Derivatives of Systems in Divergence form in the Presence of Shocks

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September 9, 2002

Abstract

In this paper we present a synthetic method to differentiate with respect to a parameter partial differential equations in divergence form with shocks. We show that the usual derivatives contain the differentiated interface conditions if interpreted by the theory of distributions. We apply the method to 3 problems: the Burger equation, Darcy's law and the shallow water equations.

Calcul de la dérivée d'un système sous forme divergencielle avec chocs

Résumé*On présente une méthode synthétique pour calculer les équations aux dérivées partielles que vérifient la dérivée par rapport à un paramètre de la solution d'un système sous forme* $\nabla \cdot v = 0$*. On montre, pour les équations de Burger, Darcy et Saint-Venant que la dérivée au sens usuel, mais interpretée au sens des distributions, contient les conditions de saut, c'est à dire les dérivées des conditions de transmission aux chocs. On retrouve ainsi les résultats de Godlewski-Raviart et al.*

Rapport de recherche R002??

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1 Introduction

Sensitivity and stability analyses lead to the differentiation of partial differential equations. For instance, if the solution of Burger's equation, u, depends on a parameter a, then, one would like to write that

$$\partial_t u + \partial_x \frac{u^2}{2} = 0 \quad \text{yields } \partial_t u' + \partial_x (uu') = 0 \tag{1}$$

where u' denotes the derivative of u with respect to a. We will show that it is correct, but in the sense of distribution theory. Indeed in the usual sense there are serious problems when there are shocks which depend upon a. This is because u has a jump at the shock, so its derivative u' is likely to have a Dirac mass at the shock and so uu' is void of sense when u is discontinuous.

This problem has been identified in a pionnering paper by Alazabal, Godlewski and Raviart[8] for (1) and also Hafez (see [4]) who pointed out that usual optimality conditions for control problems with discontinuous states would be incomplete since no variation of shock positions are allowed by standard calculus of variations; Giles[5] proved later that it is not so if the criteria of the optimization problem does not involve the shock position explicitly. From the numerical point of view, it was also found that computer program implementing (1) compute Dirac masses indeed [14][4].

For Burger's equation and to some extend for the Euler equations of fluids the problem has been analyzed completely by Godlewski et al[10][9][14] and for Darcy's law by Bernardi et al [3]. So what we propose here is only a new interpretation of the results; however this is done with a general tool to compute the divergence of distributions which have singularities tangent to the lines of singularities (Corollary 1).

The plan of the paper is as follows:

First we recall that a "simple" differentiation of the partial differential equation and of its interface conditions lead to the result. However such results would be difficult to justify mathematically; also they do not appear in a synthetic and compact form.

In the next session we show that by using the definition of derivatives in the sense of distributions, we can differentiate a function v which has a line/surface discontinuity of normal n and we can also compute the divergence of the result when $v \cdot n = 0$.

Finally in the third part we apply the result to 3 problems: Burger's equation, Darcy's law and the shallow water equations. We recover the results of [8] and [3] and show among other thing that the differentiated equations contain the jump conditions necessary for the well posedness of the differentiated problem.

2 Standard Analysis

2.1 Standard Sensitivity Analysis for Burger's equation

Burger's equation for u is

$$\partial_t u + \partial_x(\frac{u^2}{2}) = 0 \ \ \forall\{x,t\} \in Q := R \times (0,+\infty),$$
$$u(x,0) = u^0(x,a) \ \ \forall x \in R \tag{2}$$

It may have discontinuous solutions (shocks) depending on the initial data u^0. With compatible (entropy) discontinuous initial data at, say $x = 0$, the solution has a discontinuity at $x = x(t,a)$, which propagates at velocity

$$\dot{x} := \partial_t x = \bar{u} := \frac{1}{2}(u^+(x(t,a),t,a) + u^-(x(t,a),t,a)). \tag{3}$$

Here we assume that the initial data is function of a parameter a and we wish to find an equation for u'_a, the derivative of u with respect to a. We also assume that the shock is a unique curve $x(t,a)$ defined at $t = 0$ by the discontinuity of u^0 at 0.

Proposition 1 *Burger's equation differentiated with respect to a parameter, a, in the initial data is*

$$\partial_t u'_a + \partial_x(uu'_a) = 0 \quad \textit{in } \Omega\backslash S$$
$$\dot{x}' = \bar{u}'_a + x'\partial_x\bar{u} \quad \textit{on } S \tag{4}$$

where x' is the derivative with respect to a of the shock position $x(t)$, $\bar{u}' = (u'^+ + u'^-)/2$ and $\partial_x\bar{u}$ is the half sum of the left x of the right x derivative of u on the shock curve.

