{16}\int_0^T \langle Bu', u'\rangle_{W',W}. \end{aligned}$$

Thus, using (3.6), there exists a constant $c > 0$ (independent of T) such that

$$E(T) \le \frac{1}{c}(T^{-3} + T^{-1})\int_0^T \|u'\|_W^2 + \frac{1}{c}MT^{-1}\int_0^T \langle Bu', u'\rangle_{W',W}.$$

Using (3.1) and (3.8), we find

$$\begin{aligned} E(T) &\le \frac{1}{cm}(T^{-3} + T^{-1})\int_0^T \langle Bu', u'\rangle_{W',W} + \frac{1}{c}MT^{-1}\int_0^T \langle Bu', u'\rangle_{W',W} \\ &= \frac{1}{cm}(T^{-3} + T^{-1} + MmT^{-1})(E(0) - E(T)). \end{aligned}$$

Hence

$$E(T) \ \le \ \frac{1}{1 + c\,\frac{m}{T^{-3}+T^{-1}+MmT^{-1}}}\ E(0). \qquad \square$$

3.2. A condition for asymptotic stability. Assume that (2.1)–(2.5) hold for any $T > 0$. Then (2.6)–(2.7) is well-posed and it follows from Theorem 3.1 that the result of Smith [27] may be extended to the case of problem (2.6)–(2.7).

THEOREM 3.2. *Consider a sequence $(I_n)_{n\ge 0}$ of disjoint open intervals in $(0, +\infty)$, denoted by $I_n = (a_n, b_n)$, where $b_n \le a_{n+1}$ for all $n \in \mathbb{N}$, and assume that, for all $n \ge 0$, there exist $M_n, m_n > 0$ such that*

$$\forall t \in I_n, \quad \forall v \in W, \quad \langle B(t)v, v\rangle_{W',W} \ge m_n\|v\|_W^2, \tag{3.9}$$

and

$$\forall t \in I_n, \quad \forall v \in W, \quad \|B(t)v\|_{W'}^2 \le M_n\langle B(t)v, v\rangle_{W',W}. \tag{3.10}$$

Assume that the following condition holds:

$$\sum_{n=0}^{\infty} m_n T_n \min\left(T_n^2, \frac{1}{1 + m_n M_n}\right) = +\infty, \tag{3.11}$$

where T_n denotes the length of I_n. Then (2.6)–(2.7) is asymptotically stable; i.e., for all $(u_0, u_1) \in V \times H$, the solution u of (2.6)–(2.7) satisfies $E_u(t) \to 0$ as $t \to \infty$.

Remark 1. Moreover, the proof of Theorem 3.2 also provides an estimate of the decay of the energy: if there exists $C > 0$ such that

$$\forall n \in \mathbb{N}, \quad u_n := m_n T_n \min\left(T_n^2, \frac{1}{1 + m_n M_n}\right) \le C,$$

then there exists $\omega > 0$ such that

$$\forall n \in \mathbb{N}, \forall t \geq b_n, \quad E(t) \leq E(0) \exp\left(-\omega \sum_{p=0}^{n} u_p \right).$$

Remark 2. Note that condition (1.2) implies condition (3.11). Note also that, in the case of ordinary differential equations, Pucci and Serrin [25] improved the condition of [27] and proved asymptotic stability under the following condition:

$$\sum_{n=0}^{\infty} m_n T_n \min\left(T_n^2, \frac{1}{1 + \frac{m_n}{T_n} \int_{I_n} a} \right) = +\infty.$$

We do not know if this weaker condition is also sufficient in the case of the partial differential equations (2.6).

Proof of Theorem 3.2. For all $n \geq 0$, we denote $I_n = (a_n, b_n)$ and we apply Theorem 3.1 to the time interval I_n instead of $(0, T)$, which implies

$$E(b_n) \leq \frac{1}{1 + ck_n} E(a_n),$$

where, for all $n \geq 0$,

$$k_n := \frac{m_n}{T_n^{-3} + T_n^{-1} + M_n m_n T_n^{-1}} > 0.$$

Using that the energy is nonincreasing, we deduce, for all $n \geq 0$,

$$\begin{aligned} E(a_{n+1}) \leq E(b_n) &\leq \frac{1}{1 + ck_n} E(a_n) \\ &\leq \left(\prod_{p=0}^{n} \frac{1}{1 + ck_p} \right) E(a_0) \leq \left(\prod_{p=0}^{n} \frac{1}{1 + ck_p} \right) E(0). \end{aligned}$$

