HOW TO RECOGNIZE CONSTANT FUNCTIONS. CONNECTIONS WITH SOBOLEV SPACES

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Dedicated to Mark Visik with esteem and friendship

1. Introduction

Most of the ideas in this paper are coming from a series of recent collaborations with J. Bourgain, Y. Li, P. Mironescu and L. Nirenberg (see J. Bourgain, H. Brezis and P. Mironescu [1], [2], [3], [4], H. Brezis and L. Nirenberg [1], H. Brezis, Y. Li, P. Mironescu and L. Nirenberg [1]). However we will adopt here on slightly different presentation and provide some simplified proofs.

The starting point is the following

Proposition 1. *Let Ω be a connected open set in $\mathbb{R}^N$ and let $f : \Omega \to \mathbb{R}$ be a measurable function such that*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|^{N+1}} dx\, dy < \infty, \tag{1}$$

then f is a constant.

The original motivation for such a proposition was twofold:

(i) *Uniqueness of lifting.* Given a (measurable) function $u : \Omega \to \mathbb{C}$ such that $|u| = 1$ a.e., there are many liftings φ, i.e., $u = e^{i\varphi}$. If φ_1, φ_2 are 2 liftings then

$$k(x) = \frac{1}{2\pi} (\varphi_1(x) - \varphi_2(x)) : \Omega \to \mathbb{Z}.$$

Under further assumptions one may hope to prove that k is a *constant* function. For example, if φ_1, φ_2 are continuous and Ω is connected, then k is constant. The message I wish to convey is that the continuity assumption can be replaced by a different type of condition, such as (1), which is much more natural in the framework of Sobolev spaces (see Remark 3).

(ii) *A degree theory for classes of discontinuous maps.* The possibility of defining a degree for maps in Sobolev spaces (see H. Brezis and J.M. Coron [1], H. Brezis, Y. Li, P. Mironescu and L. Nirenberg [1]), is based on the fact deg $h_t(\cdot)$ remains constant along a homotopy $h_t(\cdot)$, as t varies in $[0, 1]$ (or more generally in a connected parameter space Λ). Such a conclusion holds possibly in situations where the dependence in t need not be continuous.

Remark 1. The conclusion of Proposition 1 is easy to state, but I do not know a direct, elementary, proof. Our proof is not very complicated but requires an "excursion" via the Sobolev spaces.

Remark 2. The connectedness assumption is of course needed. The conclusion of Proposition 1 still holds if in (1) $N+1$ is replaced by $q \geq N+1$. Indeed, it suffices to prove Proposition 1 when Ω is a ball B (and complete the general case via connectedness); then

$$\frac{1}{|x-y|^{N+1}} \leq \frac{C}{|x-y|^q} \qquad \forall x, y \in B.$$

(However the conclusion still holds in some non connected domains, for example $\Omega = G\backslash\Sigma$ where G is connected and Σ is closed with meas $\Sigma = 0$. It would be interesting to study non connected domains where the conclusion of Proposition 1 holds).

On the other hand, if in (1) $N+1$ is replaced by $q < N+1$, then the conclusion fails. Indeed, for *any* Lipschitz function on B one has

$$\int_B \int_B \frac{|f(x)-f(y)|}{|x-y|^q} dx\, dy \leq C \int_B \int_B \frac{dx\, dy}{|x-y|^{q-1}} < \infty$$

since $q < N+1$.

There are many consequences and variants of Proposition 1. Here are a few.

Corollary 1. *Assume Ω is a connected open set in $\mathbb{R}^N$, and let $f : \Omega \to \mathbb{Z}$ be a measurable function such that*

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+1}} dx\, dy < \infty, \tag{2}$$

for some $1 \leq p < \infty$, then f is a constant.

Proof. Observe that

$$|f(x)-f(y)|^p \geq |f(x)-f(y)|$$

since $f(x) - f(y) \in \mathbb{Z}$.

Remark 3. When $p > 1$, condition (2) says that f belongs to the fractional Sobolev space $W^{s,p}$ (see e.g. Adams [1]) with $s = 1/p$. Therefore, we may assert that any function in $W^{s,p}(\Omega;\mathbb{Z})$ with $sp \geq 1$ is a constant. Note that the condition $sp \geq 1$ is *considerably weaker* than the condition $sp > N$ which implies (via the Sobolev embedding theorem) that f is continuous. Corollary 1 is originally due to R. Hardt, D. Kinderlehrer and F.H. Lin [1] (Lemma 1.1) when $p = 2$ and $s = 1/2$ (they attribute it to Wiener when $N = 2$). Bethuel and Demengel [1] had obtained a similar conclusion under the stronger assumption $sp > 1$.

