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REMARKS ON THE STRONG MAXIMUM PRINCIPLE

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1. Introduction

The strong maximum principle asserts that if u is smooth, $u \geq 0$ and $-\Delta u \geq 0$ in a connected domain $\Omega \subset \mathbb{R}^N$, then either $u \equiv 0$ or $u > 0$ in Ω. The same conclusion holds when $-\Delta$ is replaced by $-\Delta + a(x)$ with $a \in L^p(\Omega)$, $p > \frac{N}{2}$ (this is a consequence of Harnack's inequality; see e.g. [6], and also [7, Corollary 5.3]. Another formulation of the same fact says that if $u(x_0) = 0$ for some point $x_0 \in \Omega$, then $u \equiv 0$ in Ω. A similar conclusion fails, however, when $a \notin L^p(\Omega)$, for any $p > \frac{N}{2}$. For instance, $u(x) = |x|^2$ satisfies $-\Delta u + a(x)u = 0$ in B_1 with $a = \frac{2N}{|x|^2} \notin L^{N/2}(\Omega)$.

If u vanishes on a larger set, one may still hope to conclude, under some weaker condition on a, that $u \equiv 0$ in Ω. Such a result was obtained by Bénilan-Brezis [2, Appendix D] (with a contribution by R. Jensen) in the case where $a \in L^1(\Omega)$ and $\operatorname{supp} u$ is a compact subset of Ω. Their maximum principle has been further extended by Ancona [1], who proved Theorem 1 below.

We recall that a function $v : \Omega \to \mathbb{R}$ is quasicontinuous if there exists a sequence of open subsets (ω_n) of Ω such that $v|_{\Omega \setminus \omega_n}$ is continuous $\forall n \geq 1$ and $\operatorname{cap} \omega_n \to 0$ as $n \to \infty$, where $\operatorname{cap} \omega_n$ denotes the H^1-capacity of ω_n.

Theorem 1 ([1])**.** *Assume $\Omega \subset \mathbb{R}^N$ is an open bounded set. Let $u \in L^1(\Omega)$, $u \geq 0$ a.e. in Ω, be such that Δu is a Radon measure on Ω. Then there exists $\tilde{u} : \Omega \to \mathbb{R}$ quasicontinuous such that $u = \tilde{u}$ a.e. in Ω.*

Let $a \in L^1(\Omega)$, $a \geq 0$ a.e. in Ω. If

$$-\Delta u + au \geq 0 \quad \text{in } \Omega,$$

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in the following sense

$$\int_E \Delta u \leq \int_E au \quad \textit{for every Borel set } E \subset \Omega, \tag{1.1}$$

and if $\tilde{u} = 0$ *on a set of positive* H^1*-capacity in* Ω*, then* $u = 0$ *a.e. in* Ω*.*

The proof given by Ancona is purely based on Potential Theory, while ours is more direct in the spirit of PDE's. We also discuss carefully the meaning of the condition $-\Delta u + au \geq 0$ in Ω.

The next two corollaries follow immediately from the theorem above:

Corollary 2. *Let* u *and* a *be as in Theorem* 1, *and suppose* (1.1) *is satisfied. If* $u = 0$ *on a subset of* Ω *with positive measure, then* $u = 0$ *a.e. in* Ω*. If* u *is continuous in* Ω *and* $u = 0$ *on a subset of* Ω *with positive* H^1*-capacity, then* $u \equiv 0$ *in* Ω*.*

Corollary 3. *Let* u *and* a *be as in Theorem* 1*. Suppose that* $\Delta u \in L^1(\Omega)$*. If*

$$-\Delta u + au \geq 0 \quad \textit{a.e. in } \Omega$$

and $u = 0$ *on a subset of* Ω *with positive measure, then* $u = 0$ *a.e. in* Ω*.*

The next corollary follows from Theorem 1 and Remark 3:

Corollary 4. *Let* u *and* a *be as in Theorem* 1*. Suppose that* $au \in L^1_{\text{loc}}(\Omega)$*. If*

$$-\Delta u + au \geq 0 \quad \textit{in } \mathcal{D}'(\Omega),$$

i.e.,

$$\int u\Delta\varphi \leq \int au\varphi \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0 \textit{ in } \Omega,$$

and $u = 0$ *on a subset of* Ω *with positive measure, then* $u = 0$ *a.e. in* Ω*.*