Proof:
Equation (4-a) is straightforward. To obtain (4-b) we must differentiate (3) with respect to a. All primes meaning derivatives with respect to a, (3) comes from the fact that $\partial_a f(x(t,a),t,a) = f'_a + x'\partial_x f$.
The fact that the original solution $u(x,a)$ satisfies the entropy condition has for consequence that u'_a and (in most cases) x' are solutions of a well posed problem. This is the object of the proposition (see Tai-Ping-Liu for instance [12]):

Proposition 2 *Let $u(x,t,a)$ be the entropic solution of the Burger equation with initial data $u_0(x,a)$. Assume that it has a "simple" structure: there is a closed set S which is the union of a finite number of shocks curves and away from this set u*

is continuously differentiable in x and t. Then the solution of (4-a) is uniquely and continuously determined in term of its initial value

$$u'_a(x,a,0) = v(x) := \frac{\partial u^0}{\partial a}(x,a) \tag{5}$$

If in addition S is the union of a finite set of distincts (no intersection) shock curves $x_i(t,a)$ with origin at $t = 0$: $x_i(0,a)$; then the derivatives with respect to a, $\partial x_i(t,a)/\partial a$ depends continuously on $v(x)$ and $\partial x_i(0,a)/\partial a$ according to the formula:

$$x'_i(t) = e^{\int_0^t \partial_x \bar{u}(x_i(s),s)ds} x'_i(0) + \int_0^t e^{\int_s^t \partial_x \bar{u}(x_i(\tau),\tau)d\tau} \bar{u}'_a(x_i(s),s)ds \tag{6}$$

Proof

For any $(x,t) \notin S$ the solution of the ode

$$\frac{dx(s)}{ds} = u(x(s),s)\,, \quad x(t) = x \tag{7}$$

is well defined as long as it does not intersect S. Since u satisfies also the entropy condition there are no characteristic coming from the shocks curves therefore $x(s)$ does intersect S for $0 \leq s \leq t$. Along this curve, equation (4-a) implies the relation:

$$\frac{du'_a}{ds}(x(s),s) + \partial_x u(x(s),s)(u'_a(x(s),s) = 0 \tag{8}$$

or after integration from 0 to t:

$$u'_a(x,t) = u'_a(x(0),0)e^{-\int_0^t \partial_x u(x(s),s)ds} \tag{9}$$

which established the first part of the proposition.

Consider a shock curve $x_i(t,a)$. Having determined $\bar{u}'_a(x_i(t),t,a)$ one obtains $x_i(t,a)$, from the relation

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial a}(t,a)\right) = \bar{u}'_a(x_i(t),t,a) + \frac{\partial x_i}{\partial a}(t,a)\partial_x \bar{u}(x_i(t),t,a) \tag{10}$$

Then one uses the hypothesis that the shock curves are distincts and that they starts from $x_i(0,a)$ at $t = 0$ i.e. from points where the initial data $u(x,0)$ is discontinuous.

With an integration of (10) from 0 to t one obtains (6).

2.2 Standard Sensitivity Analysis for Darcy's Law

For hydrostatic flows through porous media,

$$v = \kappa\nabla\phi \quad \text{and} \quad \nabla\cdot v = 0 \quad \text{in } \Omega\subset R^d \tag{11}$$

As κ is discontinuous from one geological layer to the next let S be such an interface. When κ is in $L^\infty(\Omega)$ ϕ then is in $H^1(\Omega)$ and continuous in Ω when κ is piecewise C^1, even if it is discontinuous across a surface S.

Proposition 3 *Let ϕ be the unique solution of (11) and assume that κ has a discontinuity, $S(a)$, function of a parameter a then the derivative of ϕ, ϕ' with respect to a, is discontinuous across $S(a) = \{x(s)\ :\ s\in I_S\subset R^{d-1}\}$ and satisifies*

$$\begin{aligned}
&\nabla\cdot(\kappa\nabla\phi') = -\nabla\cdot(\kappa'\nabla\phi) \ \textit{ in } \ \Omega\backslash S(a)\\
&[\phi'] = -x'\cdot[\nabla\phi] \ \textit{ on } S\\
&[\kappa\frac{\partial\phi'}{\partial n}] = -x'\cdot\nabla[\kappa\frac{\partial\phi}{\partial n}] - [\frac{\kappa'}{\kappa}](\kappa\frac{\partial\phi}{\partial n}) - [\kappa\nabla\phi]\cdot n' \ \textit{ on } S
\end{aligned} \tag{12}$$

where all primes indicate a derivative with respect to a and n is the normal to S oriented from minus to plus while the jump bracket is $[\phi'] = \phi'^{+} - \phi'^{-}$.