Corollary 2. *Assume Ω is a connected open set in $\mathbb{R}^N$ and A is any measurable subset such that*

$$\int_A \int_{{}^cA} \frac{dx\, dy}{|x-y|^{N+1}} < \infty \tag{3}$$

then either meas$(A) = 0$ or meas$(\Omega\backslash A) = 0$.

It suffices to apply Proposition 1 to $f = \chi_A$, the characteristic function of A. Note that in (3), $(N+1)$ is again optimal. If A is any subset of Ω with smooth boundary, then (3) holds if $(N+1)$ is replaced by any $q < N+1$ (it suffices to consider the case where ∂A is flat and to make an explicit computation).

Now some variants of Proposition 1.

Proposition 2. *Assume Ω is a connected open set in $\mathbb{R}^N$ and $f : \Omega \to \mathbb{R}$ is a measurable function such that*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+p}} dx\, dy < \infty, \tag{4}$$

for some $1 \le p < \infty$, then f is constant.

[Proposition 1 corresponds to the case $p = 1$].

Still a further generalization

Proposition 3. *Assume Ω is a connected open set in $\mathbb{R}^N$ and $f : \Omega \to \mathbb{R}$ is a measurable function such that*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \psi(|x-y|) dx\, dy < \infty, \tag{5}$$

where $p \ge 1$ and $\psi \in L^1_{loc}(0,\infty)$, $\psi \ge 0$ satisfies

$$\int_0^1 \psi(r) r^{N-1} dr = \infty, \tag{6}$$

then f is a constant.

[Proposition 2 corresponds to the case $\psi(r) = r^{-N}$].

Here is one important generalization of Proposition 2.

Proposition 4. *Assume Ω is a connected open set in $\mathbb{R}^N$ and $f : \Omega \to \mathbb{R}$ is a measurable function such that*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+p-\varepsilon}} dx\, dy = o\left(\frac{1}{\varepsilon}\right) \text{ as } \varepsilon \to 0, \tag{7}$$

i.e.,

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+p-\varepsilon}} dx\, dy = 0 \tag{7'}$$

for some $p \ge 1$, then f is a constant.

Remark 4. Assumption (7) is clearly much weaker than (4) (when Ω is bounded) which says that

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+p-\varepsilon}} dx\, dy = 0(1) \text{ as } \varepsilon \to 0,$$

On the other hand (7) is optimal since for any Lipschitz function f on Ω

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+p-\varepsilon}} dx\, dy = 0\left(\frac{1}{\varepsilon}\right) \tag{8}$$

because

$$\int_0^1 \frac{1}{r^{N-\varepsilon}} r^{N-1} dr = \frac{1}{\varepsilon}.$$

Here is a final generalization, which brings us closer to the connection with Sobolev spaces.

Theorem 1. *Assume Ω is a connected open set in $\mathbb{R}^N$ and $f : \Omega \to \mathbb{R}$ is a measurable function. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be a sequence of radial mollifiers, i.e.*

$$\rho_\varepsilon \in L^1_{loc}(0,\infty), \quad \rho_\varepsilon \geq 0, \tag{9}$$

$$\int_0^\infty \rho_\varepsilon(r) r^{N-1} dr = 1 \qquad \forall \varepsilon > 0, \tag{10}$$

$$\text{for every } \delta > 0, \qquad \lim_{\varepsilon\to 0} \int_\delta^\infty \rho_\varepsilon(r) r^{N-1} dr = 0. \tag{11}$$

Assume that, for some $p \geq 1$,

$$\lim_{\varepsilon\to 0} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|) dx\, dy = 0. \tag{12}$$

Then f is a constant.

Note that Proposition 4 is a consequence of Theorem 1 when choosing

$$\rho_\varepsilon(r) = \begin{cases} \varepsilon r^{-N+\varepsilon}, & r < 1 \\ 0 \quad , & r > 1. \end{cases}$$

And Proposition 3 is also a consequence of Theorem 1 when choosing

$$\rho_\varepsilon(r) = \begin{cases} 0 & \text{if } r < \varepsilon \\ a_\varepsilon \psi(r) & \text{if } \varepsilon < r < 1 \\ 0 & \text{if } r > 1, \end{cases}$$

where

$$a_\varepsilon = \left(\int_\varepsilon^1 \psi(r) r^{N-1} dr\right)^{-1} \to 0 \quad \text{as } \varepsilon \to 0. \tag{13}$$

Note that, in view of (5),

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|) dx\, dy \le C a_\varepsilon \to 0 \text{ as } \varepsilon \to 0, \text{ by (13)}.$$

The proof of Theorem 1 involves an excursion into Sobolev spaces which we will now describe.

