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Symmetrization of dissipative-dispersive traveling waves for systems of conservation laws

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Abstract

This work addresses existence of traveling waves for systems of conservation laws through an Hamiltonian formalism. A related Hamilton–Jacobi equation is given and studied. Our proofs rely on the construction of a cost functional and require a smallness condition for the dispersion with respect to diffusion. Some numerical applications are also presented, concerning a model phase transition problem.

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1. Introduction

In this paper, we consider a system of one-dimensional conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \tag{1}$$

where the flux $f : \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be a smooth function, and $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^N$ is the unknown, see [22,26,34]. A shock is a discontinuous function of the form

$$u(x,t) = U^- \quad \text{for } x < \sigma t, \qquad u(x,t) = U^+ \quad \text{for } x \geq \sigma t, \tag{2}$$

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where $\sigma \in \mathbb{R}$ and U^+ and U^- in $\mathbb{R}^N$. The function given by (2) is a solution to (1) in a weak sense if and only if U^+ and U^- satisfy the Rankine–Hugoniot condition:

$$-\sigma(U^+ - U^-) + f(U^+) - f(U^-) = 0. \tag{3}$$

Among solutions satisfying (3), entropy solutions are the one which verify the additional entropy inequality

$$-\sigma(S(U^+) - S(U^-)) + F(U^+) - F(U^-) \leq 0. \tag{4}$$

The entropy function $S : \mathbb{R}^N \to \mathbb{R}$ and the entropy flux $F : \mathbb{R}^N \to \mathbb{R}$ are *given* functions (provided by the underlying physics aimed to be described) and satisfy the conditions

$$\frac{\partial S^t}{\partial u}\frac{\partial f}{\partial u} = \frac{\partial F^t}{\partial u} \quad \text{and} \quad \frac{\partial^2 S}{\partial u^2} > 0. \tag{5}$$

However, in some problems this selection principle seems not to be sufficient, i.e. (1), (3) and (4) may provide shocks which are not observed.[1] A reason is that (1) is only a rough approximation and one has to take into account phenomena at lower scales. In some models for instance it is appropriate to add dispersion as well as diffusion in (1). In this direction, we will study the system

$$\partial_t u + \partial_x f(u) = \mu \partial_x \left(\frac{\partial^2 S}{\partial u^2}\partial_x u\right) + \varepsilon \partial_{xx}\left(\frac{\partial^2 S}{\partial u^2}\partial_x u\right), \tag{6}$$

where $\mu > 0$ and $\varepsilon \in \mathbb{R}$. Introducing the new variable $v = \frac{\partial S}{\partial u}$, (6) becomes

$$\partial_t u + \partial_x f(u) = \mu \partial_{xx} v + \varepsilon \partial_{xxx} v. \tag{7}$$

The specific form of the diffusion and dispersion terms have been considered in [24] (see also [10]). The physical and numerical relevance of this model will be discussed in Section 2.

A natural approach to define admissible solutions to (1), (3) and (4) could be to single out those arising as limits as $\mu \to 0$ and $\varepsilon \to 0$ of solutions to (6). Since shock solutions are traveling waves, we will restrict ourselves to traveling wave solutions to (6). More precisely, a traveling wave solution to (6) is of the form

$$u_\mu(x,t) = U\left(\frac{x - \sigma t}{\mu}\right), \tag{8}$$

so that the function u_μ satisfies (6) if and only if U is a solution of the third-order differential equation

$$-\sigma U' + f(U)' = \left(\frac{\partial^2 S}{\partial u^2}U'\right)' + \tau\left(\frac{\partial^2 S}{\partial u^2}U'\right)'', \tag{9}$$

where we introduced the constant

$$\tau \equiv \frac{\varepsilon}{\mu^2}.$$

In view of the above discussion, we supplement this equation with the boundary condition at infinity

$$U(\pm\infty) = U^\pm, \qquad U'(\pm\infty) = 0. \tag{10}$$

[1] A beautiful account on this issue can be found in [30].

For given σ and τ kept constant, if U is a solution to (9) and (10), then u_μ converges to a "shock" solution of (1) as $\mu \to 0$. However, since Eq. (8) is non-degenerate, its solutions U are smooth, and therefore this limiting approach gives some "fine" structure[2] to the shock.

The purely dissipative case $\tau = \varepsilon = 0$ has been extensively studied in the past. The general theory in this case is broadly covered in [10] and [36]. Major references for abstract viscous shocks are [29] and [39, Chapter 24]. Concerning gas dynamics, the investigation of shocks dates back to [42,13]. Later works on more general fluids include in particular [28,33]. In this case, integrating (9) once yields a first-order equation of gradient type (see Remark 1). To our knowledge, this structure was first pointed out by Kulikovsky [21], see also [15], as a consequence of the existence of an entropy pair (S, F). Another issue is the study of the stability of shocks for non-convex Hugoniot curves: a modern review for compressible gas dynamics is given in [30] (see also the references therein). See also [25,8,3], with some applications to numerical approximations. Slow shocks in numerics are discussed in [20,19]. Dynamical phase transitions of the p-system are discussed in [43]. An up-to-date reference is [10] and therein.

In the case $\tau \neq 0$, i. e. $\varepsilon \neq 0$, let us mention [24] where a model dissipative-dispersive model is also studied. Application of the dissipative-dispersive approach to the study of multiphase flows may be found in [40,41,37,4]. Other references in a very related context are [6] for the linear stability of phase boundaries, [2] for profiles of some hyperbolic–elliptic problems. Shocks allowed in this context are called non-classical shocks [2,23]. For phase boundaries the existence of internal structures can be recast in a purely dissipative framework [38].

We propose the following approach.

