Mortar element coupling between global scalar and local vector potentials to solve eddy current problems

Yvon Maday[1], Francesca Rapetti[1], and Barbara I. Wohlmuth[2]

[1] Laboratoire J.-L. Lions, CNRS & Paris 6 University, Boîte courrier 187, 75252 Paris cedex 05, France
[2] Laboratoire J. A. Dieudonne, CNRS & Nice et Sophia Antipolis University, Parc Valrose, 06108 Nice cedex 2, France
[3] Mathematisches Institut A, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

Abstract. The $T - \Omega$ formulation of the magnetic field has been introduced in many papers for the approximation of the magnetic quantities modeled by the eddy current equations. This decomposition allows to use a scalar function in the main part of the computational domain, reducing the use of vector quantities in the conducting parts. We propose here to approximate these two quantities on different and non-matching grids so as to be able for instance to tackle a problem where the conducting part can move in the global domain. The connection between the two grids is managed with the mortar element tools. The numerical analysis will be presented resulting in error bounds of the solution.

1 Introduction

The numerical simulation of low frequency electromagnetic devices can be based on the eddy current model, see [1,2,4]. Two main families of formulations are widely used, the one based on magnetic and the one based on electric fields. Here, we restrict ourselves to the magnetic field approach and we extend what has been briefly presented in [8]. The entire space $\mathbb{R}^3$ is decomposed in the conducting region V_c and the external region $\mathbb{R}^3 \setminus V_c$. Then, the quasi-stationary Maxwell equations restricted to the conducting region read:

$$\nabla \times H = J, \qquad \nabla \times E = -\partial_t B, \qquad \nabla \cdot B = 0\,, \qquad \text{in} \quad V_c\,, \tag{1}$$

where H, B, J and E denote the magnetic field, the magnetic flux density, the current density and the electric field. The densities and the fields are linked by the constitutive properties, i.e., $J = \sigma E$, $B = \mu H$, where μ is the magnetic permeability and $\sigma \geq \sigma_0 > 0$ stands for the electric conductivity. Moreover, we assume that the material parameters are time independent and associated with a linear isotropic material. In the external insulating region the conductivity is equal to zero and thus, we obtain the following field equations:

$$\nabla \times H = J_s, \qquad \nabla \cdot B = 0, \qquad \text{in} \quad V \setminus V_c \,, \tag{2}$$

with $B = \mu H$, and the external source J_s. We assume that no external source is situated within the conduction regions. The source J_s is extended by zero onto $\mathbb{R}^3$ and still denoted by J_s. The problem is well posed by imposing regularity conditions at infinity and the following interface conditions:

$$[H] \times n_c = 0, \qquad [B] \cdot n_c = 0, \qquad \text{on} \quad \partial V_c \,, \tag{3}$$

where n_c is the outer normal to ∂V_c and $[v]$ stands for the jump of the quantity v at the interface. Clearly, such a type of interface conditions is to be verified at any surface where σ or μ is discontinuous. Additionally to the boundary conditions, we have to impose suitable initial values for the vector fields at a given time t_0. In particular, the initial condition on B has to give $\nabla \cdot B = 0$ and $[B] \cdot n = 0$ at any interface.

It is worth noting that the calculation of the electric field E in the external region $\mathbb{R}^3 \setminus V_c$ is not required to determine the eddy currents in V_c, but can be readily obtained afterwards from the knowledge of its curl, its divergence, specified by the charge distribution outside V_c that we suppose to be zero, and its tangential component at ∂V_c:

$$\nabla \times E = -\partial_t B, \qquad \nabla \cdot E = 0, \qquad E \times n_c = \sigma^{-1} \, J \times n_c. \tag{4}$$

In the conducting region, the electric field is computed from the current density J just applying Ohm's law $J = \sigma \, E$.