2. A new characterization of Sobolev spaces

For simplicity, we start with the case of all of $\mathbb{R}^N$. Let $f \in L^p(\mathbb{R}^N)$, $1 < p < \infty$. It is well-know (see e.g. H. Brezis [1], Proposition IX.3) that if $f \in W^{1,p}(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} |f(x+h) - f(x)|^p dx \le |h|^p \int_{\mathbb{R}^N} |\nabla f|^p dx \qquad \text{for every } h \in \mathbb{R}^N. \tag{14}$$

And conversely, if $f \in L^p(\mathbb{R}^N)$ and if there exists a constant C such that

$$\int_{\mathbb{R}^N} |f(x+h) - f(x)|^p dx \le C|h|^p \text{ as } h \to 0, \tag{15}$$

then $f \in W^{1,p}(\mathbb{R}^N)$.

When $p = 1$, $W^{1,1}$ should be replaced by BV, the space of functions in L^1 who's derivatives (in the sense of distributions) are bounded Radon measures; thus $f \in BV$ if and only if

$$\int_{\mathbb{R}^N} |f(x+h) - f(x)| dx \le C|h| \text{ as } |h| \to 0, \tag{16}$$

and then (16) holds for all $h \in \mathbb{R}^N$ with $C = \int |\nabla f| dx$. In particular, if ρ_ε satisfies (9), (10) and $f \in W^{1,p}$, we have

$$\int_{\mathbb{R}^N} \rho_\varepsilon(|h|) dh \int_{\mathbb{R}^N} \frac{|f(x+h) - f(x)|^p}{|h|^p} dx \le C \text{ as } \varepsilon \to 0, \tag{17}$$

since

$$\int_{\mathbb{R}^N} \rho_\varepsilon(|h|) dh = \sigma_N \int_0^\infty \rho_\varepsilon(r) r^{N-1} dr = \sigma_N$$

where $\sigma_N = |S^{N-1}|$.

Changing variables in (17) yields

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|) dx\, dy \le C \text{ as } \varepsilon \to 0. \tag{18}$$

Similarly, if $f \in BV$, we have

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)-f(y)|}{|x-y|} \rho_\varepsilon(|x-y|) dx\, dy \le C \text{ as } \varepsilon \to 0. \tag{19}$$

The heart of the matter is that (18) (resp. (19)) gives a characterization of $W^{1,p}$ when $p > 1$ (resp. BV).

Theorem 2. *Assume $f \in L^p(\mathbb{R}^N)$ satisfies (18) with $p > 1$. Let (ρ_ε) be as in (9)-(10)-(11). Then $f \in W^{1,p}$ and*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(|x - y|) dx\, dy = K_{p,N} \int_{\mathbb{R}^N} |\nabla f|^p dx \tag{20}$$

where $K_{p,N}$ depends only on p and N.

Similarly for $p = 1$ we have

Theorem 3. *Assume $f \in L^1(\mathbb{R}^N)$ satisfies (19). Let (ρ_ε) be as in (9)-(10)-(11). Then $f \in BV$ and*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|}{|x - y|} \rho_\varepsilon(|x - y|) dx\, dy = K_{1,N} \int_{\mathbb{R}^N} |\nabla f| dx \tag{21}$$

where the right-hand side denote the total mass of the measure ∇f.

An interesting consequence of Theorem 3 is the following

Corollary 3. *Let A be a bounded measurable set in $\mathbb{R}^N$. Then A has finite perimeter (in the sense of De Giorgi) if and only if*

$$\int_A \int_{{}^cA} \frac{1}{|x - y|} \rho_\varepsilon(|x - y|) dx\, dy \le C \quad \text{as} \quad \varepsilon \to 0$$

and then

$$\lim_{\varepsilon \to 0} \int_A \int_{{}^cA} \frac{1}{|x - y|} \rho_\varepsilon(|x - y|) dx\, dy = K_{1,N} Per(A). \tag{22}$$

Proof of Theorem 2. The original proof of Theorem 2 is to be found in Bourgain, Brezis and Mironescu [3]. We present here a simpler argument suggested by E. Stein [1]. Assume $f \in L^p$ satisfies (18) an let (γ_δ) be any sequence of smooth mollifiers. Set

$$f_\delta = \gamma_\delta \star f.$$

Note that (18) still holds when f is replaced by its translates $(\tau_h f)(x) = f(x + h)$. Also, (18) is stable under convex combinations and thus f_δ satisfies (18) with the *same* constant C, i.e., we have

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f_\delta(x) - f_\delta(y)|^p}{|x - y|^p} \rho_\varepsilon(|x - y|) dx\, dy \le C \tag{23}$$

where C is *independent* of ε *and* δ.