Proof
From (11), (12-a) is obvious. Equation (12-b) comes from the differentiation in a of the condition $[\phi]_S = 0$. Equation (12-c) is obtained by differentiating the jump condition $[\kappa\partial_n\phi]$=0:

$$[\kappa\frac{\partial\phi}{\partial n}]' + x'\cdot\nabla[\kappa\frac{\partial\phi}{\partial n}] = 0 \tag{13}$$

The first term is

$$[\kappa'\frac{\partial\phi}{\partial n}] + [\kappa\frac{\partial\phi'}{\partial n}] + [\kappa\nabla\phi\cdot n'] \tag{14}$$

and the whole can be rewritten as (12-b).

Remark 1 *This derivation is somewhat formal and difficult to justify because a lot of regularity is required form ϕ. Nevertheless intuitively it should be clear that there are enough conditions to have a unique ϕ'.*

3 Derivatives in the Sense of Distributions

We proceed now to rederive and reinterpret these two results above in the light of Distribution Theory.

3.1 Preliminaries

Let Ω be an open bounded set of R^d and A an interval of R. Consider a vector valued function from Ω to R^d function of a parameter $a \in A$:

$$(x, a) \in \Omega \times A \rightarrow v(x, a) \in R^d \tag{15}$$

Let $S(a)$ is a smooth surface which cuts Ω into two parts Ω^+, Ω^-. Assume that the restrictions of v, $v^\pm$ to $\Omega^\pm$ are continuously differentiable. We denote by C^1_S the space of such functions.
Assume also that v is differentiable with respect to a everywhere except on $S(a)$ and denote by v'_a this pointwise derivative.
We introduce the following notations:

- $x = x(s, a)$, $s \in R^{d-1}$, is a parametric representation of $S(a)$
- x' its partial derivative with respect to a.
- $Q = \Omega \times A$, $\Sigma = \{(x, a) \ : \ x \in S(a), \ \ a \in A\}$
- $Q^\pm = \{(x, a) \ : \ x \in \Omega^\pm(a), \ \ a \in A\}$
- n, n_Σ: the normals to $S(a)$ and Σ pointing inside Ω^+ and Q^+.
- $[v] := v^+ - v-$ is the jump of v across $S(a)$.

Proposition 4 *Let v be a function in C^1_S. The derivative of v with respect to a, in the sense of distribution v', is*

$$v' = v'_a - [v]x' \cdot n \, \delta_S \tag{16}$$

where v'_a is the pointwise derivative of v with respect to a and δ_S is the Dirac function on S defined by $\int_\Omega w\delta_S = \int_S w \ \forall w \in \mathcal{D}(\Omega)$.

Proof For clarity we do the proof for $d = 3$ only. By definition of derivatives in the sense of distributions,

$$\forall w \in \mathcal{D}(Q) \ : \quad \int_Q v'w = -\int_Q vw' \tag{17}$$

Let us make use of the fact that $v^\pm := v|_{\Omega^\pm}$ are smooth:

$$\int_Q vw' = \int_{Q^-} v^- w' + \int_{Q^+} v^+ w'$$

$$= -\int_{Q^-} v^{-\prime} w - \int_{Q^+} v^{+\prime} w + \int_{\Sigma} (v^+ - v^-) n_{\Sigma d+1} w \tag{18}$$

where $n_{\Sigma d+1}$ is the last component of the normal to Σ. If Σ has for equation $\sigma(x,a) = 0$, then a normal is $(\partial_x\sigma, \partial_a\sigma)^T$. Now if $x = x(s,a)$ is an parametric representation of $S(a)$ then $\sigma(x(s,a),a) = 0$ for all a so $(\partial_x\sigma)x' + \partial_a\sigma = 0$. This shows that the normal to σ is also $(n, -x'\cdot n)$ because $\partial_x\sigma$ is n, a normal to S.

Corollary 1

$$(\nabla\phi)' = \nabla\phi'_a - \delta_S[\frac{\partial\phi}{\partial n}]x'\cdot n \tag{19}$$

Proposition 5 *Let v be tangent to $S(a)$, then*

$$\nabla\cdot(v\delta_S) = \delta_S\nabla_S\cdot v \tag{20}$$

where $\nabla_S\cdot$ is the surface divergence on S.