Remark 1. In view of Corollary 4 above, it would seem natural to replace condition (1.1) in Theorem 1 by

$$\int u\Delta\varphi \leq \int au\varphi \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0 \text{ in } \Omega, \tag{1.2}$$

which makes sense even if $au \notin L^1_{\text{loc}}(\Omega)$ (note that $au\varphi \geq 0$ a.e., so that the right-hand side is always well-defined, possibly taking the value $+\infty$). However, the strong maximum principle is no longer true in general. See Remark 4.

There are several interesting questions related to Theorem 1:

Open problem 1. In the statement of Theorem 1, suppose in addition that $\operatorname{supp} u \subset \Omega$ is a compact set. Can one replace the assumption $a \in L^1_{\text{loc}}$ by a weaker condition, for example $a^{1/2} \in L^1_{\text{loc}}$ (or $a^{1/2} \in L^p_{\text{loc}}$ for some $p > 1$), and still conclude that $u = 0$ a.e. in Ω?

Note that one cannot hope to go below $L^{1/2}$. For instance the C^2-function u given by

$$u(x) = \begin{cases} \left(1 - |x|^2\right)^4 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases},$$

satisfies $-\Delta u + au \geq 0$ for some function $a(x)$ such that $a(x) \sim \frac{1}{\left(1-|x|\right)^2}$ for $|x| \lesssim 1$. Here, $a^\alpha \in L^1 \quad \forall \alpha < 1/2$, but $a^{1/2} \notin L^1$.

Here is another question:

Open problem 2. Assume $u \in C^0$, $u \geq 0$, and $a \in L^q_{\text{loc}}$ for some $q \geq 1$, $a \geq 0$ a.e., satisfy (1.1). Suppose that $u = 0$ on a set E with $\operatorname{cap}_{1,2q}(E) > 0$, where $\operatorname{cap}_{1,2q}$ refers to the capacity associated with the Sobolev space $W^{1,2q}$. Can one conclude that $u \equiv 0$?

Theorem 1 above shows that the answer is positive when $q = 1$. It is also true when $q > \frac{N}{2}$ by the strong maximum principle mentioned above (note that if $q > \frac{N}{2}$ and x_0 is any point, then $\operatorname{cap}_{1,2q}\left(\{x_0\}\right) > 0$).

2. Some comments about condition (1.1)

Since in the statement of Theorem 1 it may happen that $au \notin L^1_{\text{loc}}(\Omega)$, and so au is not necessarily a distribution, one should be careful in order to give a precise meaning to the inequality

$$\Delta u \leq au \quad \text{in } \Omega.$$

More generally, let μ be a Radon measure on Ω and f a measurable function, $f \geq 0$ a.e. in Ω. Here are two possible definitions for the inequality $\mu \leq f$ in Ω:

Definition 1. We shall write $\mu \leq_1 f$ in Ω if

$$\int_E d\mu \leq \int_E f \quad \text{for every Borel set } E \subset \Omega.$$

Definition 2. We shall write $\mu \leq_2 f$ in Ω if

$$\int \varphi \, d\mu \leq \int f\varphi \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0 \text{ in } \Omega.$$

In the first definition, we view f as the nonnegative measure $f\,dx$, while in the second one f is treated as if it were a distribution.

Remark 2. If $\mu \leq_1 f$ in Ω, then $\mu \leq_2 f$ in Ω. However, the converse is not true in general. See Remark 4 below.

Remark 3. If we assume in addition that $f \in L^1_{\mathrm{loc}}(\Omega)$, then $\mu \leq_1 f$ in Ω if, and only if, $\mu \leq_2 f$ in Ω.