Next, let $g \in C^2(\mathbb{R}^N)$ be such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)dx\,dy \le C \quad \text{as} \quad \varepsilon \to 0, \tag{24}$$

where ρ_ε satisfies (9), (10), (11). We claim that

$$\int_{\mathbb{R}^N} |\nabla g(x)|^p dx \le C/K_{p,N}, \tag{25}$$

with C taken from (24) and

$$K_{p,N} = \int_{S^{N-1}} |(\sigma \cdot e)|^p d\sigma, \quad e \in S^{N-1}. \tag{26}$$

Proof of (25). Let K be any compact subset of $\mathbb{R}^N$. For $x \in K$ and $|h| \le 1$ we have

$$|g(x+h) - g(x) - h \cdot \nabla g(x)| \le C_K |h|^2. \tag{27}$$

From (24) we have

$$\int_K dx \int_{|h|\le 1} \frac{|g(x+h)-g(x)|^p}{|h|^p}\rho_\varepsilon(|h|)dh \le C. \tag{28}$$

By (27) we have

$$|h \cdot \nabla g(x)| \le |g(x+h) - g(x)| + C_K|h|^2$$

and therefore, for every $\theta > 0$

$$|h \cdot \nabla g(x)|^p \le (1+\theta)|g(x+h)-g(x)|^p + C_{\theta,K}|h|^{2p}.$$

Combining this with (28) yields

$$\int_K dx \int_{|h|\le 1} \frac{|(h\cdot\nabla g(x))|^p}{|h|^p}\rho_\varepsilon(|h|)dh \le (1+\theta)C + C_{\theta,K}|K| \int_{|h|\le 1} |h|^p \rho_\varepsilon(|h|)dh. \tag{29}$$

But, for any vector $V \in \mathbb{R}^N$,

$$\int_{|h|\le 1} \frac{|(h\cdot V)|^p}{|h|^p}\rho_\varepsilon(|h|)dh = K_{p,N}|V|^p \int_0^1 \rho_\varepsilon(r) r^{N-1} dr.$$

On the other hand, it is clear from (10) and (11) that

$$\lim_{\varepsilon\to 0}\int_{|h|\le 1} |h|^p \rho_\varepsilon(|h|)dh = 0.$$

Passing to the limit as $\varepsilon \to 0$ in (29) we find

$$K_{p,N} \int_K |\nabla g(x)|^p dx \leq (1+\theta)C. \tag{30}$$

Since (30) holds for every $\theta > 0$ and every compact set K (with C independent of θ and K) we obtain (25), that is,

$$K_{p,N} \int_{\mathbb{R}^N} |\nabla g(x)|^p dx \leq \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|) dx\, dy. \tag{31}$$

On the other hand, if $g \in C_0^2(\mathbb{R}^N)$ we have, as above,

$$|g(x+h) - g(x)| \leq |h \cdot \nabla g(x)| + C'|h|^2 \qquad \forall\, x \in \mathbb{R}^N,\ \forall\, h \in \mathbb{R}^N.$$

Hence

$$|g(x+h) - g(x)|^p \leq (1+\theta)|h \cdot \nabla g(x)|^p + C'_\theta |h|^{2p}.$$

We multiply this by $\rho_\varepsilon(|h|)/|h|^p$ and integrate over the set $\{(x,h) \in \mathbb{R}^{2N} : x \text{ or } x+h \in \operatorname{supp} g\}$ to obtain

$$\int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{|g(x+h)-g(x)|^p}{|h|^p} \rho_\varepsilon(|h|) dh \leq$$

$$(1+\theta) \int_{\mathbb{R}^N} K_{p,N} |\nabla g(x)|^p dx + 2C'_\theta |\operatorname{supp} g| \int_{\mathbb{R}^N} |h|^p \rho_\varepsilon(|h|) dh.$$

We first let $\varepsilon \to 0$ and then $\theta \to 0$. This yields

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{|g(x+h)-g(x)|^p}{|h|^p} \rho_\varepsilon(|h|) dh \leq K_{p,N} \int_{\mathbb{R}^N} |\nabla g(x)|^p dx. \tag{32}$$

Combining (31) and (32) yields, for every $g \in C_0^2(\mathbb{R}^N)$,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|g(x)-g(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|) dx\, dy = K_{p,N} \int_{\mathbb{R}^N} |\nabla g(x)|^p dx.$$

Since $C_0^2(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, it is easy to conclude (using (14)) that (20) holds for every $f \in W^{1,p}(\mathbb{R}^N)$.

We may now complete the proof of Theorem 2. Assuming $f \in L^p(\mathbb{R}^N)$ satisfies (18) and applying Claim (25) to $g = f_\delta$ we see that

$$\int_{\mathbb{R}^N} |\nabla f_\delta|^p dx \leq C/K_{p,N}, \tag{33}$$

where C comes from (18). Finally, we pass to the limit in (33) as $\delta \to 0$ and obtain $f \in W^{1,p}$.