Proof

$$\begin{aligned}&\int_\Omega v\delta_S\cdot\nabla w = \int_S v\cdot\nabla w\\ &= \sum_{i=1}^{d-1}\int_S v\cdot s^i\partial_{s^i}w = -\sum_{i=1}^{d-1}\int_S \partial_{s^i}v\cdot s^i w\end{aligned} \tag{21}$$

Consequently from (16) we have the following property.

Corollary 2 *If $[v\cdot n] = 0$ and the pointwise derivative in a, v'_a has a trace left and right of S,*

$$\nabla\cdot v' = \nabla\cdot v'_a + [v'_a\cdot n]\delta_S - \delta_S\nabla_S\cdot([v]x'\cdot n) \tag{22}$$

Remark 2 *Notice that v'_a being discontinuous across S, its divergence contains also a Dirac mass and $\nabla\cdot v'_a + [v'_a\cdot n]\delta_S$ is $\nabla\cdot v'_a$ in the distribution sense.*

Symbolic Notation

Let f, g be functions of C^1_S, differentiable with respect to a in $\Omega^\pm$. Let

$$\bar f = f^\pm \text{ in } \Omega^\pm \text{ and } \frac{(f^+ + f^-)}{2} \text{ on } S(a) \text{ and similarly for } g. \tag{23}$$

Then

$$(fg)' = f'\bar g + \bar f g' \tag{24}$$

Justification
Derivatives of fg, f and g may have Dirac masses on $S(a)$, but by the identity $[fg] = \bar{f}[g] + [f]\bar{g}$, we have equality of the Dirac masses (see Proposition 4). Within $\Omega^{\pm}$ equation (24) is obviously true.

Remark 3 Such definition is of course symbolic because it makes no sense to define a function on a set of measure zero, but it allows to go further in a symbolic calculus of derivatives for products as we shall see below for Darcy's law and for the shallow water equations.
Furthermore note that we do not provide a rule to compute $(g\bar{h})'$ for instance, except in the case of Corollary 2, which makes it difficult to apply the above formula recursively to compute triple products such as $(fgh)'$. As is well known products of distributions are dangerous objects.

3.2 Sensitivity Analysis for Burger's equation (II)

Let $v = (v_1, v_2)^T,\ v_1 = u,\ v_2 = u^2/2$. Calling $x_1 = t,\ x_2 = x$, we see that (2) is $\nabla \cdot v = 0$.
Furthermore v is smooth except across the shock $S = (t, x(t))$, and $n = (-\dot{x}, 1)^T/\sqrt{1+\dot{x}^2}$; Proposition 4 says that

$$u' = u'_a - [u]\frac{x'}{\sqrt{1+\dot{x}^2}}\delta_S \qquad (\frac{u^2}{2})' = uu'_a - [\frac{u^2}{2}]\frac{x'}{\sqrt{1+\dot{x}^2}}\delta_S \quad (25)$$

Similarly Corollary 2 says that

$$\begin{aligned} 0 = \nabla \cdot v' &= \nabla \cdot v'_a + [v'] \cdot n\delta_S - d_t([v \cdot s]x')\delta_S \\ &= \partial_t u'_a + \partial_x(uu'_a) - [u'_a]\dot{x} + [uu'_a] - d_t(x'\frac{[u] + [\frac{u^2}{2}]\dot{x}}{1+\dot{x}^2})\delta_S \end{aligned} \quad (26)$$

where ∂_t and ∂_x are classical derivatives and $s = (1, \dot{x})^T/\sqrt{1+\dot{x}^2}$. Recall that $\dot{x} = \bar{u}$ and that $[uu'_a] = \bar{u}[u'_a] + baru'_a[u]$. We have introduced the notation

$$d_t f := \frac{\partial f}{\partial t}(x(t), t, a) = \partial_t f + \dot{x}\partial_x f \text{ for the time derivative on } x(t) \quad (27)$$

to emphasize the difference with $\partial_t f$. As $[u] + [\frac{u^2}{2}]\dot{x} = [u](1+\dot{x}^2)$ we have:

Proposition 6 *Burger's equation differentiated with respect to a parameter a in the initial data is*

$$\partial_t u'_a + \partial_x(uu'_a) = 0 \quad \textit{in } \Omega\backslash S$$

$$-[u]\bar{u}'_a + d_t([u]x') = 0 \quad on\ S \tag{28}$$

with $\bar{u}'_a = ({u^+}'_a + {u^-}'_a)/2$.