Remark 4. Theorem 1 above is no longer true in general (even for the case where $\Delta u \in L^1(\Omega)$) if we replace (1.1) by

$$-\Delta u + au \geq_2 0 \quad \text{in } \Omega,$$

i.e., if

$$\int u\Delta\varphi \leq \int au\varphi \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0 \text{ in } \Omega.$$

In fact, let $N \geq 2$. Take $v \in L^1(\mathbb{R}^N)$, $v \geq 0$ a.e. in $\mathbb{R}^N$, such that $\operatorname{supp} v \subset B_1$, $\Delta v \in L^1(\mathbb{R}^N)$, but v is unbounded (this is possible since $N \geq 2$). In particular, there exists $b \in L^1(\mathbb{R}^N)$, $b \geq 0$ a.e. in $\mathbb{R}^N$, such that $bv \notin L^1(\mathbb{R}^N)$.

Let $(x_j) \subset B_1$ be a dense sequence in B_1 and, for each $j \geq 1$, let

$$\gamma_j := \min\Big\{\frac{1}{j}, \frac{1-|x_j|}{2}\Big\}.$$

We define

$$u(x) := \sum_{j=1}^\infty \frac{1}{2^j \gamma_j^{N-2}}\, v\Big(\frac{x-x_j}{\gamma_j}\Big), \quad a(x) := \sum_{j=1}^\infty \frac{1}{2^j \gamma_j^{N}}\, b\Big(\frac{x-x_j}{\gamma_j}\Big).$$

Then

$$\begin{aligned} &u \in L^1(\mathbb{R}^N), \quad u \geq 0 \text{ a.e. in } \mathbb{R}^N,\\ &\Delta u \in L^1(\mathbb{R}^N),\\ &a \in L^1(\mathbb{R}^N), \quad a \geq 0 \text{ a.e. in } \mathbb{R}^N, \end{aligned}$$

and

$$\int u\Delta\varphi \leq \int au\varphi \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0 \text{ in } \Omega,$$

(note that the integral in the right-hand side is either 0 or $+\infty$), but $\operatorname{supp} u \subset \overline{B}_1$ and $u \not\equiv 0$ in $\mathbb{R}^N$. On the other hand, in view of Theorem 1, the inequality $\Delta u \leq_1 au$ is **not** satisfied.

From now on, we shall always consider the inequality $\Delta u \leq au$ in the sense of Definition 1. In particular we shall omit the subscript 1 in the symbol $\leq_1$.

3. Proof of the quasicontinuity statement of Theorem 1

Before proving the first part of Theorem 1 (see Lemma 1 below), we make the following remark:

Remark 5. If $v \in H^1_{\mathrm{loc}}(\Omega)$, then there exists $\tilde{v} : \Omega \to \mathbb{R}$ quasicontinuous such that $v = \tilde{v}$ a.e. in Ω (see e.g. [5]). In addition, $\tilde{v}$ is well-defined modulo polar subsets of Ω, i.e., if $\tilde{v}_1$ and $\tilde{v}_2$ are two quasicontinuous functions such that $\tilde{v}_1 = v = \tilde{v}_2$ a.e. in Ω, then there exists a polar set $P \subset \Omega$ such that $\tilde{v}_1(x) = \tilde{v}_2(x) \quad \forall x \in \Omega \backslash P$ (see [3]).

Notation. Given $k > 0$, we denote by $T_k : \mathbb{R} \to \mathbb{R}$ the truncation function

$$T_k(s) := \begin{cases} k & \text{if } s \geq k, \\ s & \text{if } -k < s < k, \\ -k & \text{if } s \leq -k. \end{cases}$$

The existence of a quasicontinuous function $\tilde{u} : \Omega \to \mathbb{R}$ such that $u = \tilde{u}$ a.e. in Ω as in the statement of Theorem 1 is a consequence of Lemma 1 below (see [1]):

Lemma 1. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Assume $u \in L^1(\Omega)$ is such that Δu is a Radon measure on Ω. Then*

$$T_k(u) \in H^1_{\mathrm{loc}}(\Omega) \quad \forall k > 0, \tag{3.1}$$

and, for each open subset $A \subset\subset \Omega$, there exists $C_A > 0$ so that

$$\int_A \left|\nabla T_k(u)\right|^2 \leq k\Big(\int_\Omega |\Delta u| + C_A \int_\Omega |u|\Big) \quad \forall k > 0. \tag{3.2}$$

Moreover, there exists $\tilde{u} : \Omega \to \mathbb{R}$ quasicontinuous such that $u = \tilde{u}$ a.e. in Ω.