Since the energy is nonincreasing, in order to prove Theorem 3.2, it is sufficient to prove that $E(a_{n+1}) \to 0$ as $n \to \infty$. Thus it is sufficient to prove that

$$\prod_{p=0}^{+\infty} \frac{1}{1 + ck_p} = 0, \quad \text{or} \quad \sum_{p=0}^{+\infty} \ln\left(\frac{1}{1 + ck_p} \right) = -\infty.$$

If $k_p \not\to 0$ as $p \to \infty$, then the result follows, and if $k_p \to 0$ as $p \to \infty$, then it reduces to prove that $\sum_{p=0}^{+\infty} k_p = +\infty$. This condition is equivalent to (3.11) since

$$\begin{aligned} \frac{1}{2} m_n T_n \min\left(T_n^2, \frac{1}{1 + m_n M_n} \right) \leq k_n &= \frac{m_n T_n}{\frac{1}{T_n^2} + 1 + m_n M_n} \\ &\leq m_n T_n \min\left(T_n^2, \frac{1}{1 + m_n M_n} \right), \end{aligned}$$

which ends the proof of Theorem 3.2. Note also that condition (1.2) implies condition (3.11) since

$$m_n T_n \delta_n^2 \leq m_n T_n \min\left(T_n^2, \frac{1}{1 + m_n M_n} \right),$$

which proves Remark 2. It remains to prove Remark 1. Using $k_p \geq u_p/2$, we have

$$E(b_n) \leq \prod_{p=0}^{n} \frac{1}{1+cu_p/2} E(0) = \exp\left(-\sum_{p=0}^{n} \ln(1+cu_p/2)\right) E(0).$$

Since $\ln(1+cx/2) \geq \ln(1+cC/2)x/C$ for all $x \in (0,C)$, we obtain

$$E(b_n) \leq \exp\left(-\frac{\ln(1+cC/2)}{C} \sum_{p=0}^{n} u_p\right) E(0). \qquad \square$$

3.3. Further comments. The main restrictive assumption of our general setting is that the damping term $B(t)u'$ is assumed to be *uniformly* distributed in space. However, this restriction is crucial if we want to consider an *on/off* damping. Our result does not apply to an on/off damping that is only *locally* distributed in space, even if the geometric condition of Bardos, Lebeau, and Rauch [2, 3] is satisfied.

Let us explain why the case of a locally distributed on/off damping, for example, $B(t)u' = a(t,x)u'$, is out of reach, at least under such a general form. Indeed, even for a very simple example, the situation is complicated and the statement of the results depends on a lot of parameters.

Let us consider the one-dimensional wave equation in $(0,1)$:

$$\begin{cases} u'' - u_{xx} = -b(t)c(x)u', & x \in (0,1),\ t > 0, \\ u(t,0) = u(t,1) = 0, & t > 0, \\ u(0,\cdot) = u_0 \in H_0^1(0,1),\ u'(0,\cdot) = u_1 \in L_0^2(0,1). \end{cases} \tag{3.12}$$

Here we consider $a(t,x) = b(t)c(x)$ and we can distinguish three cases.

1. *The locally distributed (non on/off) case.* Our result does not apply to this case. Actually, it was not the purpose of the present paper, since this case has been widely studied in the literature. Let us recall some well-known results.

First, we consider the time-independent case:

$$a(t,x) = c(x), \quad \text{i.e., } b(t) \equiv 1,$$

with

$$c(x) \geq c_0 > 0 \quad \text{for all } x \in \omega,$$

where ω is an open subset of $(0,1)$. If ω is nonempty, then asymptotic stability holds. More generally, this result is well known in higher dimension spaces, provided that ω satisfies the geometric condition of Bardos, Lebeau, and Rauch [2, 3]. On the other hand, this kind of result may be extended to the time-dependent case

$$a(t,x) = b(t)c(x),$$

provided that

$$0 < \sigma(t) \leq b(t) \leq 1/\sigma(t),$$

with the condition

$$\int_0^{+\infty} \sigma(\tau)\, d\tau = +\infty.$$

See, for example, [20, 22, 26]. Notice that this allows us to consider a time-dependent damping, but not an on/off damping, since the assumption $b(t) > 0$ is needed.