We point out the fact that the vector fields J and $\partial_t B$ are automatically forced to be soleinodal by equations (1) respectively. Of course, the condition $\nabla \cdot B = 0$ is satisfied at any time provided that it is verified by the initial condition. By introducing artificial boundary conditions, we can work on a bounded domain V. For simplicity, we assume that V_c is a bounded simply connected polyhedral subdomain of V and $\bar{V}_c \subset V$. In a weak form, equations $\nabla \cdot B = 0$ and $B = \mu H$ in V read as follows:

$$\int_V \mu \, H \, \nabla v \; = 0 \,, \qquad \forall \, v \in H^1_0(V) \tag{5}$$

where $H^1_0(V) = \{ v \in L^2(V) \,|\, \nabla v \in L^2(V), \, v_{|\partial V} = 0 \, \}$. From equations (2) in V_c, we can write

$$\int_{V_c} \nabla \times H \, \nabla \times W + \int_{V_c} \sigma \, \partial_t (\mu \, H) W = 0 \,, \qquad \forall \, W \in H_0(\text{curl}; V_c) \tag{6}$$

where $H_0(\text{curl}; V_c) = \{ W \in L^2(V_c) \,|\, \nabla \times W \in L^2(V_c), \, W \times n_c = 0 \, \}$.

For the current density J, the condition $\nabla \cdot J = 0$ suggests the introduction of a vector potential $\tilde{T}$ such that $J = \nabla \times \tilde{T}$. Then the difference between the vector potential $\tilde{T}$ and the magnetic field H can be written as the gradient of a scalar function $\tilde{\Omega}$, i.e., $H = \tilde{T} - \nabla \tilde{\Omega}$. A similar argument holds for the insulating region where we assume knowing a vector potential T_s such that $J_s = \nabla \times T_s$. Combining external and conducting regions we write H in V as

$$H := \begin{cases} \tilde{T} - \nabla \tilde{\Omega}, & \text{in } V_c, \\ T_s - \nabla \tilde{\Omega}, & \text{in } V \setminus V_c. \end{cases} \tag{7}$$

By eliminating the magnetic field H and the density B, we obtain a coupled eddy current problem in terms of the electric potential $\tilde{T}$ and the scalar potential $\tilde{\Omega}$. We note that $\tilde{T}$ is defined only in the conducting region V_c whereas $\tilde{\Omega}$ is defined everywhere in V. This system is completed with appropriate interface conditions over ∂V_c stating e.g. that $\tilde{\Omega}$ is continuous. This is nevertheless not enough to define $\tilde{\Omega}$ and $\tilde{T}$ uniquely. The divergence of $\tilde{T}$ is not specified and thus there are many different gauge possibilities. One of them is to require that $\tilde{T}$ has the same divergenc e as H in V_c but this eliminates $\tilde{\Omega}$ on V_c. We prefer another condition, stated in the next section.

2 Variational problem

In this section, we define a new variational formulation. It is based on the decomposition (7) of H into $\tilde{T}$ and $\nabla \tilde{\Omega}$. We restrict ourselves to the case of a boundary value problem which is obtained from the degenerated parabolic initial-boundary value problem (1) and (2). Only implicit time discretization schemes can satisfy the stability condition. Having discretized the time derivative by means of a finite difference scheme of time step Δt, then at each time step, we have to face the boundary value problem

$$\begin{aligned} \int_V \nabla \tilde{\Omega} \nabla v - \int_{V_c} \tilde{T} \nabla v &= \int_{V\setminus V_c} f \nabla v, \ \forall v \in H_0^1(V) \\ \int_{V_c} \alpha \nabla \times \tilde{T} \nabla \times W + \tilde{T} W - \int_{V_c} \nabla \Omega W &= \int_{V_c} f_c W, \\ &\qquad \forall W \in H_0(\mathrm{curl}; V_c) \end{aligned} \tag{8}$$

where the coefficient $\alpha > 0$ is constant. In the more general approach, it is uniformly positive definite and depends on the material parameters σ, μ as well as on the time step (e.g. $\alpha = \frac{\Delta t}{\mu \sigma}$ in V_c). Here, the unknowns $\tilde{T}$ and $\tilde{\Omega}$ denote the approximation at the current time-step, f_c depends on the approximations of $\tilde{T}$ and $\tilde{\Omega}$ at the previous time-step, and f denotes the scaled source term depending on T_s. Additionally, $\tilde{T}$ and $\tilde{\Omega}$ have to satisfy the interface conditions at each time step: we choose here $\tilde{T} \in H_0(\mathrm{curl}; V_c)$ and $\tilde{\Omega} \in H_0^1(V)$. In our approach, the strong coupling between $\tilde{T}$ and $\tilde{\Omega}$ at the interface will be replaced by a weaker form. We then consider the following time discrete variational problem: find $(\tilde{T}, \tilde{\Omega}) \in H_0(\mathrm{curl}; V_c) \times H_0^1(V)$ such that