Proof. We shall split the proof of Lemma 1 into two steps:
Step 1. Proof of (3.1) and (3.2). We first extend u to the whole $\mathbb{R}^N$ so that $u \equiv 0$ outside Ω. Let $\rho \in C_0^\infty(B_1)$ be a radial, nonnegative, mollifier. Set

$$u_\varepsilon(x) := \rho_\varepsilon * u(x) = \int_\Omega \rho_\varepsilon(x-y)u(y)\,dy \quad \forall x \in \Omega.$$

For $k > 0$ fixed, we have $T_k(u_\varepsilon) \in H^1(\Omega)$ and

$$\nabla T_k(u_\varepsilon) = \nabla u_\varepsilon\ \chi_{[|u_\varepsilon|<k]}, \tag{3.3}$$

where $\chi_{[|u_\varepsilon|<k]}$ denotes the characteristic function of the set $[|u_\varepsilon| < k]$.

Given an open set $A \subset\subset \Omega$, let $\varphi \in C_0^\infty(\Omega)$ be such that $0 \le \varphi \le 1$ in Ω and $\varphi \equiv 1$ on A. On the one hand, using (3.3) and integrating by parts, we have

$$\begin{aligned}\int \big|\nabla T_k(u_\varepsilon)\big|^2 \varphi &= \int \nabla T_k(u_\varepsilon)\cdot(\nabla u_\varepsilon)\,\varphi \\ &= -\int T_k(u_\varepsilon)(\Delta u_\varepsilon)\,\varphi - \int T_k(u_\varepsilon)\,\nabla u_\varepsilon\cdot\nabla\varphi.\end{aligned} \tag{3.4}$$

On the other hand,

$$\begin{aligned}\int T_k(u_\varepsilon)\nabla u_\varepsilon\cdot\nabla\varphi &= -\int u_\varepsilon \nabla T_k(u_\varepsilon)\cdot\nabla\varphi - \int u_\varepsilon T_k(u_\varepsilon)\Delta\varphi \\ &= -\int T_k(u_\varepsilon)\nabla T_k(u_\varepsilon)\cdot\nabla\varphi - \int u_\varepsilon T_k(u_\varepsilon)\Delta\varphi \\ &= -\frac{1}{2}\int \nabla\big[T_k(u_\varepsilon)\big]^2\cdot\nabla\varphi - \int u_\varepsilon T_k(u_\varepsilon)\Delta\varphi \\ &= \frac{1}{2}\int \big[T_k(u_\varepsilon)\big]^2\Delta\varphi - \int u_\varepsilon T_k(u_\varepsilon)\Delta\varphi \\ &= -\int T_k(u_\varepsilon)\big(u_\varepsilon - \frac{1}{2}T_k(u_\varepsilon)\big)\Delta\varphi \ge -k\int |u_\varepsilon||\Delta\varphi|.\end{aligned} \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}\int_A \big|\nabla T_k(u_\varepsilon)\big|^2 &\le \int \big|\nabla T_k(u_\varepsilon)\big|^2\varphi \\ &\le k\Big(\int_{\operatorname{supp}\varphi} |\Delta u_\varepsilon| + \|\Delta\varphi\|_{L^\infty}\int_{\operatorname{supp}\varphi}|u_\varepsilon|\Big).\end{aligned}$$

In particular, for every $0 < \varepsilon < \operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega)$,

$$\int_A \big|\nabla T_k(u_\varepsilon)\big|^2 \le k\Big(\int_\Omega |\Delta u| + \|\Delta\varphi\|_{L^\infty}\int_\Omega |u|\Big).$$

Letting $\varepsilon \downarrow 0$, we conclude that $T_k(u) \in H^1(A)$ and (3.2) holds with $C_A = \|\Delta\varphi\|_{L^\infty}$.