2. *The uniformly distributed on/off case.* In the present paper, we consider a damping that is uniformly distributed in space, but is allowed to be on/off in time. For example, we assume

$$a(t, x) = b(t)c(x) \geq c_0 b(t),$$

with $c_0 > 0$ and where $b(t) = 0$ on an infinite union of intervals. In this case, Theorem 3.2 gives a sharp condition of asymptotic stability. (See section 4 for several examples of application of Theorem 3.2.)

3. *The locally distributed on/off case.* Now let us turn to the more general case of a locally distributed on/off damping, and let us see why its study is out of reach, at least in a general setting.

We consider the following "simple" example:

$$c(x) = \chi_\omega(x), \quad \text{where } \ \omega = (1/2 - \lambda, 1/2 + \lambda),$$

with $0 < \lambda \leq 1/2$, and $b : \mathbb{R}_+ \to \mathbb{R}_+$ is T-periodic such that

$$b(t) = 1 \quad \text{on} \quad [0, T) \quad \text{and} \quad b(t) = 0 \quad \text{on} \quad [T, 2T).$$

Notice that, for all $0 < \lambda \leq 1/2$, ω satisfies the geometric condition of Bardos, Lebeau, and Rauch [2, 3]. However, this is not sufficient to insure asymptotic stability in this case. Indeed the following result holds.

THEOREM 3.3. [21, Theorem 2.3, p. 340].

(i) *If*

$$\left(\frac{1}{T} \in 2\mathbb{N} \quad \textit{and} \quad 2\lambda < T\right),$$

then there exists initial condition $(u_0, u_1) \in H_0^1(0,1) \times L^2(0,1)$ *such that the energy of the solution of* (3.12) *remains constant with time:* $E(t) = E(0) > 0$ *for all* $t > 0$.

(ii) *If*

$$\left(\frac{1}{T} \in 2\mathbb{N} \quad \textit{and} \quad 2\lambda > T\right), \quad \textit{or} \quad \left(\frac{1}{T} \notin 2\mathbb{N}\right),$$

then the energy of the solutions of (3.12) *decays uniformly exponentially to* 0.

This result shows that the situation is much more complicated because both difficulties (coming from the fact the damping is *locally* distributed in space and is *on/off* in time) are considered. Notice that Theorem 3.3 is proved in [21] without using the notion of optic rays. However, the results can be explained with the following comments related to optic rays propagation.

First consider the case of a uniformly distributed damping, i.e., $\lambda = 1/2$. In this case, Theorem 3.3 insures that asymptotic stability holds for all $T > 0$. Notice that, in this case, it is clear that each optic ray crosses the damping space region during a period when the damping is effective.

Next consider the more interesting case of a locally distributed damping: $0 < \lambda < 1/2$. Now there are some values of T and some values of λ for which *some optic rays cross the space damping region only when the feedback is not active.* For example, take $T = 1/2$, $\lambda < T/2 = 1/4$, and consider the optic ray that leaves the point $x = T/2 = 1/4$ and that goes to the left (toward the point $x = 0$) at time $t = 0$. This ray describes the segment $[1/4, 3/4]$ (that contains the damping region) during the time intervals $[T, 2T]$, $[3T, 4T], \ldots$, thus during periods when $b(t) = 0$. The same situation occurs as soon as $1/T \in 2\mathbb{N}$ with $2\lambda < T$. In these cases, Theorem 3.3 provides negative results of stability, while it provides positive results in the other cases.

We see that all these results are coherent with the optic ray condition known for time-independent feedbacks [2, 3]. But now, the fact that the feedback depends on time has to be taken into account. And it seems to be crucial that *each optic ray crosses the damping space region during a period when the damping is effective.*

4. Some examples. We first consider the case of *ordinary differential equations* that has been widely studied (see, for example, [1, 9, 10, 25, 27, 28]).

Example 1 (the oscillator equation). Assume $a \in L^\infty_{\text{loc}}(\mathbb{R}_+)$ is a nonnegative function whose minimum and maximum values in I_n are denoted by α_n and A_n.

With $H = V = D(A) = \mathbb{R}$, $Au = u$, and $B(t) = a(t)Id$, Theorem 3.2 applies to the oscillator equation (1.1) (with $m_n := \alpha_n$ and $M_n := A_n$). Since (1.2) implies condition (3.11), this again gives Theorem 1.1 with a small improvement of the sufficient condition. (In particular, this gives a new proof of the result of Smith, very different from the original proof based on monotonicity properties.)