1.1. The Hamilton–Jacobi formalism

Assume the entropy function S verifies the condition[3]

$$(\mathrm{H}_1) \qquad \lim_{|u|\to+\infty} \frac{S(u)}{|u|} = +\infty.$$

Then, since S is assumed to be convex by (5), the change of variable $\Phi : \mathbb{R}^N \to \mathbb{R}^N$,

$$u \mapsto v = \Phi(u) \equiv \nabla S(u)$$

is a diffeomorphism of $\mathbb{R}^N$. Let $\sigma \in \mathbb{R}$, $U^+, U^- \in \mathbb{R}^N$ satisfying the Rankine–Hugoniot condition (3). For $v \in \mathbb{R}^N$, we set

$$R_0(v) = v \cdot \Phi^{-1}(v) - S(\Phi^{-1}(v)) - v \cdot U^+ + S(U^+), \qquad R_1(v) = v \cdot f(\Phi^{-1}(v)) - F(\Phi^{-1}(v)) - v \cdot f(U^+) + F(U^+), \qquad R(v) = R_1(v) - \sigma R_0(v) \quad \text{and} \quad V^\pm = \Phi(U^\pm).$$

The function R is sometimes called the generating function in the sense of Kulikovsky[4] [21,15]. A classical computation, based on (5), yields

$$\nabla R(v) = -\sigma(u - U^+) + f(u) - f(U^+) = -\sigma(u - U^-) + f(u) - f(U^-), \tag{11}$$

where $u = \Phi^{-1}(v)$, so that

$$\nabla R(V^\pm) = 0, \tag{12}$$

[2] We do not claim this structure to be unique.
[3] This condition is not essential, but convenient for expository purposes.
[4] Obtained, following Godunov [14], through the use of the entropy variable v.

in view of the Rankine–Hugoniot condition.[5] Relation (11) shows that changing U^+ by U^- in the definition of R would not affect its gradient. We have the following result of local nature.

Proposition 1. *Let $I \in \mathcal{C}^2(\mathbb{R}^N, \mathbb{R})$ and assume that I satisfies the Hamilton–Jacobi equation, for $\tau > 0$,*

$$\frac{\tau}{2}|\nabla I|^2(v) = R(v) - I(v) \quad \forall\, v \in \mathbb{R}^N. \tag{13}$$

Let V be a solution, on some interval $]a, b[$, of the gradient flow

$$V'(s) = \nabla I(V(s)) \quad \text{for } s \in]a, b[. \tag{14}$$

Then, the function $U(s) = \Phi^{-1}(V(s))$ is a solution of Eq. (9) *on the interval $]a, b[$.*

Proof. In the V variable, Eq. (9) becomes

$$\tau V'' + V' = \nabla R(V). \tag{15}$$

Differentiating (13) with respect to V, we are led to

$$\tau D^2 I \cdot \nabla I = \nabla R - \nabla I \text{ on } \mathbb{R}^N.$$

If V is a solution of (14), differentiating (14) with respect to s we obtain

$$V''(s) = D^2 I(V(s)) \cdot V'(s) = D^2 I(V(s)) \cdot \nabla I(V(s)) = \frac{1}{\tau}(\nabla R(V(s)) - \nabla I(V(s))) = \frac{1}{\tau}(\nabla R(V(s)) - V'(s))$$

so that (15) is satisfied. □

Remark 1. In the case $\tau = 0$, Eq. (9) directly becomes

$$V'(s) = \nabla R(V(s)), \tag{16}$$

which is also of clear gradient type. In this respect, our approach allows to analyze (9) as a gradient flow, even in the presence of dispersion.

1.2. The main result

The main goal of this paper is to construct a solution to (13), and which takes into accounts the conditions at infinity (10) for U, that is

$$V(\pm\infty) = V^{\pm}, \qquad V'(\pm\infty) = 0 \tag{17}$$

for the function V. Notice in particular that, if V is globally defined on $]a, b[= \mathbb{R}$, and satisfies condition (17), then necessarily

$$\nabla I(V^{\pm}) = 0, \tag{18}$$

i.e. $V^{\pm}$ are critical points of I. In our opinion, an important advantage of Eq. (14) is that it is of the first order, and moreover of gradient type. In particular, if I is a solution of (13) and V^- is a critical point of I, then the states V^+ that can be connected to V^- (in the sense of Eqs. (9) and (10)) correspond to the critical points of I which are in the unstable manifold of V^-.

[5] In particular, in view of (12) R is not globally strictly convex if $V^+ \neq V^-$.

In view of (14) and (18) a necessary[6] condition for the existence of a global solution V satisfying (17) is that

$$I(V^{\pm}) = R(V^{\pm}). \tag{19}$$

Notice that (19) implies the entropy condition (4). Indeed, by (14)

$$0 \leq \int_{\mathbb{R}} |V'(s)|^2 \, \mathrm{d}s = I(V^+) - I(V^-) = R(V^+) - R(V^-) = +\sigma(S(U^+) - S(U^-)) - F(U^+) + F(U^-), \tag{20}$$

where we have used the Rankine–Hugoniot condition and the definition of R.

Theorem 1. *Assume that R satisfies the assumptions*[7]

$$(\mathrm{H}_2) \qquad D^2 R \geq -c\,\mathrm{Id},$$

for some constant $c \in \mathbb{R}$, and
(H_3) *there exists an increasing family $\{K_n\}_{n\in\mathbb{N}}$ of bounded compact convex non-empty sets such that $\cup K_n = \mathbb{R}^N$ and*

$$R(P_n(u)) \leq R(u) \quad \forall u \in R^N, \tag{21}$$

where P_n is the orthogonal projector on K_n.
There exists a constant τ_0, which depends only on σ, f and S, such that if

$$0 < \tau < \tau_0, \tag{22}$$

then there exists a $\mathcal{C}^\infty$ solution I of (13) and (19). Moreover $s \mapsto V(s)$ is a solution of the classical traveling wave equation (15) *supplemented with the condition at infinity* (17) *if and only if it satisfies* (14) *on $\mathbb{R}$ with the same condition at infinity.*

The meaning of condition (22) is that dispersion is small as compared to diffusion. We therefore expect that classical results of the purely dissipative case can be extended in this range. A typical example we have in mind is the existence of traveling waves related to shocks of small amplitude for pairs satisfying the Rankine–Hugoniot condition (see [36, Chapter 7]). We hope nevertheless that some of the tools used here will also be useful to study shocks in the presence of large dispersion. We deduce from Theorem 1 the following corollary.

Corollary 1. *Assume that τ satisfies* (22). *Let U_1 and U_2 be two solutions of the third-order traveling wave equation* (9) *such that $U_1^{\pm}$ and $U_2^{\pm}$ exist but may possibly be different. Then the orbits*[8] *of U_1 and U_2 are either identical or disjoint.*

The solution I will be given explicitly as the minimum of a strictly convex cost functional. Section 2 is devoted to examples, Section 3 to the proof of Theorem 1. Additional properties of I are presented in Section 4, and additional references in Section 5. A numerical example is treated in Section 6.

[6] Assuming as above $I \in \mathcal{C}^2$.
[7] Assumption (H_3) is in particular verified if the level sets $R^c = \{z \in \mathbb{R}^N,\ R(z) \leq c\}$ are compact convex for an unbounded sequence $c_n \to +\infty$.
[8] The orbit of U is $\{U(s);\ -\infty < s < +\infty\} \subset \mathbb{R}^N$.