$$\begin{aligned} a(\tilde{\Omega}, v) + \hat{b}_c(\tilde{T}, v) &= \int_{V\setminus V_c} f \nabla v, \quad \forall v \in H_0^1(V) \\ a_c(\tilde{T}, W) + \hat{b}_c(W, \tilde{\Omega}) &= \int_{V_c} f_c W, \qquad \forall W \in H_0(\mathrm{curl}; V_c). \end{aligned} \tag{9}$$

Here, the bilinear forms are defined by

$$\begin{aligned} a_c(\tilde{T},W) &:= \textstyle\int_{V_c} (\alpha\,\nabla\times\tilde{T}\,\nabla\times W + \tilde{T}\,W)\,, \forall T,\, W \in H_0(\mathrm{curl};V_c), \\ \hat{b}_c(W,v) &:= -\textstyle\int_{V_c} W\,\nabla\,v\,, \qquad \forall\, W \in H_0(\mathrm{curl};V_c),\ \forall\, v \in H_0^1(V)\,, \\ a(\tilde{\Omega},v) &:= \textstyle\int_V \nabla\,\tilde{\Omega}\,\nabla\,v\,, \qquad \forall\,\Omega,\, v \in H_0^1(V). \end{aligned}$$

The bilinear forms are continuous with respect to the corresponding norms. We associate to $H_0(\mathrm{curl};V_c)$ the energy norm $|||\cdot|||_{V_c}$

$$|||W|||_{V_c}^2 := \|W\|_{0;V_c}^2 + \alpha\|\nabla\times W\|_{0;V_c}^2, \qquad \forall\, W \in H_0(\mathrm{curl};V_c)$$

where $\|\cdot\|_{0;D}$ stands for the L^2-norm on the open bounded set D. This norm is equivalent to the standard Hilbert space norm.

Now, it is easy to see that if $(\tilde{T},\tilde{\Omega})$ is a solution of (9), then $(\tilde{T}+\nabla\phi, \tilde{\Omega}+\phi)$, $\phi \in H_0^1(V_c)$, is a solution as well. To achieve uniqueness, we choose ϕ such that $\Omega = \tilde{\Omega}+\phi$ is harmonic on V_c; to this purpose, we introduce the harmonic extension operator $\mathcal{H} : H^1(V_c) \longrightarrow H^1(V_c)$ verifying

$$\mathcal{H}v_{|\partial V_c} := v_{|\partial V_c}, \qquad \int_{V_c} \nabla\,\mathcal{H}v\ \nabla\,w\ = 0, \qquad \forall\, w \in H_0^1(V_c). \tag{10}$$

In terms of this harmonic extension, we can guarantee the unique solvability of our modified variational problem: find $(T,\Omega) \in H_0(\mathrm{curl};V_c)\times H_0^1(V)$ such that

$$\begin{aligned} a(\Omega,v) + b_c(T,v) &= \textstyle\int_{V\setminus V_c} f\,\nabla\,v, \ \ \forall\, v \in H_0^1(V) \\ a_c(T,W) + b_c(W,\Omega) &= \textstyle\int_{V_c} f_c\,W, \qquad \forall\, W \in H_0(\mathrm{curl};V_c). \end{aligned} \tag{11}$$

Here, the bilinear form $b_c(\cdot,\cdot)$ is defined by

$$b_c(W,v) := -\int_{V_c} W\,\nabla\,\mathcal{H}\,v, \quad \forall\, v \in H_0^1(V),\ \forall\, W \in H_0(\mathrm{curl};V_c).$$

Lemma 1. *There exists a constant* $0 < \gamma_1 < 1$ *depending only on the geometry of* V_c *and* V *such that*

$$\|\nabla\,\mathcal{H}v\|_{0;V_c} \le \gamma_1\|\nabla\,v\|_{0;V}, \quad \forall\, v \in H_0^1(V). \tag{12}$$