Proof of Theorem 3. If $f \in L^1(\mathbb{R}^N)$ and satisfies (19) and we proceed as above we are led to

$$\int_{\mathbb{R}^N} |\nabla f_\delta| dx \le C/K_{1,N}.$$

Therefore $f \in BV$ and

$$\int_{\mathbb{R}^N} |\nabla f| dx \le C/K_{1,N}.$$

In other words we have proved that

$$K_{1,N} \int_{\mathbb{R}^N} |\nabla f| dx \le \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|}{|x - y|} \rho_\varepsilon(|x - y|) dx\, dy. \tag{34}$$

On the other hand it is easy to see, using (16), that for $f \in BV$

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|}{|x - y|} \rho_\varepsilon(|x - y|) dx\, dy \le \tilde{K}_N \int_{\mathbb{R}^N} |\nabla f| dx. \tag{35}$$

Unfortunately the constant $\tilde{K}_N$ in (35) is not the same as $K_{1,N}$. It is also clear that (21) holds when $f \in C_0^2(\mathbb{R}^N)$. However we cannot conclude easily that (21) holds for every $f \in BV$ since $C_0^2(\mathbb{R}^N)$ is *not* dense in BV.

It remains to be shown that, for every $f \in BV(\mathbb{R}^N)$

$$\limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x) - f(y)|}{|x - y|} \rho_\varepsilon(|x - y|) dx\, dy \le K_{1,N} \int_{\mathbb{R}^N} |\nabla f| dx.$$

This has been established by J. Davila [1] using new ideas which are not presented here.

Remark 5. There are statements similar to Theorem 2 and Theorem 3 when $\mathbb{R}^N$ is replaced by a *smooth* bounded domain Ω in $\mathbb{R}^N$. However the same conclusion *fails* for a general bounded domain Ω if $\partial\Omega$ is *not smooth*. It is still *true* (for a general Ω) that

$$K_{p,N} \int_\Omega |\nabla f|^p \le \liminf_{\varepsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^p} \rho_\varepsilon(|x - y|) dx\, dy. \tag{36}$$

However, it may happen for $p > 1$ that $f \in W^{1,p}(\Omega)$ (so that the left hand side in (36) is *finite*) while the right-hand side in (36) is *infinite*. Here is such an example. Let $\Omega = D \backslash \Sigma$ where D is a disc (in $\mathbb{R}^2$) and Σ is a slit. Let f be a smooth function in Ω which is discontinuous across the slit (for example two different constants on each side of the slit). Clearly $f \in W^{1,p}(\Omega)$, but the RHS in (36) is infinite. This is so because

$$\int_\Omega \int_\Omega \ldots = \int_D \int_D \ldots$$

and if the RHS in (36) were finite we would conclude that $f \in W^{1,p}(D)$ (by Theorem 2), which is obviously wrong. This example suggests the following

Open problem 1. Let $\Omega \subset \mathbb{R}^N$ be a bounded connected set (not necessarily smooth). Let $\delta(x,y)$ denote the *geodesic distance* in Ω. Let $f \in L^p(\Omega)$ be such that

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{\delta(x,y)^p} \rho_\varepsilon(\delta(x,y))dx\,dy \le C \text{ as } \varepsilon \to 0.$$

Does it follow that $f \in W^{1,p}$ and if so, does one have

$$\lim_{\varepsilon\to 0} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{\delta(x,y)^p} \rho_\varepsilon(\delta(x,y))dx\,dy = K_{p,N} \int_\Omega |\nabla f|^p dx?$$

Remark 6. The characterization of $W^{1,p}$ (resp. BV) given by Theorem 2 (resp. 3) suggests a definition of Sobolev spaces for maps $f : M \to \tilde{M}$ between metric spaces, where M is equipped with a measure μ, namely

$$\int \int \frac{\tilde{d}(f(x),f(y))^p}{d(x,y)^p} \rho_\varepsilon(d(x,y))d\mu(x)d\mu(y) \le C \text{ as } \varepsilon \to 0.$$

Note that assumptions (10) and (11) involve the notion of a dimension N but this can be done easily by considering $\lim_{r\to 0} |log\, \mu(B_r(x))|/|log\, r|$. It would be interesting to study the properties of such maps (Sobolev imbeddings, etc...) and to compare this notion with other definitions (see Korevaar and Schoen [1], P. Hajlasz and P.Koskela [1], L. Ambrosio and P.Tilli [1] and the numerous references in these works).

Remark 7. There are variants of Theorems 2 and 3 when Ω is a *smooth* bounded domain in $\mathbb{R}^N$. For example, we have

Theorem 2'. *Assume $f \in L^p(\Omega)$ satisfies*

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|)dx\,dy \le C \text{ as } \varepsilon \to 0, \tag{37}$$

with ρ_ε as in (9), (10), (11). Then $f \in W^{1,p}(\Omega)$ and

$$\lim_{\varepsilon\to 0} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|)dx\,dy = K_{p,N} \int_\Omega |\nabla f|^p. \tag{38}$$

Sketch of proof. First assume that (37) holds. By a standard technique of reflection across the boundary and multiplication by a cut-off one constructs a function $\tilde{f}$ on $\mathbb{R}^N$, with compact support, such that $\tilde{f} = f$ on Ω and satisfying

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{f}(x)-\tilde{f}(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|)dx\,dy \le C' \text{ as } \varepsilon \to 0, \tag{39}$$

By Theorem 2 we conclude that $\tilde{f} \in W^{1,p}(\mathbb{R}^N)$ and thus $f \in W^{1,p}(\Omega)$.