2. Some relevant examples

2.1. Scalar equations

We consider first the case $N = 1$. In this case $S(u) = \frac{u^2}{2}$ is always an admissible entropy (admitting as entropy flux $F(u) = \int u f'(u)$, so that $v = u$). With such an entropy, Eq. (6) becomes

$$\partial_t u + \partial_x f(u) = \mu \partial_{xx} u + \varepsilon \partial_{xxx} u \tag{23}$$

and the traveling wave equation is

$$-\sigma u' + f(u)' = u'' + \tau u'''. \tag{24}$$

An interesting particular case of non-convex flux is given by the cubic one $f(u) = u^3$ (see e.g. [23,24], which also covers the case of large dispersion). In this case, R is explicitly given by

$$R(u) = -\sigma \frac{(u-u^+)^2}{2} + \frac{1}{4}(u-u^+)^2(u^2 + 2uu^+ + 3u^{+2}).$$

2.2. Nonlinear elastodynamics

This system (see [37,40]) is

$$\partial_t v + \partial_x p(w) = \mu \partial_{xx} v - \varepsilon \partial_{xxx} w, \qquad \partial_t w - \partial_x w = 0$$

and is assumed to model some one-dimensional fluid flow: v stands for the velocity, w for the specific volume, whereas $p(w)$ denotes the pressure (assumed to be a function of w). It is of special interest in the case of non-convex pressure laws p, that model van der Waals equations of states. Even though the above system is not of the form (6), the related traveling wave system

$$-\sigma v' + p(w)' = v'' - \tau w'', \qquad -\sigma w' - v' = 0,$$

can be recast, after elimination of the v variable, into the scalar equation

$$-\sigma^2 w' - p(w)' = \sigma w'' + \tau w''',$$

which is of the form (24) considered above.

In view of problems with variable coefficients it is worthwhile to notice that the approach proposed in this work admits an extension to

$$\partial_t v + \partial_x p(w) = \partial_x (\mu(w) \partial_x v) - \partial_x \left(K(w) \partial_{xx} w + \frac{K'(w)}{2} (\partial_x w)^2 \right), \qquad \partial_t w - \partial_x w = 0, \tag{25}$$

which corresponds to the isothermal system for a compressible fluid written in Lagrangian coordinate, see [7] and therein. The variable viscosity is $\mu(w) > 0$ and the variable capillarity is $K(w)$. It is reasonable to assume that $K(w) > 0$. The corresponding traveling wave equation is

$$-\sigma^2 w' - p(w)' = \left(\sigma \mu(w) w' + K(w) w'' + \frac{K'(w)}{2} (w')^2 \right)', \tag{26}$$

i.e. after integration from $+\infty$

$$-\sigma^2(w-w^+)-(p(w)-p(w^+))=\sigma\mu(w)w'+K(w)w''+\frac{K'(w)}{2}(w')^2. \tag{27}$$

Let us define

$$R(w)=-\sigma^2\frac{(w-w^+)^2}{2}-\int_{w^+}^{w}p(z)\,\mathrm{d}z+(w-w^+)p(w^+).$$

The extension of Proposition 1 to (27) is given by

Proposition 2. *Set $\tau_\sigma(w)=\frac{K(w)}{\sigma^2\mu(w)^2}$. Let I be a C^2 solution of the Hamilton Jacobi equation*

$$\frac{\tau_\sigma(w)}{2}|I'(w)|^2=R(w)-I(w) \tag{28}$$

and let w be a solution of the gradient flow

$$\sigma\mu(w)w'(s)=I'(w(s)). \tag{29}$$

Then w is a solution of the traveling wave equation (27).

The proof is similar to the proof of Proposition 1. First, we differentiate (28) with respect to w. This yields

$$K(w)\left(\frac{I'}{\sigma\mu}\right)'(w)\left(\frac{I'}{\sigma\mu}\right)(w)+\frac{K'(w)}{2}\left(\frac{I'}{\sigma\mu}\right)^2(w)=R'(w)-I'(w).$$

Next, we differentiate (29) with respect to s. We obtain

$$w''(s)=\left(\frac{I'}{\sigma\mu}\right)'(w(s))w'(s)=\left(\frac{I'}{\sigma\mu}\right)'(w(s))\left(\frac{I'}{\sigma\mu}\right)(w(s)).$$

Combining the two equations leads to

$$K'(w(s))w''(s)+\frac{K(w)}{2}(w'(s))^2+\sigma\mu(w)w'(s)=R'(w)=-\sigma^2(w-w^+)-p(w)+p(w^+).$$

The proof is complete.

2.3. Numerical methods

Independently of the specific context considered so far, we show next that dispersion terms naturally occur in the discretization of Eq. (16). A simple first-order scheme for (16) is given by

$$\frac{V_{i+1}-V_i}{h}=\nabla R(V_i),\quad i\in\mathbb{Z}. \tag{30}$$

It is well known that its implementation as a shooting method for the search of heteroclinic solutions presents substantial difficulties[9] [5]. Here we suggest a slightly different method. The starting point is the observation that,

[9] Except in the scalar case!

whereas (30) is a first-order approximation of (16), it is a second-order approximation of

$$\frac{h}{2}V'' + V' = \nabla R(V),$$

which is of the form (15) for $\tau = \frac{h}{2}$. For fixed (but small) mesh size h, our analysis shows that the trajectories arise directly through the minimization of a strictly convex functional (see Theorem 1 and comments thereafter), for which robust numerical methods are available.

3. Proof of Theorem 1

In order to present the proof of Theorem 1 and to construct I, we first go back to a Lagrangian formalism.

3.1. Lagrangian formalism

To give a variational flavor to Eq. (15)

$$\tau V''(s) + V'(s) = \nabla R(V(s))$$

we perform the change of variable

$$s \mapsto \exp(\gamma s) = t, \quad \text{where } \gamma = \frac{1}{\tau} > 0.$$

The function V is a solution to (17) if and only the function W defined by $W(t) = V(s)$ satisfies

$$(t^2 W'(t))' = \tau \nabla R(W(t)) \text{ on }]0, +\infty[. \tag{31}$$

In view of the factor t^2 on the left-hand side of (31), there is no direct variational formulation of heteroclinic solutions of (31). Instead, we look at trajectories on $]0, \alpha[$, for arbitrary $\alpha > 0$, and whose value v at the point $t = \alpha$ is prescribed. For that purpose let $\mathfrak{v} \in \mathbb{R}^N$ and $\alpha > 0$ be given. We define the functional $\mathcal{C}_{\mathfrak{v},\alpha}$ on the space

$$X_\alpha = \left\{ w \in H^1_{\text{loc}}(]0, \alpha], \mathbb{R}^N) \text{ s.t. } \int_0^\alpha t^2 |w'(t)|^2 \,\mathrm{d}t < +\infty,\ w(\alpha) = 0 \right\}$$

by

$$\mathcal{C}_{\mathfrak{v},\alpha}(w) = \int_0^\alpha \frac{t^2}{2\tau} |w'(t)|^2 + R(\mathfrak{v} + w(t)) \,\mathrm{d}t.$$

The following elementary lemma justifies the definition of $\mathcal{C}_{\mathfrak{v},\alpha}$.