Proof. Using the definition of the harmonic extension operator, we find that there exists a constant such that

$$\|\nabla\mathcal{H}v\|_{0;V_c} \le C|v|_{\frac{1}{2};\partial V_c},$$

where $|\cdot|_{1/2;\partial V_c}$ stands for the $H^{1/2}$-semi norm on ∂V_c. On the other hand, the continuity of the trace operator yields

$$|v|_{\frac{1}{2};\partial V_c} \le C\|\nabla v\|_{0;V\setminus V_c}.$$

Combining these two upper bounds and observing that $\|\nabla\mathcal{H}v\|_{0;V_c} \le \|\nabla v\|_{0;V_c}$, we get (12).

We remark that the constant γ_1 does not depend on the shape regularity of $V \setminus V_c$ and V_c. In particular for time dependent problems where V_c may be moving, this might be important. More precisely, γ_1 does not depend on the distance between ∂V_c and ∂V. In the limit case that this distance tends to zero, γ_1 is still bounded away from one. Figure 1 illustrates the situation that the distance between ∂V_c and ∂V is small. In this situation, we work with a fictitious domain V_g such that $\bar{V} \subset V_g$ and $V_g \setminus V$ is shape regular. Due to the homogeneous Dirichlet boundary conditions of $v \in H_0^1(V)$, we extend v by zero to an element in $H_0^1(V_g)$. For homogeneous Dirichlet boundary conditions, we can establish the ellipticity of the variational problem (11) in terms of Lemma 1. Moreover, the ellipticity constant does not degenerate if the distance between ∂V_c and ∂V tends to zero.

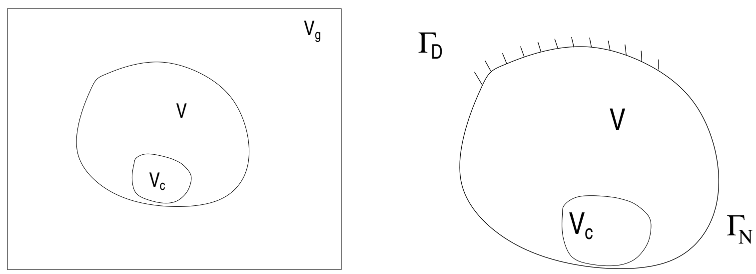

Fig. 1. Embedding of V in V_g (left) and degenerated limit case (right)

However if we have more general boundary conditions as, e.g., Neumann boundary conditions on a part Γ_N of ∂V, γ_1 tends to one if the distance between Γ_N and ∂V_c tends to zero. In the limit case that $\partial V_c \cap \Gamma_N$ has a non zero measure, γ_1 would be one, and the ellipticity constant degenerates. Therefore, we consider an element $W \in H_0(\mathrm{curl}; V_c)$ more in detail. According to Theorem 2.4 in [6], each element $W \in H_0(\mathrm{curl}; V_c)$ can be uniquely decomposed into

$$W = \nabla \Phi + \mathcal{E} W, \quad \Phi \in H_0^1(V_c),$$

and where $\mathcal{E} W := \nabla \times \Psi$ with $\Psi \in (H_0^1(V_c))^3$ satisfying $\nabla \cdot \Psi = 0$ and $\Psi \cdot n_c = 0$ on ∂V_c. Applying integration by parts and using the boundary conditions, it can be easily seen that this decomposition is orthogonal with respect to the L^2-inner product. The following lemma shows that the operator $\mathcal{E}$ is a contraction in suitable norms.