Taking $B(t) = a(t)f$, we may also consider the nonlinear oscillator

$$u'' + u + a(t)f(u') = 0, \qquad t > 0,$$

where we assume that $f \in \mathcal{C}^1(\mathbb{R})$ is such that $f(0) = 0$ and $0 < \beta \le f' \le B$. Hence B defined by $B(t)v := a(t)f(v)$ satisfies (3.9) and (3.10) with $m_n := \beta\alpha_n$ and $M_n := BA_n$.

Next we assume that Ω is a bounded open set of $\mathbb{R}^N$ with regular boundary and we turn to the case of *uniformly damped partial differential equations* with a *bounded* damping operator.

Example 2 (a wave equation). Let $a_1,\ a_2 \in L^\infty_{\text{loc}}(\mathbb{R}_+)$ be two nonnegative functions such that *either*

$$a_2(t) = 0 \quad \text{a.e. on } \mathbb{R}_+$$

or

$$\exists C > 0, \quad a_1(t) \le Ca_2(t) \quad \text{ for a.e. } t \in \mathbb{R}_+$$

and

$$\forall n \in \mathbb{N}, \quad \alpha_n \le a_1(t) + a_2(t) \le A_n \quad \text{ for a.e. } t \in I_n.$$

Consider also $f \in \mathbb{C}^1(\mathbb{R})$ such that $f(0) = 0$ and $0 < \beta \le f' \le B$.

Then we study the following damped wave equation:

$$\begin{cases} u'' - \Delta u + a_1(t)f(u') - a_2(t)\Delta u' = 0, & x \in \Omega, t > 0, \\ u = 0, & x \in \partial\Omega, t > 0, \\ u(t=0) \in H^1_0(\Omega), \quad u'(t=0) \in L^2(\Omega). \end{cases} \tag{4.1}$$

We choose $H = L^2(\Omega)$, $A = -\Delta$ (with Dirichlet boundary conditions), $D(A) = H^2 \cap H^1_0(\Omega)$, and $B(t)v := a_1(t)f(v) - a_2(t)\Delta v$. The choice of W depends on a_2: in the case $a_2 \equiv 0$, $W = H$; otherwise $W = V$.

Consider $J = (0, T)$ with $T > 0$ and let us verify that the assumptions of Theorem 2.1 are satisfied. First (2.1) and (2.2) clearly hold. Then we write, for $w, z \in W$,

$$\langle B(t)w - B(t)z, w - z\rangle_{W',W} = a_1(t)\int_\Omega (f(w) - f(z))(w - z)\,dx + a_2(t)\int_\Omega |\nabla(w - z)|^2\,dx \geq 0,$$

which gives (2.3). On the other hand, for $w \in W$, we have

$$\begin{aligned}\langle B(t)w, w\rangle_{W',W} &= a_1(t)\int_\Omega f(w)w\,dx + a_2(t)\int_\Omega |\nabla w|^2\,dx\\ &\geq a_1(t)\beta\|w\|_H^2 + a_2(t)\|w\|_V^2.\end{aligned}$$

If $a_2 \equiv 0$, then $W = H$ and we choose $b(t) = \sqrt{\beta a_1(t)}$. In the other case, $W = V$ and we choose $b(t) = \sqrt{a_2(t)}$. Thus (2.4) is also satisfied. Finally, we write, for $w, z \in W$,

$$\begin{aligned}\|B(t)w - B(t)z\|_{W'}^2 &\leq 2a_1(t)^2\|f(w) - f(z)\|_{W'}^2 + 2a_2(t)^2\|\Delta w - \Delta z\|_{W'}^2\\ &\leq 2a_1(t)^2B^2\|w - z\|_H^2 + 2a_2(t)^2\|\Delta w - \Delta z\|_{W'}^2.\end{aligned}$$

In the case $a_2 \equiv 0$, we deduce

$$\|B(t)w - B(t)z\|_H^2 \leq 2a_1(t)^2B^2\|w - z\|_H^2 \leq 2\frac{B^2}{\beta^2}b(t)^4\|w - z\|_H^2.$$

In the case $a_2 \not\equiv 0$, we deduce

$$\begin{aligned}\|B(t)w - B(t)z\|_{H^{-1}(\Omega)}^2 &\leq 2[B^2C^2a_2(t)^2\|w - z\|_{L^2(\Omega)}^2 + a_2(t)^2\|w - z\|_{H_0^1(\Omega)}^2]\\ &\leq Kb^4(t)\|w - z\|_{H_0^1(\Omega)}^2,\end{aligned}$$

where $K > 0$ is a constant. Thus (2.5) proved in both cases. Hence Theorem 2.1 insures the well-posedness of (2.6)–(2.7).