Step 2. We prove that, under the assumptions of the lemma, there exists a function $\tilde{u} : \Omega \to \mathbb{R}$ quasicontinuous such that $u = \tilde{u}$ a.e. in Ω.

By (3.1) and Remark 5, for each $k > 0$ there exists $\widetilde{T_k(u)} : \Omega \to \mathbb{R}$ quasicontinuous such that $T_k(u) = \widetilde{T_k(u)}$ a.e. in Ω.

Let $v_k := \frac{1}{k}T_k(u)$, so that

$$v_k \to 0 \quad \text{in } L^q(\Omega) \quad \forall q \in [1, \infty)$$

and, by (3.2),

$$\int_A |\nabla v_k|^2 \to 0 \quad \forall A \subset\subset \Omega.$$

In particular, $v_k \to 0$ in $H^1_{\mathrm{loc}}(\Omega)$, which implies there exists a polar set $P \subset \Omega$ such that

$$\tilde{v}_k(x) = \frac{1}{k}\,\widetilde{T_k(u)}(x) \to 0 \quad \forall x \in \Omega\backslash P.$$

We conclude that

$$\operatorname{cap}\left[\left|\widetilde{T_k(u)}\right| > \frac{k}{2}\right] = \operatorname{cap}\left[|\tilde{v}_k| > \frac{1}{2}\right] \to 0. \tag{3.6}$$

Set

$$w(x) := \begin{cases} \sup\limits_{k\in\mathbb{N}} \left\{\widetilde{T_k(u)}(x)\right\} & \text{if } \sup\limits_{k\in\mathbb{N}} \left|\widetilde{T_k(u)}(x)\right| < \infty, \\ 0 & \text{otherwise,} \end{cases} \tag{3.7}$$

so that $w = u$ a.e. in Ω. By (3.6) and the quasicontinuity of the functions $\widetilde{T_k(u)}$, it is easy to see that w is quasicontinuous in Ω. This concludes the proof of the lemma.

4. A variant of Kato's inequality when Δu is a Radon measure

We start with the following (see [1])

Lemma 2. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Assume $u \in L^1(\Omega)$, $u \geq 0$ a.e. in Ω, is such that Δu is a Radon measure on Ω. Then*

$$\Delta T_k(u) \quad \text{is a Radon measure} \quad \forall k > 0.$$

Moreover, for any $a \in L^\infty(\Omega)$, $a \geq 0$ a.e. in Ω, we have

$$\Delta T_k(u) - aT_k(u) \leq \big(\Delta u - au\big)^+ \quad \text{in } \mathcal{D}'(\Omega). \tag{4.1}$$

Proof. We shall use the same notation as in the proof of Lemma 1. By the standard L^1-version of Kato's inequality (see [4]) we have (note that $T_k|_{\mathbb{R}_+}$ is concave)

$$\Delta T_k(u_\varepsilon) \leq t_k(u_\varepsilon)\Delta u_\varepsilon \quad \text{in } \Omega \quad \forall \varepsilon > 0, \tag{4.2}$$

where the function $t_k : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$t_k(s) := \begin{cases} 1 & \text{if } 0 \leq s \leq k, \\ 0 & \text{if } s > k. \end{cases}$$

Since $T_k(s) \geq t_k(s)s \quad \forall s \geq 0$ and $a \geq 0$ a.e. in Ω, it follows from (4.2) that

$$\Delta T_k(u_\varepsilon) - aT_k(u_\varepsilon) \leq t_k(u_\varepsilon)\big(\Delta u_\varepsilon - au_\varepsilon\big) \leq (\Delta u_\varepsilon - au_\varepsilon\big)^+ \quad \text{in } \mathcal{D}'(\Omega).$$

In other words, we have

$$\int T_k(u_\varepsilon)\Delta\varphi - aT_k(u_\varepsilon)\varphi \le \int \big(\Delta u_\varepsilon - au_\varepsilon\big)^+\varphi \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \ge 0 \text{ in } \Omega. \tag{4.3}$$