Lemma 1. *Assume R satisfies (H_2). Then,*

(1) *The functional $\mathcal{C}_{\mathfrak{v},\alpha}$ is well defined on X_α, with values in $\mathbb{R} \cup \{+\infty\}$.*
(2) *$\mathcal{C}_{\mathfrak{v},\alpha}$ is Fréchet differentiable on $X_\alpha \cap L^\infty(0, \alpha)$.*
(3) *The critical points of $\mathcal{C}_{\mathfrak{v},\alpha}$ on $X_\alpha \cap L^\infty(0, \alpha)$ are solutions of (31) on $]0, \alpha[$.*

Proof. The only point which requires some care is (1). First, by Hardy's inequality, we have the embedding

$$X_\alpha \to L^2(0,\alpha) \tag{32}$$

(see Theorem 2). Second, by assumption (H_2)

$$R(w) \geq -C(\|w\|^2+1) \quad \forall\, w \in \mathbb{R}^N \tag{33}$$

for some constant $C>0$. Let $R^-(w)=\min(0,R(w))$ be the non-positive part of R. Combining (32) and (33) we deduce that the negative term $\int_0^\alpha R^-(w(t))\,\mathrm{d}t$ is bounded and the conclusion (1) follows. Deriving (2) and (3) is then fairly standard and we omit it. □

Remark 2. (1) If a traveling wave solution of (9) exists and satisfies the conditions at infinity (10), the corresponding function $W-W(\alpha)$ belongs to X_α. Indeed, since $V'(s)=\gamma t W'(t)$ and since by (10), V' tends to zero at $-\infty$, we have $tW'(t)\to 0$ as $t\to 0$.

(2) In terms of the function v, the functional $\mathcal{C}_{\mathfrak{v},\alpha}$ becomes

$$\mathcal{C}_{\mathfrak{v},\alpha}(w)=\int_{-\infty}^{\tau\log\alpha}\exp(\gamma s)\left(\frac{|v'(s)|^2}{2}+\gamma R(v(s)+\mathfrak{v})\right)\mathrm{d}s$$

for $v(s)=w(t)$, and therefore $v(\tau\log\alpha)=w(\alpha)=0$.

(3) Since Eq. (17) is autonomous, it is invariant by translation, and this fact reflects on the following property of the function $\mathcal{C}_{\mathfrak{v},\alpha}$:

$$\mathcal{C}_{\mathfrak{v},\alpha_1}(w(\cdot))=\frac{\alpha_1}{\alpha_2}\mathcal{C}_{\mathfrak{v},\alpha_2}\left(w\left(\frac{\alpha_2}{\alpha_1}\cdot\right)\right) \quad \forall w\in X_1. \tag{34}$$

Therefore, in the sequel we will mainly consider the case $\alpha=1$ and omit the α-subscript setting

$$\mathcal{C}_{\mathfrak{v}}=\mathcal{C}_{\mathfrak{v},1} \quad \text{and} \quad X=X_1.$$

3.2. Convexity of $\mathcal{C}_{\mathfrak{v}}$

In this section, we will show the following proposition.

Proposition 3. *Assume R satisfies (H_2). There exists $\tau_0>0$ such that if*

$$0<\tau<\tau_0 \tag{35}$$

then $\mathcal{C}_{\mathfrak{v}}$ is strictly convex and coercive on X. Therefore, there exists a unique minimizer $m(\mathfrak{v})$ of $\mathcal{C}_{\mathfrak{v}}$ on X. Moreover, the function $\mathfrak{v}\mapsto m(\mathfrak{v})$ is of class $\mathcal{C}^2$.

The main tool in the proof of Proposition 3 is the following inequality due to Hardy (see [16], see also [12,31,32]).

Theorem 2 (Hardy). *Let $w\in X$ (so that $w(1)=0$). We have*

$$\int_0^1|w(t)|^2\,\mathrm{d}t\leq 4\int_0^1 t^2|w'(t)|^2\,\mathrm{d}t=4\|w\|_X^2.$$

Proof of Proposition 3. We show that $\mathcal{C}_{\mathfrak{v}}$ is α-convex, that is, that for some $\alpha>0$,

$$\mathcal{C}_{\mathfrak{v}}(\theta w_1+(1-\theta)w_2)\leq\theta\mathcal{C}_{\mathfrak{v}}(w_1)+(1-\theta)\mathcal{C}_{\mathfrak{v}}(w_2)-\alpha\theta(1-\theta)\|w_1-w_2\|_X^2. \tag{36}$$

To establish the previous inequality, we distinguish two cases.

Case 1. One of the quantities $\mathcal{C}_{\mathfrak{v}}(w_1)$ or $\mathcal{C}_{\mathfrak{v}}(w_2)$ is infinite. Then, there is nothing to prove.

Case 2. If both of those quantities are bounded, we write

$$\mathcal{C}_{\mathfrak{v}}(\theta w_1 + (1-\theta)w_2) - \theta\mathcal{C}_{\mathfrak{v}}(w_1) - (1-\theta)\mathcal{C}_{\mathfrak{v}}(w_2) = -\frac{\theta(1-\theta)}{\tau}\|w_1 - w_2\|_X^2 + \int_0^1 [R(\theta w_1 + (1-\theta)w_2) - \theta R(w_1) - (1-\theta)R(w_2)]\,\mathrm{d}t. \tag{37}$$

On the other hand, by assumption (H_2), $D^2(R) \geq -c\,\mathrm{Id}$, so that

$$R(\theta w_1 + (1-\theta)w_2) - \theta R(w_1) - (1-\theta)R(w_2) \leq 2c\theta(1-\theta)|w_1 - w_2|^2.$$

Therefore, by Hardy's inequality we obtain

$$\mathcal{C}_{\mathfrak{v}}(\theta w_1 + (1-\theta)w_2) - \theta\mathcal{C}_{\mathfrak{v}}(w_1) - (1-\theta)\mathcal{C}_{\mathfrak{v}}(w_2) \leq -\theta(1-\theta)\frac{1-8\tau c}{\tau}\|w_1 - w_2\|_X^2.$$