Lemma 2. *There exists a constant* $0 < \gamma_2 < 1$ *depending only on the geometry of* V_c *and the coefficient* α *such that*

$$\|\mathcal{E} W\|_{0;V_c} \le \gamma_2 |||W|||_{V_c}, \quad \forall\, W \in H_0(\mathrm{curl}; V_c). \tag{13}$$

Proof. We start by observing that $\nabla \times W = \nabla \times \mathcal{E} W$. Moreover using Theorem 2.3 in [6], we find that the H^1-norm of Ψ is equivalent to the L^2-norm of its curl. Here, it is essential that Ψ is a divergence free vector field satisfying homogeneous boundary conditions in normal direction. This equivalence guarantees the existence of a constant $0 < \hat{c} < \infty$ such that the L^2-norm of Ψ is bounded by that of its curl, i.e.,

$$\|\Psi\|_{0;V_c} \leq \hat{c}\|\nabla \times \Psi\|_{0;V_c}.$$

Using the orthogonality of $\nabla \times \Psi$ and $\nabla \Phi$ and the zero boundary conditions of W, $n_c \times W = 0$, we find an upper bound for $\|\nabla \times \Psi\|_{0;V_c}$ in terms of $\|\nabla \times W\|_{0;V_c}$ that is

$$\begin{aligned} \|\nabla \times \Psi\|^2_{0;V_c} &= \int_{V_c} W \; \nabla \times \Psi \;\; = \;\; \int_{V_c} \nabla \times W \; \Psi \\ &\leq \|\nabla \times W\|_{0;V_c}\|\Psi\|_{0;V_c} \;\leq\; \hat{c}\; \|\nabla \times W\|_{0;V_c}\|\nabla \times \Psi\|_{0;V_c}. \end{aligned}$$

Observing $\|\nabla \times \Psi\|_{0;V_c} \leq \|W\|_{0;V_c}$, we obtain for each $s \in [0,1]$

$$\begin{aligned} \|\mathcal{E}W\|^2_{0;V_c} &\leq s\|W\|^2_{0;V_c} + (1-s)\hat{c}^2 \; \|\nabla \times W\|^2_{0;V_c} \\ &\leq \left(\tfrac{\hat{c}^2}{\alpha+\hat{c}^2}\right)|||W|||^2_{V_c}. \end{aligned}$$

Thus (13) holds with some γ_2 depen ding only on the geometry of V_c and on the coefficient α.

We remark that the proof is based on a Helmholtz decomposition. Now, we define the bilinear form $a_g(\cdot,\cdot)$ which is associated with the variational problem (11) by

$$a_g((W,w),(V,v)) := a_c(W,V) + b_c(W,v) + b_c(V,w) + a(w,v),$$

where $V, W \in H_0(\text{curl}; V_c)$ and $v, w \in H^1_0(V)$.

Lemma 3. *The bilinear form $a_g(\cdot,\cdot)$ is continuous and elliptic on the space $H_0(\text{curl}; V_c) \times H^1_0(V)$.*

Proof. The proof of the ellipticity is based on Lemma 1 and Lemma 2. The ellipticity constant only depends on the constants in (12) and (13). We note that γ_2 but not γ_1 depends on the coefficient α and γ_1 but not γ_2 depends on the geometry of $V \setminus V_c$ in the case of more general boundary conditions. We start by considering $a_g((W,w),(W,w))$, $(W,w) \in H_0(\text{curl}; V_c) \times H^1_0(V)$ in more detail. The L^2-orthogonality of $\nabla \Phi$ and $\mathcal{H}w$ yields

$$\begin{aligned} a_g((W,w),(W,w)) &= |||W|||^2_{V_c} + \|\nabla w\|^2_{0;V} - 2\int_{V_c} W \nabla \mathcal{H}w \\ &= |||W|||^2_{V_c} + \|\nabla w\|^2_{0;V} - 2\int_{V_c} \mathcal{E}W \nabla \mathcal{H}w \\ &\geq |||W|||^2_{V_c} + \|\nabla w\|^2_{0;V} - 2\|\mathcal{E}W\|_{0;V_c}\|\nabla \mathcal{H}w\|_{0;V_c}. \end{aligned}$$

Now, we can use (12) and (13) to bound $\|\nabla \mathcal{H}w\|_{0;V_c}$ and $\|\mathcal{E}W\|_{0;V_c}$ in terms of $\|\nabla w\|_{0;V}$ and $|||W|||_{V_c}$, and we find

$$\begin{aligned} a_g((W,w),(W,w)) &\geq |||W|||^2_{V_c} + \|\nabla w\|^2_{0;V} - 2\gamma_1\gamma_2|||W|||_{V_c}\|\nabla w\|_{0;V} \\ &\geq (1-\gamma_1\gamma_2)(|||W|||^2_{V_c} + \|\nabla w\|^2_{0;V}). \end{aligned}$$

The last inequality shows that to obtain a "good" ellipticity constant, it is sufficient that γ_1 or γ_2 is not too close to one. This is achieved if, e.g., homogeneous Dirichlet boundary conditions are imposed over Ω, or if α is not too small.