For $\lambda > 0$, let $\Omega_\lambda := \big\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \lambda\big\}$. Thus, if $0 < \varepsilon < \lambda$, we get

$$\begin{aligned}\Delta u_\varepsilon - au_\varepsilon &= \big(\Delta u - au\big)_\varepsilon + (au)_\varepsilon - au_\varepsilon \\ &\le \rho_\varepsilon * \big(\Delta u - au\big)^+ + \big|(au)_\varepsilon - au\big| + \big|au - au_\varepsilon\big| \quad \text{in } \Omega_\lambda.\end{aligned}$$

Therefore, for any $\varphi \in C_0^\infty(\Omega)$, $\varphi \ge 0$ in Ω, and $0 < \varepsilon < \operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega)$, we may write

$$\begin{aligned}\int \big(\Delta u_\varepsilon - au_\varepsilon\big)^+\varphi &\le \int \rho_\varepsilon * \big(\Delta u - au\big)^+\varphi \\ &\quad + \|\varphi\|_{L^\infty}\Big\{\big\|(au)_\varepsilon - au\big\|_{L^1} + \|a\|_{L^\infty}\big\|u - u_\varepsilon\big\|_{L^1}\Big\} \\ &= \int (\rho_\varepsilon * \varphi)\big(\Delta u - au\big)^+ + o(1).\end{aligned} \tag{4.4}$$

Since $\rho_\varepsilon * \varphi \to \varphi$ uniformly in Ω and $\big(\Delta u - au\big)^+$ is a Radon measure in Ω, by letting $\varepsilon \downarrow 0$ in (4.3) and (4.4), we conclude that

$$\int T_k(u)\Delta\varphi - aT_k(u)\varphi \le \int \big(\Delta u - au\big)^+\varphi \quad \forall \varphi \in C_0^\infty(\Omega),\ \varphi \ge 0 \text{ in } \Omega,$$

so that $T_k(u)$ is a Radon measure (take for instance $a \equiv 0$) and (4.1) holds.

Lemma 3. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Assume $u \in L^1(\Omega)$, $u \ge 0$ a.e. in Ω, is such that Δu is a Radon measure on Ω. Let $a \in L^1(\Omega)$, $a \ge 0$ a.e. in Ω. If*

$$-\Delta u + au \ge 0 \quad \textit{in } \Omega,$$

in the following sense

$$\int_E \Delta u \le \int_E au \quad \textit{for every Borel set } E \subset \Omega, \tag{4.5}$$

then

$$-\Delta T_k(u) + aT_k(u) \ge 0 \quad \textit{in } \mathcal{D}'(\Omega) \quad \forall k > 0. \tag{4.6}$$

Proof. By the preceding lemma applied with $a_i := T_i(a)$, where i is a positive integer, we know that

$$\Delta T_k(u) - a_iT_k(u) \le \big(\Delta u - a_iu\big)^+ \quad \text{in } \mathcal{D}'(\Omega). \tag{4.7}$$

On the other hand, from (4.5) we get

$$\int_E (\Delta u - a_i u) \leq \int_E (a - a_i) u \quad \text{for every Borel set } E \subset \Omega. \tag{4.8}$$

Since $(a - a_i)u \geq 0$ a.e. in Ω, (4.8) implies that

$$0 \leq \int_E (\Delta u - a_i u)^+ \leq \int_E (a - a_i) u \quad \text{for every Borel set } E \subset \Omega. \tag{4.9}$$

Hence, $(\Delta u - a_i u)^+$ is a nonnegative measure which is absolutely continuous with respect to the Lebesgue measure. Therefore, we have

$$(\Delta u - a_i u)^+ \in L^1(\Omega) \quad \forall i = 1, 2, \dots. \tag{4.10}$$

We now return to (4.9) to conclude that

$$0 \leq (\Delta u - a_i u)^+ \leq (a - a_i) u \quad \text{a.e. in } \Omega.$$

In particular,

$$(\Delta u - a_i u)^+ \downarrow 0 \quad \text{a.e. in } \Omega \text{ as } i \uparrow \infty. \tag{4.11}$$

It follows from (4.10) and (4.11) that

$$(\Delta u - a_i u)^+ \to 0 \quad \text{in } L^1(\Omega) \text{ as } i \to \infty. \tag{4.12}$$

Finally, for any $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ in Ω, by (4.7) and (4.12) we have

$$\int T_k(u)\Delta\varphi - aT_k(u)\varphi \leq \int T_k(u)\Delta\varphi - a_i T_k(u)\varphi \leq \int \big(\Delta u - a_i u\big)^+ \varphi \to 0$$

as $i \to \infty$, so that (4.6) holds.