It suffices then to choose τ such that $\tau < \tau_0 = \frac{1}{8c}$. □

We are now in position to define the cost functional I by

$$I(\mathfrak{v}) = \mathcal{C}_{\mathfrak{v}}(m(\mathfrak{v})). \tag{38}$$

In view of Proposition 1, our next purpose is to derive regularity properties for I. Although it may not necessary, the boundedness of the trajectory $m(\mathfrak{v})$ will be helpful to obtain such results. In that direction however, assumption (H_2) is not sufficient to guaranty the boundedness as the following example shows. Consider the equation

$$\tau V'' + V' = \nabla\left(-\frac{|V|^2}{2}\right) = -V \tag{39}$$

whose solutions are generated by $\exp(r_\pm s)$, with

$$r_\pm = \frac{-1 \pm \sqrt{1-4\tau}}{2\tau} < 0.$$

Since the trajectories of solutions of (38) (with $R(V) = -\frac{|V|^2}{2}$) are given by solutions of (39) for $s < 0$, it follows that the solutions $m(\mathfrak{v})$ of (38) are not bounded.[10]

Lemma 2. *Assume R satisfies (H_2) and (H_3), and τ satisfies (35). Then for any $\mathfrak{v} \in \mathbb{R}^N$, $m(\mathfrak{v})$ is a bounded function.*

Proof. Let K_n be such that $\mathfrak{v} \in K_n$ and define

$$u = P_n(m(\mathfrak{v}) + \mathfrak{v}) - \mathfrak{v}.$$

By convexity of K_n, we have

$$|u'(t)| \leq m'(\mathfrak{v}(t)) \quad \forall t \in]0,1[. \tag{40}$$

On the other hand, $P_n(\mathfrak{v}) = \mathfrak{v}$, so that $u(1) = 0$, and hence $u \in X$. By (21) and (40), we derive

$$C_{\mathfrak{v}}(u) \leq C_{\mathfrak{v}}(m(\mathfrak{v})).$$

So that by uniqueness $m(\mathfrak{v}) = u$ is bounded. □

[10] In any case, such unbounded solutions are not relevant in the quest for heteroclinic solutions.

3.3. End of the proof of Theorem 1

The proof is completed in five steps.

Step 1. *The function I given by* (38) *is* C^∞. We rely on the implicit function theorem, in order to avoid explicit computations. It is technically convenient to replace the original functional $C_{\mathfrak{v}}$ by one where the potential R is quadratic at infinity. Since we know that the minimizer $m(\mathfrak{v})$ is bounded, for $\mathfrak{v}$ in a fixed compact set, this does not affect the result. In this case the function $(\mathfrak{v}, w) \to C_{\mathfrak{v}}(w)$ is C^∞ from $\mathbb{R}^N \times X$ into $\mathbb{R}$, and therefore $g : (\mathfrak{v}, w) \to \nabla_w C_{\mathfrak{v}}(w)$ is C^∞ from $\mathbb{R}^N \times X$ into the Hilbert space X. By strict convexity the minimizer $m(\mathfrak{v})$ is the unique solution to the equation

$$g(\mathfrak{v}, m(\mathfrak{v})) = 0.$$

Moreover, invoking α-convexity $\partial_w g(\mathfrak{v}_0, m(\mathfrak{v}_0))$ is invertible for every $\mathfrak{v}_0 \in \mathbb{R}^N$. Applying the implicit function theorem, we deduce that locally the function $\mathfrak{v} \to m(\mathfrak{v})$ is well defined and C^∞. Going back to the identity $I(\mathfrak{v}) = C_{\mathfrak{v}}(m(\mathfrak{v}))$, we deduce that I is C^∞.

Step 2. We claim that

$$\nabla I(\mathfrak{v}) = \frac{m(\mathfrak{v})'(1)}{\tau}. \tag{41}$$

We first note that in view of Proposition 3, one has the equality

$$\langle \nabla_w C_{\mathfrak{v}}(m(\mathfrak{v})), \varphi \rangle = 0 \qquad \forall \varphi \in X, \tag{42}$$

which writes

$$\int_0^1 \left[\frac{t^2}{\tau} m(\mathfrak{v})'(t)\varphi'(t) + \nabla R(\mathfrak{v} + m(\mathfrak{v})(t))\varphi(t) \right] \mathrm{d}t = 0 \qquad \forall \varphi \in X. \tag{43}$$

Thus, in view of (42)

$$\nabla I(\mathfrak{v}) = \nabla(C_{\mathfrak{v}}(m(\mathfrak{v}))) = (\partial_{\mathfrak{v}} C_{\mathfrak{v}})(m(\mathfrak{v})) = \int_0^1 \nabla R(\mathfrak{v} + m(\mathfrak{v})(t))\, \mathrm{d}t = \lim_{\varepsilon \to 0} \int_0^1 \nabla R(\mathfrak{v} + m(\mathfrak{v})(t))\varphi_\varepsilon(t)\, \mathrm{d}t, \tag{44}$$

where the function φ_ε is defined by $\varphi_\varepsilon(t) \equiv 1$ for $0 \leq t \leq 1 - \varepsilon$ and $\varphi_\varepsilon(t) = \frac{1-t}{\varepsilon}$ for $1 - \varepsilon \leq t \leq 1$, so that φ_ε is continuous, piecewise linear and belongs to X. In particular $\varphi_\varepsilon(1) = 0$. Going back to (43) and passing to the limit $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon \to 0} \int_0^1 \nabla R(\mathfrak{v} + m(\mathfrak{v})(t))\varphi_\varepsilon(t)\, \mathrm{d}t = -\lim_{\varepsilon \to 0} \int_0^1 \frac{t^2}{\tau} m(\mathfrak{v})'(t)\varphi_\varepsilon'(t)\, \mathrm{d}t = \frac{m(\mathfrak{v})'(1)}{\tau}.$$

This establishes (41).