Proposition 7 *Equation (28-b) is the derivative of the Rankine-Hugoniot conditions with respect to the parameter in the initial data.*

Proof The Rankine-Hugoniot condition is

$$\dot{x} = \bar{u}|_{x=x_S(t,a)}.\ \text{Differentiated in } a \text{ it is: } \dot{x}' = \bar{u}'_a + x'\partial_x\bar{u}. \tag{29}$$

Recall the identities

$$\partial_x(\bar{u}[u]) = \bar{u}\partial_x[u] + [u]\partial_x\bar{u} = \partial_x[\frac{u^2}{2}] = -\partial_t[u] = -d_t[u] + \bar{u}\partial_x[u] \tag{30}$$

Multiplied by $[u]$, (29) gives

$$\begin{aligned} &[u]\dot{x}' = [u]\bar{u}'_a + [u]x'\partial_x\bar{u} = [u]\bar{u}'_a - x'd_t[u] \\ &\Rightarrow \ d_t([u]x') = [u]\bar{u}'_a \end{aligned} \tag{31}$$

which is (4-b).

Proposition 8 *Equation (4-a) read in the sense of Distribution theory contains (4-b).*

Proof
As explained above,

$$\partial_t u' + \partial_x(uu') = 0 \quad \text{is} \quad \nabla\cdot\begin{pmatrix} u' \\ uu' \end{pmatrix} = 0 \quad \text{and also } \nabla\cdot\begin{pmatrix} u \\ \frac{u^2}{2} \end{pmatrix}' = 0 \tag{32}$$

Now by Proposition 2

$$0 = \nabla\cdot\begin{pmatrix} u \\ \frac{u^2}{2} \end{pmatrix}' = \nabla\cdot\begin{pmatrix} u' \\ uu' \end{pmatrix} + ([u']\dot{x} - [uu'] - d_t([u]x'))\delta_S \tag{33}$$

which is also

$$0 = \partial_t u' + \partial_x(uu') - ([u]\bar{u}' + d_t([u]x'))\delta_S \tag{34}$$

So we must have (4-a) in $\Omega\backslash S$ and by density this shows that (4-b) holds on S.

Remark 4 *Uniform BV estimates for nonlinear hyperbolic systems in one space variable with convenient hypotheses show that for any continuous test function with compact support* φ*,*

$$\lim_{\delta a\to 0}\int_Q \frac{u(x,t,a+\delta a) - u(x,t,a)}{\delta a}\varphi = \int_Q u'\varphi \tag{35}$$

Numerical Consequences Finite volume methods are based on the weak form of the Burger equation and a finite difference approximation of the time derivative. From the above we learn that a space-time approximation such as used by Johnson[2] would be preferable because it would be potentially capable of handling the singularities of u'. A formulation based on

$$\int_Q u'\partial_t w + uu'\partial_x w = 0 \tag{36}$$

would contain the differentiated Rankine-Hugoniot condition. Still u' has a Dirac singularity on $S(a)$ and special numerical care must be applied to handle it, such as explicitly writing the Dirac masses hidden in (36)

$$\int_Q u'\partial_t w + uu'\partial_x w - \int_0^T [u]x'\frac{dw}{dt}(x(t),t) = 0 \tag{37}$$

This is an immediate consequence of Proposition 4. More generally:

Corollary 3 *Let uv be C^1_S functions of $(x,t) \in Q := \Omega \times (0,T)$; let u', v' be their derivative with respect to a in the sense of distribution. Assume that, in the sense of distribution,*

$$\partial_t u + \partial_x v = 0, \quad \partial_t u' + \partial_x v' = 0. \tag{38}$$

Then

$$\int_Q (u'_a\partial_t w + v'_a\partial_x w) - \int_0^T [u]x'\frac{dw}{dt}(x(t),t) = 0 \tag{39}$$

Proof: By definition $\int_Q (u'\partial_t w + v'\partial_x w = 0$, therefore, by Proposition 4

$$\int_Q (u'_a\partial_t w + v'_a\partial_x w - \int_\Sigma [u](\partial_t w + [v]\partial_x w)x' \cdot n = 0. \tag{40}$$

Now by (41-a) $[v] = [u]\dot{x}$ and $x' \cdot n d\sigma = x' dt$.