For the study of asymptotic stability, we verify that (3.9) and (3.10) are satisfied.

First case: $a_2 \equiv 0$ *(*$W = H$*).* We write for all $n \geq 0$, $t \in I_n$, $v \in W$,

$$(B(t)v, v)_H = a_1(t)\int_\Omega f(v)v\,dx \geq \alpha_n\beta\|v\|_H^2,$$

and

$$\begin{aligned}\|B(t)v\|_H^2 &= a_1(t)^2\int_\Omega f(v)^2\,dx\\ &\leq A_nBa_1(t)\int_\Omega f(v)v\,dx = A_nB(B(t)v, v)_H.\end{aligned}$$

Thus (3.9) and (3.10) are satisfied for $m_n := \beta\alpha_n$ and $M_n := BA_n$.

Second case: $a_2 \not\equiv 0$ *(*$W = V = H_0^1(\Omega)$*).* We write for all $n \geq 0$, $t \in I_n$, $v \in W$,

$$\langle B(t)v, v\rangle_{V',V} \geq a_2(t)\|w\|_V^2 \geq \alpha_n\|w\|_V^2,$$

and

$$\begin{aligned}\|B(t)v\|^2_{H^{-1}(\Omega)} &= \|a_1(t)f(v) - a_2(t)\Delta v\|^2_{H^{-1}(\Omega)} \\ &\le 2a_1(t)^2 \int_\Omega f(v)^2\,dx + 2a_2(t)^2\|\Delta v\|^2_{H^{-1}(\Omega)} \\ &\le 2Ca_2(t)a_1(t)B\int_\Omega f(v)v\,dx + 2a_2(t)^2\int_\Omega \nabla v^2\,dx \\ &\le 2(BC+1)a_2(t)\left(a_1(t)\int_\Omega f(v)v\,dx + a_2(t)\int_\Omega \nabla v^2\,dx\right) \\ &\le 2(BC+1)A_n\langle B(t)v, v\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}.\end{aligned}$$

Thus (3.9) and (3.10) are satisfied for $m_n := \alpha_n$ and $M_n := 2(BC+1)A_n$.

Applying Theorem 3.2, we deduce that (3.11) is a sufficient condition of asymptotic stability for (2.6).

Note that we may also consider the more general case

$$B(t)v := a_1(t)f(v) + a_2(t)(-\Delta)^{1/2}g((-\Delta)^{1/2}v).$$

Note also that in the particular case of a linear bounded damping $B(t)v := a(t)v$, this completes the work done in [21], where we studied the case of a locally damped wave equation with a periodic on/off damping.

Example 3 (some plate equations). In the same spirit, taking $H = L^2(\Omega)$, $A = \Delta^2$, $V = H^2_0(\Omega) = \{v \in H^2(\Omega) \mid v = 0 \text{ and } \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\}$ and $D(A) = H^4 \cap H^2_0(\Omega)$, we may consider the following damped plate equation:

$$(4.2)\quad \begin{cases} u'' + \Delta^2 u + a_1(t)f(u') - a_2(t)\Delta u' + a_3(t)\Delta g(\Delta u') = 0, & x \in \Omega, t > 0, \\ u = 0, \quad \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(t=0) = u_0 \in H^2_0(\Omega), \quad u'(t=0) = u_1 \in L^2(\Omega). \end{cases}$$

Here the damping operator is defined by

$$B(t)v := a_1(t)f(v) - a_2(t)\Delta v + a_3(t)\Delta g(\Delta v),$$

and the choice of W depends on the assumptions on the functions a_i. There are 3 cases : $W = H$, $W = H^1_0(\Omega)$ and $W = V$. For the applications of Theorems 2.1 and 3.2, we leave the details to the reader.

5. Another result: The case of a positive-negative damping. In this part, we study the case of a "positive-negative" damping. We assume that H, $D(A)$, and V are defined as in section 2 and that A still satisfies assumption (2.1) with $W = H$. And we consider a time-dependent operator B such that $B \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathrm{Lip}(H))$. (Note that we only consider the case of a bounded operator B, so we assume in this part that $W = H$.)