Step 3. *I satisfies* Eq. (13). The computations are quite classical [11], we specify them in our problem. First for $0 < \alpha < 1$, consider the integral of the action along the trajectory $m(\mathfrak{v})$

$$A(\alpha) = \int_0^\alpha \left(\frac{t^2}{2\tau} |m(\mathfrak{v})'(t)|^2 + R(\mathfrak{v} + m(\mathfrak{v}(t))) \right) \mathrm{d}t,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}A(\alpha)=\frac{\alpha^2}{2\tau}|m(\mathfrak{v})'(\alpha)|^2+R(\mathfrak{v}+m(\mathfrak{v}(\alpha))).$$

On the other hand one has the identity

$$A(\alpha)=C_{\mathfrak{v}+m(\mathfrak{v})(\alpha),\alpha}(z_\alpha(\mathfrak{v})),$$

where $z_\alpha\in X_\alpha$ is defined by $z_\alpha(\mathfrak{v})=m(\mathfrak{v})-m(\mathfrak{v})(\alpha)$. Invoking relation (34), for $\alpha_1=\alpha$ and $\alpha_2=1$, we obtain

$$A(\alpha)=\alpha C_{\mathfrak{v}+m(\mathfrak{v})(\alpha),A}\left(z_\alpha(\mathfrak{v})\left(\frac{\cdot}{\alpha}\right)\right).$$

Since $z_\alpha(\mathfrak{v})$ is the minimizer for the problem $\min_{w\in X_\alpha}C_{\mathfrak{v}+m(\mathfrak{v})(\alpha),\alpha}(w)$, it follows that $z_\alpha(\mathfrak{v})\left(\frac{\cdot}{\alpha}\right)$ is the minimzer for the problem $\min_{w\in X}C_{\mathfrak{v}+m(\mathfrak{v})(\alpha),1}(w)$: hence

$$z_\alpha(\mathfrak{v})\left(\frac{\cdot}{\alpha}\right)=m(\mathfrak{v}+m(\mathfrak{v})(\alpha))(\cdot)$$

and

$$A(\alpha)=\alpha I(\mathfrak{v}+m(\mathfrak{v})(\alpha)).$$

We use this expression to provide a different computation for $\frac{\mathrm{d}}{\mathrm{d}\alpha}A(\alpha)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}A(\alpha)=I(\mathfrak{v}+m(\mathfrak{v})(\alpha))+\alpha\langle\nabla I(\mathfrak{v}+m(\mathfrak{v})(\alpha)),m(\mathfrak{v})'(\alpha)\rangle.$$

Combining the two expressions for $\frac{\mathrm{d}}{\mathrm{d}\alpha}A(\alpha)$, for $\alpha=1$, we are led to

$$\frac{1}{2\tau}|m(\mathfrak{v})'(1)|^2+R(\mathfrak{v})=I(\mathfrak{v})+\langle\nabla I(\mathfrak{v}),m(\mathfrak{v})'(1)\rangle.$$

Finally since $m(\mathfrak{v})'(1)$ is given by (41), we obtain (13).

Step 4. *I satisfies the condition* (19). It suffices to show that if $\mathfrak{v}$ is a critical point for R then $m(\mathfrak{v})=0$. If $\nabla R(\mathfrak{v})=0$ then the constant function $w\equiv 0$ is obviously solution to (43), that is a critical point for the functional $C_\mathfrak{v}(w)$. By convexity the critical point is unique, so that

$$I(\mathfrak{v})=C_\mathfrak{v}(0)=R(\mathfrak{v}).$$

Step 5. *Classical traveling waves correspond to gradient flow for I*. This is somehow the converse to Proposition 1, which states that gradient flow solutions satisfy the traveling wave equation. Consider a solution V to Eq. (15) i.e.

$$\tau V''(s)+V'(s)=\nabla R(V(s)) \tag{45}$$

supplemented with the condition at infinity (17):

$$V(\pm\infty)=V^\pm,\qquad V'(\pm\infty)=0.$$

Our aim is to establish that

$$V'(s) = \nabla I(V(s)) \quad \forall s \in \mathbb{R}$$

and actually, by translation invariance, it suffices to show that

$$V'(0) = \nabla I(V(0)). \tag{46}$$

For that purpose, we introduce the function u defined by $u(t) = V(\tau \log t) - V(0)$ for $0 < t \leq 1$. We are going to show that

$$u = m(V(0)). \tag{47}$$

Let us show first that (47) implies (46). Assuming (47) we obtain by (41)

$$\frac{u'(1)}{\tau} = \frac{m(\mathfrak{v})'(1)}{\tau} = \nabla I(V(0)).$$

On the other hand, by definition $u'(1) = \tau V'(0)$, which yields (46).

It remains to prove (47). Since obviously $u \in X$, it suffices to verify the Euler–Lagrange equation (42), i.e.

$$\langle \nabla_w C_{\mathfrak{v}}(m(\mathfrak{v})), \varphi \rangle = 0 \quad \forall \varphi \in X. \tag{48}$$

Since $C_c^\infty(]0, 1])$ is dense in X, it suffices to check (48) for $\varphi \in C_c^\infty(]0, 1])$ (in particular such a φ vanishes on a full neighborhood of zero). This follows directly multiplying the equation for u

$$\left(-\frac{t^2 u'}{\tau}\right)' + \nabla R(V(0) + u(t)) = 0 \text{ on }]0, 1]$$

by φ and integrating by parts.

The proof of Theorem 1 is completed.

4. Additional properties of I

In this section we briefly present additional properties of I. These properties are fairly natural consequences of the fact that I is the cost functional of a variational problem.

(a) We assume all the hypotheses of Theorem 1. Then

$$\|\nabla I\|_{L^\infty(K_n)} \leq \|\nabla R\|_{L^\infty(K_n)}. \tag{49}$$

This inequality is a consequence of the formula (44), i.e. $\nabla I(\mathfrak{v}) = \int_0^1 \nabla R(\mathfrak{v} + m(\mathfrak{v})(t))\, \mathrm{d}t$ and of the boundedness of the trajectory: indeed $m(\mathfrak{v})(t) \in K_n$ for $\mathfrak{v} \in K_n$, see Lemma 2.

(b) We assume all the hypotheses of Theorem 1. Consider two values of the parameter τ, covered by our restriction in Theorem 1, namely $\tau_0 > \tau_1 > \tau_2 > 0$. Let I_1 be the cost function for the parameter τ_1, that is $I_1(\mathfrak{v}) = \min_{w \in X} \int_0^1 \frac{t^2}{2\tau_1} |w'(t)|^2 + R(\mathfrak{v} + w(t))$. Let I_2 be the cost functional for the parameter τ_2, that is $I_2(\mathfrak{v}) = \min_{w \in X} \int_0^1 \frac{t^2}{2\tau_2} |w'(t)|^2 + R(\mathfrak{v} + w(t))$. Then

$$I_1(\mathfrak{v}) \leq I_2(\mathfrak{v}) \quad \forall \mathfrak{v} \in \mathbb{R}^N. \tag{50}$$

(c) Theorem 1 has an extension for negative values of the parameter $-\tau_0 < \tau < 0$. For this task one considers the maximization problem

$$I(\mathfrak{v}) = \max_{w\in X} \int_0^1 \frac{t^2}{2\tau}|w'(t)|^2 + R(\mathfrak{v} + w(t)), \quad \tau < 0, \tag{51}$$

instead of the minimization problem of Proposition 3. This problem is equivalent to $\min_{w\in X} \int_0^1 -\frac{t^2}{2\tau}|w'(t)|^2 - R(\mathfrak{v} + w(t))$. Thus assuming that the hypotheses (H_2) and (H_3) of Theorem 1 are satisfied for $-R$, instead of R, then Theorem 1 holds for $-\tau_0 < \tau < 0$.