3.3 Sensitivity Analysis for Darcy's Law (II)

Proposition 9 *If ϕ satisfies $\nabla \cdot (\kappa\nabla\phi) = 0$ and κ is discontinuous on $S(a)$ then the derivative of ϕ' with respect to a is discontinuous across $S(a)$ and satisifies*

$$\nabla \cdot (\bar{\kappa}\nabla\phi' + \kappa'\overline{\nabla\phi}) = 0 \tag{41}$$

with $\kappa' = \kappa'_a - \delta_S[\kappa]x' \cdot n$ where $x = x(s,a)$ is the equation of S. The meaning of $\nabla \cdot ([\kappa]x' \cdot n\nabla\phi'\delta_S)$ is given by Corollary 2.

Proof
By Proposition 4 we see that, with $v = \kappa\nabla\phi$,

$$v' = (\kappa\nabla\phi)'_a - [\kappa\nabla\phi \cdot n]x'\delta_S = (\kappa\nabla\phi)'_a \tag{42}$$

because the normal composent of $\kappa\nabla\phi$ does not jump across S.
With the symbolic notations (24),

$$v' = (\kappa\nabla\phi)' = \kappa'\overline{\nabla\phi} + \bar{\kappa}\nabla\phi' \tag{43}$$

which, by Proposition 4 and Corollary 1 is

$$\begin{aligned} v' &= \kappa'_a\nabla\phi + \kappa\nabla\phi'_a - \delta_S x' \cdot n([\kappa]\bar{\nabla}\phi \cdot n + \bar{\kappa}[\nabla\phi] \cdot n) \\ &= \kappa'_a\nabla\phi + \kappa\nabla\phi'_a - \delta_S x' \cdot n[\kappa\nabla\phi] \end{aligned} \tag{44}$$

but the last term is zero. By Corollary 2:

$$0 = \nabla \cdot v' = \nabla \cdot v'_a - \partial_s([\kappa\partial_s\phi]x') \tag{45}$$

which can be summed up as

$$\nabla \cdot (\bar{\kappa}\nabla\phi') = -\nabla \cdot (\kappa'\overline{\nabla\phi}) \tag{46}$$

If κ is piecewise constant, then $\kappa'_a = 0$ and $[\phi] = 0 \Rightarrow [\partial_s\phi] = 0 \Rightarrow [\kappa\partial_s\phi] = 0$ therefore $\nabla \cdot v' = 0$ and $v' = \kappa\nabla\phi'$. However it is not a standard problem because the solution is in $L^2(\Omega)$ only and it jumps across S of a known quantity. The problem can be written in mixed form with the usual understanding of derivatives:

$$v' \in H(\text{div}, \Omega),\ \ \phi' \in L^2(\Omega)\ \ :\ \nabla \cdot v' = 0,\ \ \nabla\phi' - \frac{1}{\kappa}v' = [\frac{1}{\kappa}]v \cdot nx' \cdot n\delta_S. \tag{47}$$

So we have the following

Corollary 4 *Equation (41) contains the derivative with respect to a of the transmission conditions on $S(a)$: $\frac{d}{da}([\phi]_{x=x(a)}) = 0$ and $\frac{d}{da}([\kappa\frac{\partial\phi}{\partial n}]_{x=x(a)}) = 0$*

To illustrate the usefulness of the symbolic notation introduced with (24) let us show that (47) can be derived directly.
From the definition of v, $v = \kappa\nabla\phi$, when κ is piecewise constant on both sides of $S(a)$, we find that

$$v' = \bar{\kappa}\nabla\phi' + \kappa'\overline{\nabla\phi} = \bar{\kappa}\nabla\phi' - [\kappa]\overline{\nabla\phi}x'\delta_S \tag{48}$$

Now

$$0 = [\kappa\nabla\phi] = [\kappa]\overline{\nabla\phi} + \bar{\kappa}[\nabla\phi] \tag{49}$$

and

$$[\nabla\phi] = [\frac{\kappa\nabla\phi}{\kappa}] = \overline{(\frac{1}{\kappa})}[\kappa\nabla\phi] + [\frac{1}{\kappa}]\overline{\kappa\nabla\phi} = [\frac{1}{\kappa}]\overline{\kappa\nabla\phi} \tag{50}$$

so,

$$[\kappa]\overline{\nabla\phi} = [\frac{1}{\kappa}]\bar{\kappa}\overline{\kappa\nabla\phi} \tag{51}$$

and (47-b) is recovered.