Now we assume that B is a "positive-negative" feedback: let $(t_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of $\mathbb{R}^+$ such that $t_n \to +\infty$ as $n \to +\infty$. For all $n \in \mathbb{N}$, we define $I_{2n} := (t_{2n}, t_{2n+1})$ and $I_{2n+1} := (t_{2n+1}, t_{2n+2})$, and we assume that B is positive on I_{2n} and negative on I_{2n+1}. *Hence the energy decays on the time intervals* I_{2n} *and increases on the time intervals* I_{2n+1}.

We also assume that, for all $n \in \mathbb{N}$, there exist three positive constants m_{2n}, M_{2n}, M_{2n+1} such that

$$(5.1)\qquad \forall t \in I_{2n},\quad \forall v \in H,\quad (B(t)v, v)_H \ \ge\ m_{2n}\|v\|^2_H,$$

$$\forall t \in I_{2n}, \quad \forall v \in H, \quad \|B(t)v\|_H^2 \ \leq \ M_{2n}(B(t)v, v)_H, \tag{5.2}$$

$$\forall t \in I_{2n+1}, \quad \forall v \in H, \quad -M_{2n+1}\|v\|_H^2 \ \leq \ (B(t)v, v)_H \ \leq \ 0. \tag{5.3}$$

Note that the well-posedness of (2.6)–(2.7) is classical (using standard arguments on Lipschitz perturbations of contraction semigroups).

Then, from Theorem 3.2, we deduce the following sufficient condition of asymptotic stability.

THEOREM 5.1. *Assume* (2.1), (5.1), (5.2), (5.3). *Assume that the following condition holds:*

$$\begin{cases} \sum_{p=0}^{+\infty} M_{2p+1}T_{2p+1} < \infty, \\ \sum_{p=0}^{+\infty} m_{2p}T_{2p}\delta_{2p}^2 = +\infty, \end{cases}$$

where T_p denotes the length of I_p and $\delta_p = \min(T_p, (1 + M_p)^{-1})$. Then equation (2.6)*–*(2.7) *is asymptotically stable; i.e., for all $(u_0, u_1) \in V \times H$, the solution u of* (2.6)*–*(2.7) *satisfies $E_u(t) \to 0$ as $t \to \infty$.*

Remark. This gives a result of stability in the case of a globally distributed time-dependent feedback of indefinite sign. This completes [21], where we studied the wave equation damped by a time-dependent boundary feedback of indefinite sign. See also [4, 6] for results in the case of space-dependent feedback of indefinite sign.

Proof of Theorem 5.1. For all $n \in \mathbb{N}$, applying Theorem 3.1 on the time intervals I_{2n}, we obtain

$$E(t_{2n+1}) \leq \frac{1}{1 + c\frac{m_{2n}}{T_{2n}^{-3}+T_{2n}^{-1}+M_{2n}m_{2n}T_{2n}^{-1}}} E(t_{2n}).$$

On the other hand, on the time intervals I_{2n+1}, we can write

$$0 \leq E'(t) = -\langle B(t)u', u' \rangle_{W',W} \leq M_{2n+1}\|u'(t)\|_H^2 \leq 2M_{2n+1}E(t).$$

Thus

$$E(t_{2n+2}) \leq E(t_{2n+1})e^{2M_{2n+1}T_{2n+1}}.$$

Hence

$$E(t_{2n+2}) \leq \left(\prod_{p=0}^{n} e^{2M_{2p+1}T_{2p+1}} \frac{1}{1 + c\frac{m_{2p}}{T_{2p}^{-3}+T_{2p}^{-1}+M_{2p}m_{2p}T_{2p}^{-1}}} \right) E(0).$$

In particular, we deduce a condition of asymptotic stability:

$$\sum_{p=0}^{+\infty} \left(2M_{2p+1}T_{2p+1} - \ln\left(1 + c\frac{m_{2p}}{T_{2p}^{-3} + T_{2p}^{-1} + M_{2p}m_{2p}T_{2p}^{-1}}\right)\right) = -\infty.$$

For example, we may assume

$$\begin{cases} \sum_{p=0}^{+\infty} M_{2p+1}T_{2p+1} < \infty, \\ \sum_{p=0}^{+\infty} m_{2p}T_{2p}\delta_{2p}^2 = +\infty, \end{cases}$$

or we may assume

$$\begin{cases} \sum_{p=0}^{+\infty} m_{2p} T_{2p} \delta_{2p}^2 = +\infty, \\ M_{2p+1} T_{2p+1} = o(\ln(1 + m_{2p} T_{2p} \delta_{2p}^2)). \end{cases} \qquad \square$$

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