Next one shows that if $f \in C^2(\overline{\Omega})$, then

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|) dx\, dy \le C(\Omega) \int_\Omega |\nabla f|^p dx. \tag{40}$$

Finally one proves that if $f \in C^2(\overline{\Omega})$

$$\lim_{\varepsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|) dx\, dy = K_{p,N} \int_\Omega |\nabla f|^p dx. \tag{41}$$

The conclusion of Theorem 2' follows from an easy density argument.

Remark 8. There are several choices for ρ_ε which are of interest. Here are a few

A) *Choice 1*

$$\rho_\varepsilon(r) = \begin{cases} \dfrac{\varepsilon}{r^{N-\varepsilon}} & 0 < r < 1 \\ 0 & r > 1. \end{cases}$$

This choice yields

Corollary 4. *Assume Ω is a smooth bounded domain in $\mathbb{R}^N$. Let $f \in L^p(\Omega)$ be such that*

$$\varepsilon \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+p-\varepsilon}} dx\, dy \le C \text{ as } \varepsilon \to 0,$$

then $f \in W^{1,p}(\Omega)$ and

$$\lim_{\varepsilon \to 0} \varepsilon \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+p-\varepsilon}} dx\, dy = K_{p,N} \int_\Omega |\nabla f|^p. \tag{42}$$

Recall that the standard fractional Sobolev space $W^{s,p}$, $0 < s < 1$, $1 < p < \infty$, is equipped with Gagliardo (semi) norm

$$\|f\|^p_{W^{s,p}} = \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+sp}} dx\, dy. \tag{43}$$

It is well-known that $\|f\|_{W^{s,p}}$ does *not* converge to $\|f\|_{W^{1,p}}$ as $s \uparrow 1$; in fact it converges to ∞ (unless f is constant) by Proposition 2. However in view of Corollary 4 we may now assert that

$$\lim_{s \uparrow 1} (1-s) \|f\|^p_{W^{s,p}} = \frac{K_{p,N}}{p} \int_\Omega |\nabla f|^p. \tag{44}$$

This "reinstates" $W^{1,p}$ as a continuous limit of $W^{s,p}$ as $s \uparrow 1$ provided one uses the norm $(1-s)^{1/p} \|f\|_{W^{s,p}}$ on $W^{s,p}$.

B) *Choice 2*

$$\rho_\varepsilon(r) = \begin{cases} \dfrac{N}{\varepsilon^N} & \text{if } r < \varepsilon \\ 0 & \text{if } r > \varepsilon \end{cases}$$

This choice yields

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \int_\Omega \int_\Omega_{|x-y|<\varepsilon} \frac{|f(x)-f(y)|^p}{|x-y|^p} dx\, dy = \frac{K_{p,N}}{N} \int_\Omega |\nabla f|^p. \tag{45}$$

A variant is

$$\rho_\varepsilon(r) = \begin{cases} \dfrac{(N+p)r^p}{\varepsilon^{N+p}} & r < \varepsilon \\ 0 & r > \varepsilon \end{cases}$$

and then we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N+p}} \int_\Omega \int_\Omega_{|x-y|<\varepsilon} |f(x)-f(y)|^p dx\, dy = \frac{K_{p,N}}{(N+p)} \int_\Omega |\nabla f|^p. \tag{46}$$

Still another choice yields

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N+p}} \int_\Omega \int_\Omega_{\varepsilon<|x-y|<2\varepsilon} |f(x)-f(y)|^p dx\, dy = \tilde{K}_{p,N} \int_\Omega |\nabla f|^p. \tag{47}$$

C) *Choice 3*

$$\rho_\varepsilon(r) = \begin{cases} 0 & r < \varepsilon \\ \dfrac{1}{|log\, \varepsilon| r^N} & \varepsilon < r < 1 \\ 0 & r > 1. \end{cases}$$

This choice yields

$$\lim_{\varepsilon\to 0} \frac{1}{|log\, \varepsilon|} \int_\Omega \int_\Omega_{|x-y|>\varepsilon} \frac{|f(x)-f(y)|^p}{|x-y|^{N+p}} dx\, dy = K_{p,N} \int_\Omega |\nabla f|^p. \tag{48}$$