The unique solvability of the variational problem (11) is now an easy consequence of Lemma 3 and the continuity of the bilinear forms. The first equation in (11) and the definition of the harmonic extension yields

$$b_c(T, v) = 0, \qquad \forall\, v \in H^1_0(V_c)\,.$$

Moreover, taking $W = \nabla v$, for a function $v \in H^1_0(V_c)$, in the second equation of (11), we get that T is divergence free as soon as f_c is divergence free. Hence, T is implicitly gauged, and Ω restricted to V_c is harmonic.

3 Discretization

For the discretization of the vector field T, we use the lowest order Nédélec finite elements, also known as edge elements; see [5,7,9]. The associated finite element functions are curl-conforming. Let K be a simplicial element in $\mathbb{R}^3$. Then, the local space of Nédélec finite elements is given by

$$\mathcal{R}(K) := P_0(K)^3 + P_0(K)^3 \times \mathbf{x}.$$

The global space is defined in terms of the local ones:

$$X_{0;h}(V_c) := \left\{T \in H_0(\mathrm{curl}; V_c) | \quad T_{|_K} \in \mathcal{R}(K),\, K \in \mathcal{T}_h\right\},$$

where $\mathcal{T}_h$ is a given quasi-uniform simplicial triangulation of V_c. The degrees of freedom of the vector T are given by averages of the tangential component of T on the mesh edges e, i.e., $\lambda_e(T) := (\int_e t_e \cdot T)/h_e$, where h_e is the length of the edge e and t_e its tangent vector. We note that $T \in X_{0;h}(V_c)$ has a vanishing tangential component on ∂V_c. Thus the degrees of freedom of $X_{0;h}(V_c)$ are associated with the interior edges of the triangulation $\mathcal{T}_h$.

The domain V is associated with a second quasi-uniform simplicial triangulation denoted by $\mathcal{T}_H$. Now, the discretization of Ω on V is based on standard conforming elements of lowest order. These finite elements are characterised by basis functions which are piecewise linear and H^1-conforming. The degrees of freedom are associated with the interior vertices of the triangulation $\mathcal{T}_H$. We denote the corresponding finite element space with $S_{0;H}(V)$. The space of standard conforming elements of first order associated with $\mathcal{T}_h$ on V_c will be denoted by $S_h(V_c)$. We note that no boundary conditions are imposed on $S_h(V_c)$. Moreover, we assume that ∂V_c can be written as union of faces in $\mathcal{T}_h$. The trace space of $S_h(V_c)$ on ∂V_c is called $W_h(\partial V_c)$.

Figure 2 illustrates the situation in 2D. In general, the triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$ do not coincide on V_c. Moreover, V_c cannot be written as the union

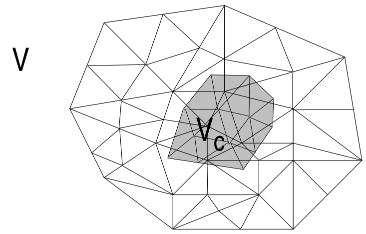

Fig. 2. Overlapping decomposition and non-matching triangulations on V_c in 2D

of elements in $\mathcal{T}_H$, i.e., the two triangolations are not inbedded one into the other.