5. Additional references

In this section we single out additional references that could serve to get another interpretation of the cost functional I.

(a) We notice that the Hamilton–Jacobi equation

$$\frac{\tau}{2}|\nabla I(v)|^2 = R(v) - I(v) \tag{52}$$

is very similar to the one encountered in the Shape-from-Shading approach, see [35,27,1]. An open issue is to relate I to the viscosity solution defined in [35,27,1], in particular in the case of larger values of the parameter τ.

(b) Note that $I(v)$ can be related to the production of entropy along the trajectory $I(V(s)) = V^- + \int_{-\infty} |V'(s)|^2\,\mathrm{d}s$, see e.g. (20). Hence, it could be possible to get another interpretation of I as an application of a principle of minimum production of entropy. In that direction we quote [9,17,18].

6. Numerical example

In this section, we take advantage of the theoretical framework previously introduced to study numerically the model phase transition problem already introduced in Section 2.2. We will observe the behavior of the different objects as the parameter $\tau = \frac{\varepsilon}{\mu^2}$ gets larger.[11] Physically, one expects that non-classical shocks occur in this regime.

The van der Waals pressure law

$$p(w) = \frac{c}{w-a} - \frac{b}{w^2}, \quad w = \frac{1}{\rho} > a \tag{53}$$

is non-convex and is often used to model phase transitions. We consider here the case $a = 1$, $b = \frac{125}{36}$, $c = 1$ and plot the graph of p for $1.8 \le w \le 7$ (Fig. 1).

With this choice of parameters, the model problem

$$\partial_t w - \partial_x v = 0, \qquad \partial_t v + \partial_x p(w) = 0$$

is non-hyperbolic in the spinodal region defined by $p'(w) > 0$, i.e. $\frac{5}{2} < w < \frac{5}{9}(4+\sqrt{7})$. For $w < \frac{5}{2}$, the system is hyperbolic and the fluid is the liquid phase, while for $w > \frac{5}{9}(4+\sqrt{7})$ the system is also hyperbolic but the fluid is in the gas phase.

[11] Larger than our restriction in Theorem 1.

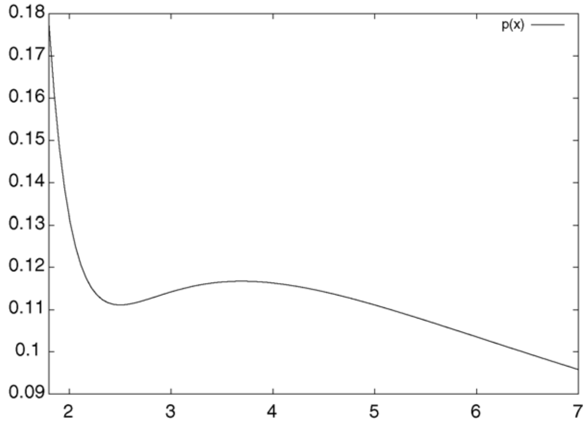

Fig. 1. van der Waals non-convex pressure law $p(w) = \frac{1}{w-1} - \frac{125}{36w^2}$.

It is known or at least expected that as τ becomes larger, non-classical shocks may occur between the liquid and gas phases. Our point of view is to fix a priori and arbitrarily $w^- = 2.1$ and $w^+ = 6$ and to let τ vary until we find numerically a smooth solution I of the Hamilton–Jacobi equation (28) such that w^- and w^+ are the only two critical points of I.

First, the Rankine–Hugoniot condition yields the value of the shock velocity with respect to the given states w^- and w^+,

$$\sigma = -\sqrt{-\frac{p(w^+) - p(w^-)}{w^+ - w^-}}.$$

We next construct the potential R. Taking the entropy $S = \frac{w^2}{2}$, we have $R(w) = -\sigma^2 R_0(w) + R_1(w)$ where $R_0(w) = \frac{(w-w^+)^2}{2}$, $R_1(w) = -q(w) + wp(w^+) + q(w^+) - w^+ p(w^+)$, and by definition

$$q(w) = \int_0^w p(z)\,\mathrm{d}z = \log(w-1) + \frac{125}{36w}.$$

The potential $R(v)$ is plotted in Fig. 2. The Rayleigh line joining $(w^-, p(w^-))$ and $(w^+, p(w^+))$ intersects the graph of p at an intermediate point w_s in the spinodal region. Since $R'(w_s) = 0$, no solution of $w' = R(w)$ may connect w^- and w^+, and we therefore add dispersion and look for solutions of the Hamilton–Jacobi equation:

$$\frac{\tau}{2}|I'(w)|^2 = R(w) - I(w), \quad I(w^\pm) = R(w^\pm). \tag{54}$$

Since we have two boundary conditions, it is not appropriate to study (54) by a shooting method. Instead our numerical solution of (54) is obtained by computing the stationary solution $I(\infty, w)$ of

$$\frac{\partial}{\partial\lambda} I(\lambda, w) = R(w) - I(w) - \frac{\tau}{2}|I'(w)|^2, \quad I(0, w) = R(w). \tag{55}$$

This is a non-stationary Hamilton–Jacobi equation for which many simple schemes are available in the literature. We avoid the analysis of these schemes and of their convergence, but observe convergence empirically to a stationary solution. This stationary discrete solution is a solution of the discretized version of (55). We next compute the

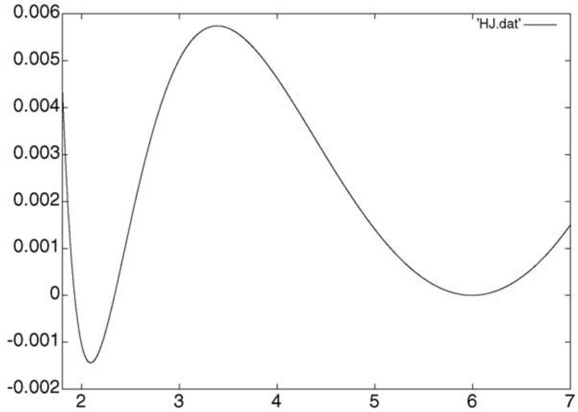

Fig. 2. The potential $R(v)$.

numerical stationary solution I_1 (resp. I_2, I_3, I_4, I_5) of (55) for the particular choices $\tau_1 = 1$ (resp. $\tau_2 = 10$, $\tau_3 = 100$, $\tau_4 = 4000$, $\tau_5 = 10\,000$). We have taken 600 cells in the interval $[w^-, w^+]$.