D) *Choice 4*

Let $\gamma \in L^1_{loc}(0, +\infty)$, $\gamma \geq 0$, be such that

$$\int_0^\infty \gamma(r) r^{N+p-1} dr = 1.$$

Choosing

$$\rho_\varepsilon(r) = \frac{1}{\varepsilon^{N+p}}\, \gamma\left(\frac{r}{\varepsilon}\right) r^p$$

yields

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N+p}}\int_\Omega\int_\Omega |f(x)-f(y)|^p\,\gamma\left(\frac{|x-y|}{\varepsilon}\right)dx\,dy = K_{p,N}\int_\Omega |\nabla f|^p,$$

for every $f\in W^{1,p}$ (with $p>1$) and for every $f\in BV$ (with $p=1$). Applying this in the BV case with $f=\chi_A$ we obtain a new *characterization* of sets of *finite perimeter*. Namely a measurable set $A\subset\Omega$ has finite perimeter if and only if

$$\frac{1}{\varepsilon^{N+1}}\int_A\int_{^cA}\gamma\left(\frac{|x-y|}{\varepsilon}\right)dx\,dy \le C \text{ as } \varepsilon\to 0,$$

and then

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{N+1}}\int_A\int_{^cA}\gamma\left(\frac{|x-y|}{\varepsilon}\right)dx\,dy = K_{1,N}\mathrm{Per}(A).$$

3. Back to constant functions

All the results of Section 1 are immediate consequences of the statements of Section 2 applied in a ball $B\subset\Omega$. One concludes that f is constant on B and then that f is constant on Ω since Ω is connected.

Note that the assumption

$$\lim_{\varepsilon\to 0}\int_B\int_B\frac{|f(x)-f(y)|}{|x-y|}\rho_\varepsilon(|x-y|)dx\,dy = 0 \tag{49}$$

implies first that $f\in BV$ and then that $\nabla f=0$, so that f is a constant.

By *contrast*, when $p>1$, and f *takes its values into* $\mathbb{Z}$ it suffices to assumes that

$$\int_B\int_B\frac{|f(x)-f(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)dx\,dy \le C \text{ as } \varepsilon\to 0. \tag{50}$$

Indeed, (50) implies that $f\in W^{1,p}$ (attention when $p=1$, (50) only implies that $f\in BV$). Then, one may use the fact that f takes its values into $\mathbb{Z}$ to conclude that f is constant. The argument is the following: write

$$\Omega = \bigcup_{k\in\mathbb{Z}} A_k$$

where $A_k=\{x\in\Omega; f(x)=k\}$ and use a well-known result of Stampacchia (see e.g. Lemma 7.7 in Gilbarg–Trudinger [1]) asserting that $\nabla f=0$ a.e. on A_k. Hence $\nabla f=0$ a.e. on Ω.

Alternatively, one may deduce from (50) and assumption $f:\Omega\to\mathbb{Z}$, that

$$\int_\Omega\int_\Omega\frac{|f(x)-f(y)|}{|x-y|}\,\frac{\rho_\varepsilon(|x-y|)}{|x-y|^{p-1}}dx\,dy\le C.$$

This yields easily

$$\lim_{\varepsilon\to 0}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\rho_\varepsilon(|x-y|)dx\,dy = 0$$

and thus f is a constant.

There are interesting extensions of some of the above results where the ratio

$$\frac{|f(x)-f(y)|^p}{|x-y|^p}$$

is replaced by a more general expression

$$\omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right).$$

Here are two results due to R. Ignat, V. Lie and A. Ponce [1].

Theorem 4. *Assume $\omega : [0,\infty) \to [0,\infty)$ is a continuous function such that $\omega(0) = 0$, $\omega(t) > 0\ \forall t > 0$ and*

$$\int_1^\infty \frac{\omega(t)}{t^2}dt = \infty. \tag{51}$$

Assume $f \in L^1(\Omega)$ satisfies

$$\int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\frac{dx\,dy}{|x-y|^N} < \infty,$$

then f is a constant.

Theorem 5. *Assume $\omega : [0,\infty) \to [0,\infty)$ is a continuous function such that $\omega(0) = 0$ and*

$$\lim_{t\to\infty}\frac{\omega(t)}{t} = \alpha > 0.$$

Assume $f \in L^1(\Omega)$ satisfies

$$\int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(|x-y|)dx\,dy \le C \text{ as } \varepsilon \to 0.$$

Then $f \in BV$ and

$$\lim_{\varepsilon\to 0}\int_\Omega\int_\Omega \omega\left(\frac{|f(x)-f(y)|}{|x-y|}\right)\rho_\varepsilon(|x-y|)dx\,dy = \int_\Omega \overline{\omega}(|\nabla f_{ac}|)dx + \alpha K_{1,N}\int_\Omega |\nabla f_s|dx,$$

where $\overline{\omega}(t) = \displaystyle\int_{S^{N-1}} \omega(t|\sigma\cdot e|)d\sigma$ and $\nabla f = \nabla f_{ac} + \nabla f_s$ is the Radon–Nikodym decomposition of ∇f.