Before we can formulate the discrete version of our variational problem (11), we have to specify a discrete operator replacing the harmonic extension in (11). A natural choice is to replace the harmonic extension by the discrete harmonic extension $\mathcal{H}_h$ defined as a map $\mathcal{H}_h : W_h(\partial V_c) \longrightarrow S_h(V_c)$ verifying

$$\mathcal{H}_h v_{|\partial V_c} := v_{|\partial V_c}, \qquad \int_{V_c} \nabla \mathcal{H}_h v \; \nabla w = 0, \qquad \forall\, w \in S_h(V_c) \cap H_0^1(V_c).$$

However in general, the restriction of $v \in S_{0;H}(V)$ to ∂V_c is not an element of $W_h(\partial V_c)$. Thus, we cannot apply directly the discrete harmonic extension to the restriction of $v \in S_{0;H}(V)$ on ∂V_c. To overcome this difficulty, we introduce a projection operator on the boundary. This operator is well known in the mortar finite element context [3] and can be defined in terms of a Lagrange multiplier space $M_h(\partial V_c)$:

$$\Pi_h : L^2(\partial V_c) \longrightarrow W_h(\partial V_c)\,, \qquad \int_{\partial V_c} \Pi_h v \nu \; = \int_{\partial V_c} v \nu \;, \qquad \forall\, \nu \in M_h(\partial V_c).$$

To obtain a well defined operator Π_h, the Lagrange multiplier space $M_h(\partial V_c)$ has to be well chosen. There are many possibilities. For simplicity reasons, we restrict ourselves to two choices. A natural way is to set $M_h(\partial V_c) := W_h(\partial V_c)$ [3]. In contrast to mortar finite element methods with many subdomains, no modification of $M_h(\partial V_c)$ is required. This is due to the fact that ∂V_c is a closed surface without crosspoints. Then, the operator Π_h is a L^2-projection, and in the implementation a mass matrix system has to be solved. A different possibility is to replace the standard hat functions by piecewise linear dual basis functions in the definition of the space $M_h(\partial V_c)$ [10]. Then, Π_h is a quasi L^2-projection having the same qualitative stability properties as before. The advantage of this approach is that the mass matrix system is reduced to a diagonal one. These choices guarantee that the operator Π_h is H^s-stable for $0 \leq s \leq 1$. Furthermore, it satisfies the approximation property in the $H^{1/2}$-norm and in the $H^{-1/2}$-norm. We refer to [3,10] for details. The following lemma is the discrete version of Lemma 1.

Lemma 4. *There exist constants $\hat{\gamma}_1 < 1$ and $h^* > 0$ depending only on the geometry of V_c and $V \setminus V_c$ such that*

$$\forall\,(h/H) < h^*, \quad \forall\, v \in S_{0;H}(V), \qquad \|\nabla \mathcal{H}_h \varPi_h v\|_{0;V_c} \le \hat{\gamma}_1\, \|v\|_{1;V}\,. \qquad (14)$$

Proof. We start by writing

$$\|\nabla \mathcal{H}_h \varPi_h v\|^2_{0;V_c} = \|\nabla \mathcal{H} v + (\nabla \mathcal{H}_h \varPi_h v - \nabla \mathcal{H} v)\|^2_{0;V_c}\,.$$

Adopting the relation $(a+b)^2 = (1+\epsilon)a^2 + (1+\frac{1}{\epsilon})b^2$ for all $\epsilon > 0$, we have that

$$\|\nabla \mathcal{H}_h \varPi_h v\|^2_{0;V_c} \le (1+\epsilon)\|\nabla \mathcal{H} v\|^2_{0;V_c} + c(\epsilon)\|\nabla \mathcal{H}_h \varPi_h v - \nabla \mathcal{H} v\|^2_{0;V_c}\,.$$

In terms of Lemma 1, we have $\|\nabla \mathcal{H} v\|_{0;V_c} \le \gamma_1 \|\nabla v\|_{0;V}$. To evaluate the second norm on the right-hand side, we write

$$\|\nabla \mathcal{H}_h \varPi_h v - \nabla \mathcal{H} v\|^2_{0;V_c} = \|\nabla \mathcal{H}(\varPi_h v - v) + (\nabla \mathcal{H}_h \varPi_h v - \nabla \mathcal{H} \varPi_h v)\|^2_{0;V_c}.$$

Now, by applying an inverse inequality, we find that, for a quantity $0 < \delta < \frac{1}{2}$,

$$|v - \varPi_h v|^2_{\frac{1}{2};\partial V_c} \le c h^{2\delta} |v|^2_{\delta+\frac{1}{2};\partial V_c} \le c h^{2\delta} \|v\|^2_{\delta+1;V} \le c \left(\frac{h}{H}\right)^{2\delta} \|v\|^2_{1;V}\,.$$