Remark 5 *An equation like (48) is hard to read; with the foreknowledge that v' does not have a Dirac mass it means that ϕ' has one and its weight is given by the last term. Then, outside $S(a)$ it means that $v' = \kappa\nabla\phi'$, and so if these have traces right and left of $S(a)$ then it contains also the equation $[\kappa\nabla\phi'\cdot n] = [v']$.*

3.4 The Shallow Water Equations

In $R^d,\ d =$1,2 or 3, consider

$$\begin{aligned}&\partial_t\rho + \nabla\cdot(\rho u) = 0\\ &\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla\rho = 0.\end{aligned} \tag{52}$$

With appropriate initial and boundary conditions, the water height ρ and its velocity u are uniquely defined, at least when $d = 1$ (see Lions[1]).
Across shocks S the Rankine-Hugoniot conditions are

$$[\rho]\dot{x} + [\rho u]\cdot n = 0 \qquad [\rho u]\dot{x} + [\rho u\otimes u\cdot n] + [\rho]n = 0 \tag{53}$$

Consider the mono dimensional case only, $d = 1$.
Differentiating (52-53) is easier to do in terms of the flux velocity $v = \rho u$. For shocks normal to the flow u, let ν be the norm of v:

$$\begin{aligned}&\partial_t\rho' + \partial_x v' = 0\\ &\partial_t v' + \partial_x(\rho^{-1}(v\otimes v' + v'\otimes v) - \rho'\rho^{-2}v\otimes v) + \partial_x\rho' = 0\\ &[\rho']\dot{x} + [\rho]\dot{x}' + [\nu'] + \dot{x}x'\cdot\partial_x[\rho] + x'\cdot\partial_x[\nu] = 0\\ &[\nu']\dot{x} + [\nu]\dot{x}' + ([\rho^{-1}\nu^2 + \rho])' + \dot{x}x'\partial_x\nu\\ &+x'\partial_x(\rho^{-1}\nu^2 + \rho) = 0\end{aligned} \tag{54}$$

Alternatively, in the variable (t,x), equation (52-a) is a divergence of $V = (\rho, v)^T$ and so the same result can be obtained in the sense of Distribution theory by Corollary 2. However in order to differentiate the products we need to write the equations as:

$$\partial_t\rho + \partial_x v = 0 \qquad \partial_t v + \partial_x(vu) + \partial_x\rho = 0 \qquad v = \rho u \tag{55}$$

Then its derivative is

$$\begin{aligned}&\partial_t\rho' + \partial_x v' = 0\\ &\partial_t v' + \partial_x(v'\bar{u} + \bar{v}u' + \rho') = 0\\ &v' = \rho'\bar{u} + \bar{\rho}u'\end{aligned} \tag{56}$$

Interpretation As for Burger's equation (56) contains the standard derivative of (52) and 3 jump conditions involving $[u], [v], [\rho], x'$.
System (52) is of the form

$$\partial_t W + \partial_x F(W) = 0 \tag{57}$$

with $W = (\rho, \rho u)^T$, therefore the linearized equation for V and the differentiated equations have the same jacobian matrix $A_{ij} = \partial_{W_i} F(W)_j$, $i, j = 1, 2$. The analysis of shocks is based on the computation of eigenvalues and eigenvectors of A. For One-Shocks for the shallow water system one Riemann invariant is computed from characteristics on the right side of the shock with a transmission condition for the left side one while the other invariant is computed with left and right characteristics independently on both sides of the shock, without transmission condition across the shock while . This means that the jump of this second invariant is fixed by (56) understood in the classical sense. So we have the 3 jump conditions yielded by (56) in the distribution sense for $[u], [v], [\rho], x'$ but with one more equation dues to the knowledge of the jump of the second Riemann invariant.

Remark 6 If we had in the momentum equation ρ^γ instead of ρ, the same analysis applies. The momentum equation is the divergence of $(v, vu + \rho^\gamma)^T$ so the terms which appears in the jump condition is $[v]\dot{x} + [vu + \rho^\gamma]$ and it is also the jump found when

$$\partial_t v' + \partial_x(u'\bar{v} + \bar{u}v' + a') = 0$$

but we must find a such that $\bar{a}[rho] = \rho^\gamma$. When $\gamma = 3/2$ (air) it can be done as follows:

$$\begin{aligned}&b^2 = \rho,\ a = b\rho\ \Rightarrow\ 2\bar{b}b' = \rho',\ a' = \bar{b}\rho' + b'\bar{\rho}\\ &\Rightarrow\ a' = \overline{\sqrt{\rho}}\rho' + \bar{\rho}\frac{\rho'}{2\overline{\sqrt{\rho}}}\end{aligned} \tag{58}$$

4 Euler's Equations

Perfect inviscid fluids are governed also by (52) but p is not a function of ρ and there is an additional equation for the conservation of energy:

$$\partial_t\theta + \partial_x(v(\frac{u^2}{2} + \theta)) = 0 \quad \text{with } \theta = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} \tag{59}$$