5. Proof of Theorem 1 completed

It follows from Lemma 1 in Section 2 that, under the hypotheses of the theorem, there exists $\tilde{u} : \Omega \to \mathbb{R}$ quasicontinuous such that $u = \tilde{u}$ a.e. in Ω. Let us assume that $\tilde{u} = 0$ on a set of positive capacity $E \subset \Omega$. We shall prove that $u = 0$ a.e. in Ω.

We split the proof into two steps:

Step 1. Under the hypotheses of the theorem, if we assume in addition that $u \in L^\infty(\Omega)$, then $u = 0$ a.e. in Ω.

Since $u \in L^\infty(\Omega)$, we have $au \in L^1(\Omega)$. It follows from (1.1) and Remark 3 that

$$-\Delta u + au \geq 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Recall that for $\varepsilon, \lambda > 0$ we have defined $\Omega_\lambda := \big\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \lambda\big\}$ and

$$u_\varepsilon(x) := \rho_\varepsilon * u(x) = \int_\Omega \rho_\varepsilon(x-y)u(y)\,dy \quad \forall x \in \Omega,$$

where $\rho \in C_0^\infty(B_1)$, $\rho \geq 0$ in B_1, is a radial mollifier.

Using the above notation, for $0 < \varepsilon < \lambda$, we have in Ω_λ that

$$\begin{aligned}\Delta u_\varepsilon \leq (au)_\varepsilon = au_\varepsilon + \big[(au)_\varepsilon - au_\varepsilon\big] &\leq au_\varepsilon + \big[(au)_\varepsilon - au_\varepsilon\big]^+ \\ =: au_\varepsilon + f_\varepsilon.\end{aligned} \tag{5.1}$$

Since $(au)_\varepsilon \to au$ in $L^1(\Omega)$, $u_\varepsilon \to u$ a.e. in Ω and u is bounded,

$$f_\varepsilon \to 0 \quad \text{in } L^1(\Omega). \tag{5.2}$$

Let $\delta > 0$ be a fixed number. Multiplying (5.1) by $\dfrac{1}{u_\varepsilon + \delta}$, we get

$$\frac{\Delta u_\varepsilon}{u_\varepsilon + \delta} \leq a + \frac{f_\varepsilon}{\delta} \quad \text{in } \Omega_\lambda \quad \forall \varepsilon \in (0, \lambda). \tag{5.3}$$

We also remark that

$$\frac{\nabla u_\varepsilon}{(u_\varepsilon + \delta)^2} = -\nabla\Big(\frac{1}{u_\varepsilon + \delta}\Big) \quad \text{in } \Omega. \tag{5.4}$$

Let $\varphi \in C_0^\infty(\Omega)$ and $0 < \varepsilon < \operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega)$. We now use (5.4), integration by parts, estimate (5.3) and Cauchy-Schwarz, to get

$$\begin{aligned}\int \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + \delta)^2}\varphi^2 = -\int \nabla u_\varepsilon \cdot \nabla\Big(\frac{1}{u_\varepsilon + \delta}\Big)\varphi^2 &= \int \frac{\Delta u_\varepsilon}{u_\varepsilon + \delta}\varphi^2 + \int \frac{2\varphi\nabla\varphi \cdot \nabla u_\varepsilon}{u_\varepsilon + \delta} \\ &\leq \int \Big(a + \frac{f_\varepsilon}{\delta}\Big)\varphi^2 + \frac{1}{2}\int \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + \delta)^2}\varphi^2 + 2\int |\nabla\varphi|^2.\end{aligned}$$

Therefore,

$$\frac{1}{2}\int \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + \delta)^2}\varphi^2 \leq \int \Big(a + \frac{f_\varepsilon}{\delta}\Big)\varphi^2 + 2\int |\nabla\varphi|^2.$$