Here is still another open problem:

Open problem 2. Let Ω be a (smooth) connected, bounded domain in $\mathbb{R}^N$. Let $f : \Omega \to \mathbb{R}$ be a continuous (or even Hölder continuous) function. Let $\omega : [0, \infty) \to [0, \infty)$ be a continuous function such that $\omega(0) = 0$ and $\omega(t) > 0$ for $t > 0$.(Here (51) might fail). Assume that

$$\int_\Omega \int_\Omega \omega\left(\frac{|f(x) - f(y)|}{|x - y|}\right) \frac{1}{|x - y|^N}\, dx\, dy < \infty.$$

Can one conclude that f is a constant?

4. Another approach. Connection with VMO

We first recall the definition of VMO($\Omega;\mathbb{R}$) (= vanishing mean oscillation). We say that a function $f \in VMO(\Omega;\mathbb{R})$ if $f \in L^1_{loc}(\Omega;\mathbb{R})$ satisfies

$$\lim_{\varepsilon\to 0} \frac{1}{|B_\varepsilon(x)|^2} \int_{B_\varepsilon(x)} \int_{B_\varepsilon(x)} |f(y) - f(z)| dy\, dz = 0 \quad \text{uniformly for } x \in \Omega.$$

Let Ω be a connected (smooth) open set in $\mathbb{R}^N$ and let $f \in VMO(\Omega;\mathbb{Z})$. Then f is a constant. This was already observed in Brezis–Nirenberg [1] (Section I.5, part 2). Indeed if we set

$$\overline{f}_\varepsilon(x) = \frac{1}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} f(y) dy$$

then $\text{dist}(\overline{f}_\varepsilon(x), \mathbb{Z}) \to 0$ *uniformly* in Ω (see Brezis–Nirenberg [1], Section I.1) and thus there is some constant $k_\varepsilon \in \mathbb{Z}$ such that $|\overline{f}_\varepsilon(x) - k_\varepsilon| \to 0$ uniformly in Ω. Hence f is a constant.

Functions in $W^{s,p}(\Omega)$ belong to $VMO(\Omega)$ provided $sp \geq N$ (see Brezis–Nirenberg [1], Section I.2). Therefore one cannot apply directly this argument in our setting which corresponds roughly speaking to $sp \geq 1$. However one may use an argument of *reduction to dimension one* already used in Bourgain–Brezis–Mironescu [2].

Assume for simplicity that Ω is a square in $\mathbb{R}^2$. Let $f \in W^{s,p}(\Omega)$. Then, the restrictions $f(x_1, \cdot)$ and $f(\cdot, x_2)$ still belong to $W^{s,p}(I)$ for a.e. x_1 and a.e. x_2 (where I is an interval) (see e.g. Brezis, Li, Mironescu and Nirenberg [1], Section 2).

This observation is very useful when combined with the following measure theoretical tool:

Lemma (see e.g. Brezis, Li, Mironescu and Nirenberg [1], Lemma 2). *Assume that $f : \Omega \to \mathbb{R}$ is measurable. Suppose that for a.e. x_1, $f(x_1, \cdot)$ and for a.e. x_2, $f(\cdot, x_2)$ are constant functions. Then f is a constant.*

The considerations above yield an alternative proof of Corollary 1 when $p > 1$. Indeed, if $p > 1$, (2) says that $f \in W^{s,p}(\Omega)$ where $s = 1/p$. The restrictions of f to almost every line still belong to $W^{s,p}$ with $s = 1/p$. Hence these restrictions are VMO.

Therefore, if $f : \Omega \to \mathbb{Z}$ one may conclude that the restrictions of f to almost every line are constant. The above lemma allows to conclude that f is constant.

The preceding argument also gives

Theorem 6. *Assume* $\Omega \subset \mathbb{R}^N$ *is connected and let* $f : \Omega \to \mathbb{Z}$ *be a measurable function such that* $f = f_0 + f_1 + f_2 + ... + f_k$ *where* $f_0 \in W^{1,1}(\Omega;\mathbb{R})$ *and* $f_i \in W^{s_i,p_i}(\Omega;\mathbb{R})$ *with* $s_i p_i \geq 1$ *for* $i = 1, 2, ..., k$. *Then* f *is a constant.*

Open problem 3. Is there a simple intrinsic assumption on f which can replace the decomposition assumption $f = f_0 + f_1 + f_2 + ... + f_k$? Is there an elegant way to unify Theorem 6 with the results of Section 1?

Another interesting direction of research is

Open problem 4. Find estimates for

$$\|f - \fint f\|$$

in terms of the quantities appearing throughout the paper and which would imply that f is constant in various situations. The reader may find some results in that direction in Bourgain, Brezis and Mironescu [4] (see also Maz'ya and Shaposhnikova [1]).

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