For small dispersion ($\tau_1 \approx 1$), one gets typically Fig. 3. For $w^- \leq w \leq w_s$ one sees that $I_1(w) \leq R(w)$. I and R are almost indistinguishable for $w \geq w_s$. The computed graph suggests that the corresponding solution is smooth. Nevertheless, it does not allow for a non-classical shock.

Next we increase the dispersion to $\tau_2 = 10$. This case will happen not to be covered by Theorem 1. The discrete stationary solution of (55) is the one in Fig. 4. This function is continuous but is not C^2, and we are not in position to conclude about the existence of traveling waves.

Increasing even more the dispersion we take now $\tau_3 = 1000$. The numerical result is shown in Fig. 5, yielding the same conclusion. Notice that the point of discontinuity has moved to the right.

For $\tau_4 = 40\,000$, see Fig. 6. This is the critical value for which the point of discontinuity observed in the previous figures coincides with w^+, and its gradient flow $w'(s) = I_4'(w(s))$ connects w^- and w^+. Thus it gives numerical evidence that a shock is possible from w^+ (the gas region) to w^- (the liquid region) for an appropriate large dispersion. This is in accordance with the result of [37].

For even larger dispersion $\tau > \tau_4$ we have always observed that the boundary value $I(w^+) = R(w^+)$ is not met at the numerical level (see Fig. 7).

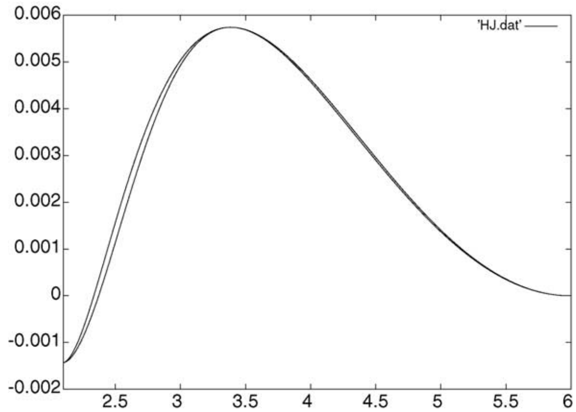

Fig. 3. $R(w)$ and $I_1(w)$ for $\tau_1 = 1$, computed with 600 cells.

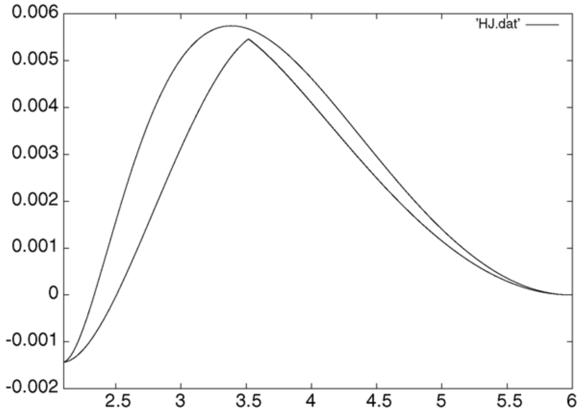

Fig. 4. $R(w)$ and $I_2(w)$ for $\tau_2 = 10$, computed with 600 cells.

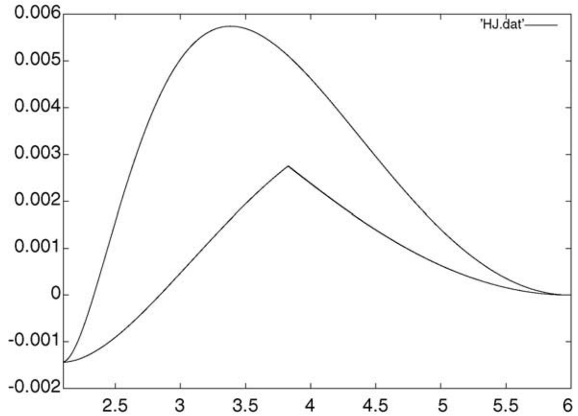

Fig. 5. $R(w)$ and $I_3(w)$ for $\tau_3 = 1000$, computed with 600 cells.

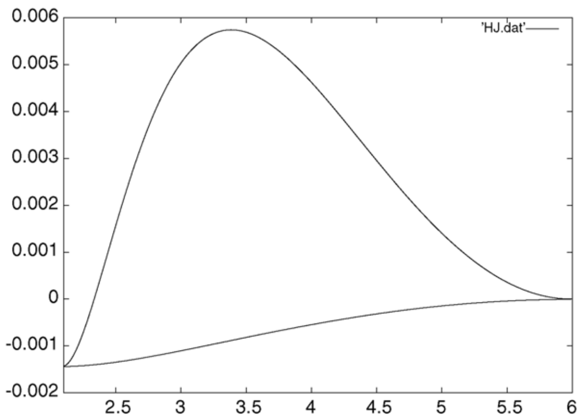

Fig. 6. $R(w)$ and $I_4(w)$ for $\tau_4 = 40\,000$, computed with 600 cells.

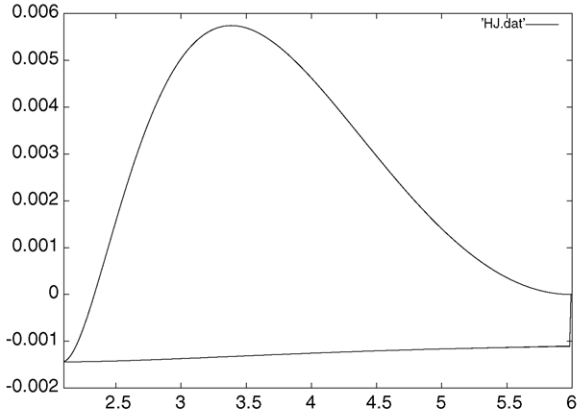

Fig. 7. $R(w)$ and $I_5(w)$ for $\tau_5 = 100\,000$, computed with 600 cells.

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