Since

$$\nabla \log\Big(\frac{u_\varepsilon}{\delta} + 1\Big) = \frac{\nabla u_\varepsilon}{u_\varepsilon + \delta},$$

the estimate above may be rewritten as

$$\frac{1}{2}\int \Big|\nabla \log\Big(\frac{u_\varepsilon}{\delta} + 1\Big)\Big|^2 \varphi^2 \leq \int \Big(a + \frac{f_\varepsilon}{\delta}\Big)\varphi^2 + 2\int |\nabla\varphi|^2. \tag{5.5}$$

We now let $\varepsilon \downarrow 0$ in (5.5). It follows from (5.2) that (see also Lemma 1)

$$\log\Big(\frac{u}{\delta}+1\Big) \in H^1_{\mathrm{loc}}(\Omega) \quad \forall \delta > 0$$

and

$$\frac{1}{2}\int \Big|\nabla \log\Big(\frac{u}{\delta}+1\Big)\Big|^2 \varphi^2 \le \int \big(a\varphi^2 + 2|\nabla\varphi|^2\big) \quad \forall \varphi \in C_0^\infty(\Omega). \tag{5.6}$$

Let $E \subset \Omega$ be a set of positive capacity such that $\tilde{u} = 0$ on E. Without any loss of generality, we may assume that $E \subset \Omega_\lambda$ for some $\lambda > 0$ sufficiently small.

Assume $\omega \subset\subset \Omega$ is an open connected set containing E. Let $\varphi_0 \in C_0^\infty(\Omega)$ be a fixed test function such that $\varphi \equiv 1$ on ω.

By (5.6), we have

$$\int_\omega \Big|\nabla \log\Big(\frac{u}{\delta}+1\Big)\Big|^2 \le 2\int \big(a\varphi_0^2 + 2|\nabla\varphi_0|^2\big). \tag{5.7}$$

Since the quasicontinuous representative

$$\widetilde{\log\big(\frac{u}{\delta}+1\big)} = \log\big(\frac{\tilde{u}}{\delta}+1\big)$$

of $\log\big(\frac{u}{\delta}+1\big)$ equals 0 on $E \subset \Omega$ with $\operatorname{cap} E > 0$, it follows from a variant of Poincaré's inequality (easily proved by contradiction) that there exists $C > 0$ (depending only on E and Ω) such that

$$\int_\omega \log^2\Big(\frac{u}{\delta}+1\Big) \le C\int_\omega \Big|\nabla \log\Big(\frac{u}{\delta}+1\Big)\Big|^2 \quad \forall \delta > 0. \tag{5.8}$$

(5.7) and (5.8) yield

$$\int_\omega \log^2\Big(\frac{u}{\delta}+1\Big) \le 2C\int \big(a\varphi_0^2 + 2|\nabla\varphi_0|^2\big) \quad \forall \delta > 0. \tag{5.9}$$

In particular, the integral in the left-hand side remains bounded as $\delta \downarrow 0$.

On the other hand,

$$\log^2\Big(\frac{u}{\delta}+1\Big) \to +\infty \quad \text{a.e. in } \omega\backslash[u=0] \text{ as } \delta \downarrow 0. \tag{5.10}$$

By (5.9) and (5.10), we conclude that $u = 0$ a.e. in ω. Since ω is an arbitrary connected neighborhood of E in Ω_λ for all $\lambda > 0$ small, we conclude that $u = 0$ a.e. in Ω.

Step 2. Proof of Theorem 1 completed.

From Lemma 3, we know that $\Delta T_1(u)$ is a Radon measure and

$$-\Delta T_1(u) + aT_1(u) \ge 0 \quad \text{in } \mathcal{D}'(\Omega).$$

In addition, $\widetilde{T_1(u)} = T_1(\tilde{u}) = 0$ on $E \subset \Omega$ with $\operatorname{cap} E > 0$.

By Step 1, we have $T_1(u) = 0$ a.e. in Ω, and so $u = 0$ a.e. in Ω.

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