To decompose all products into binary multiplications we multiply by ρ the second equation and set $w = u^2/2$. Differentiation leads to

$$\partial_t \theta' + \partial_x (v'\bar{w} + \bar{v}w' + \theta') = 0 \qquad \frac{\gamma}{\gamma - 1} p' = \bar{\theta}\rho' + \bar{\rho}\theta' \qquad w' = \bar{u}u' \tag{60}$$

Therefore the result is

$$\begin{aligned}
\partial_t \rho' + \partial_x (\bar{\rho}u' + \bar{u}\rho') = 0 \\
\partial_t (\bar{\rho}u' + \bar{u}\rho') + \partial_x (\bar{u}^2\rho' + (\bar{\rho}\bar{u} + \overline{\rho u})u' + p') = 0 \\
\frac{\gamma}{\gamma - 1}\partial_t \Big(\frac{p'}{\bar{\rho}} - \overline{(\frac{p}{\rho})}\rho'\Big) + \partial_x \Big(\frac{\gamma}{\gamma - 1}\Big(\Big(\frac{p'}{\bar{\rho}} - \overline{(\frac{p}{\rho})}\rho'\Big)\overline{\rho u} + \overline{(\frac{p}{\rho})}(\bar{\rho}u' + \bar{u}\rho')\Big) \\
+ \Big(\bar{u}\overline{\rho u} + \frac{\overline{u^2}\bar{\rho}}{2}\Big)u' + \frac{\overline{u^2}\bar{u}}{2}\rho'\Big) = 0
\end{aligned}$$

5 Conclusion

We have seen that equations in divergence form can be differentiated even in the presence of shocks. The differentiated equation, taken in the sense of distribution, contains the transmission condition across the shock which fixes its motion. Therefore integration by parts of the differentiated equation is valid and numerical methods based on space time finite volume or finite element method should work. Other equations can be analyzed also by this method, such as the transonic equation [11] [6]. This will be done in a forthcoming publication.

References

[1] Lions, P.-L. *Mathematical Topics in Fluid Mechanics* Vol 1. Oxford University Press (1996).

[2] Eriksson, K. and Johnson, C. and Larsson, S. Adaptive finite element methods for parabolic problems. VI. Analytic semigroups. SIAM J. Numer. Anal., (35), p1315–1325, (1998)

[3] Bernardi, C. and O. Pironneau. Sensitivities to discontinuities in Darcy's Law. Compte Rendus de l'Académie des Sciences. Serie I (to appear, Dec. 2002).

[4] Di Cesare N. and O. Pironneau. Shock sensitivity analysis. *Computational Fluid Dynamics Journal*, 9(2), 2000.

[5] Giles M.A. and N.A. Pierce. Analytic adjoint solutions for the quasi-one-dimensional euler equations. *Journal of Fluid Mechanics*, 426:327–345, 2001.

[6] Glowinski R. *Numerical methods for nonlinear variational problems*. Springer-Verlag, New York, 1984.

[7] Griewank, A. *Evaluating Derivatives, Principles and Techniques of Algorithmic Differentiation.* Vol 19, Frontiers in Applied Mathematics. SIAM. (2000).

[8] Olazabal M. Godlewski E. and P.A. Raviart. On the linearization of hyperbolic systems of conservation laws. Application to stability. In *Équations aux dérivées partielles et applications*, pages 549–570. Gauthier-Villars, Éd. Sci. Méd. Elsevier, Paris, 1998.

[9] Godlewski, E. and Raviart, P.-A. An introduction to the linearized stability of solutions of nonlinear hyperbolic systems of conservation laws. UPMC-J.L.Lions Lab. report R0003 (2000).

[10] Godlewski, E. and Raviart, P.-A. The linearized stability of solutions of nonlinear hyperbolic systems of conservation laws: A general numerical approach.of conservation laws. UPMC-J.L.Lions Lab. report R9850 (1998).

[11] Nečas J. *Écoulements de fluide: compacité par entropie*. Masson, Paris, 1989.

[12] Tai-Ping Liu. Hyperbolic and Viscous Conservation Laws. CBMS-NSF Regional Conference Series in Applied Mathematics, 72. (SIAM), Philadelphia, PA, 2000.

[13] O. Pironneau. *Control of Transonic Shock Positions*. To appear, 2002. ESAIM:Control, Optimisation and Calculus of Variations.

[14] Jaouen S. Étude mathématiques et numérique de stabilité pour des modèles hydrodynamiques. Thèse, Université Paris VI (2001).