By using the previous inequality, we get

$$\|\nabla \mathcal{H}(\varPi_h v - v)\|_{0;V_c} \le \|\varPi_h v - v\|_{\frac{1}{2};\partial V_c} \le c \left(\frac{h}{H}\right)^{\delta} \|v\|_{1;V}\,.$$

On the other side, the stability of the mortar operator $\varPi_h$ in the $H^{\frac{1}{2}+\delta}(\partial V_c)$-norm and the definition of $\mathcal{H}_h$ yield

$$\begin{aligned} \|\nabla \mathcal{H}_h \varPi_h v - \nabla \mathcal{H} \varPi_h v\|_{0;V_c} \;&\le\; c h^{\delta} \|\mathcal{H} \varPi_h v\|_{1+\delta;V_c} \le c h^{\delta} \|\varPi_h v\|_{\frac{1}{2}+\delta;\partial V_c} \\ &\le\; c h^{\delta} \|v\|_{\frac{1}{2}+\delta;\partial V_c} \le c h^{\delta} \|v\|_{1+\delta;V} \le c \left(\tfrac{h}{H}\right)^{\delta} \|\nabla v\|_{0;V}\,. \end{aligned}$$

By summing up all the contributions and using Lemma 1, we have

$$\begin{aligned} \|\nabla \mathcal{H}_h \varPi_h v\|^2_{0;V_c} &\le (1+\epsilon)\gamma_1^2 \|\nabla v\|^2_{0;V_c} + 2c(\epsilon) c \left(\tfrac{h}{H}\right)^{2\delta} \|\nabla v\|^2_{0;V} \\ &\le \left[(1+\epsilon)\gamma_1^2 + \tilde{c}(\epsilon) \left(\tfrac{h}{H}\right)^{2\delta}\right] \|\nabla v\|^2_{0;V}. \end{aligned}$$

If the meshsize is small enough, we can guarantee that

$$\hat{\gamma}_1^2 = \left[(1+\epsilon)\gamma_1^2 + \tilde{c}(\epsilon) \left(\frac{h}{H}\right)^{2\delta}\right] < 1.$$

In fact, we start by choosing $\delta = \frac{1}{4}$ for example and ϵ such that $\gamma_1^2(1+\epsilon) \leq (1+\gamma_1^2)/2$. Now that ϵ and δ are chosen, we select h^* such that $\tilde{c}(\epsilon)\left(\frac{h}{H}\right)^{\frac{1}{2}} < (1-\gamma_1^2)/4$ for $h/H \leq h^*$. Then

$$\left[(1+\epsilon)\gamma_1^2 + \tilde{c}(\epsilon)\left(\frac{h}{H}\right)^{2\delta}\right] < \frac{\gamma_1^2+3}{4} < 1$$

and this ends the proof.

The discrete version of Lemma 2 has exactly the same structure. It is based on a discrete Helmholtz type decomposition. To start with, we define our orthogonal splitting of $W \in X_{0;h}(V_c)$ in

$$W = \nabla \Phi_h + \mathcal{E}_h W,$$

where $\Phi_h \in S_h(V_c) \cap H_0^1(V_c)$ is uniquely defined by

$$\int_{V_c} \nabla \Phi_h \nabla v = \int_{V_c} W \nabla v, \qquad \forall\, v \in S_h(V_c) \cap H_0^1(V_c).$$

Then Theorem 2.3 in [6] yields that the L^2-norm of $\mathcal{E}_h W$ is bounded in terms of the L^2-norm of the curl of W

$$\|\mathcal{E}_h W\|_{0;V_c}^2 \leq C\|\nabla \times W\|_{0;V_c}^2, \qquad \forall\, W \in X_{0;h}(V_c). \tag{15}$$

Lemma 5. *There exists a constant $0 < \hat{\gamma}_2 < 1$ depending only on the geometry of V_c and the coefficient α such that*

$$\|\mathcal{E}_h W\|_{0;V_c} \leq \hat{\gamma}_2 |||W|||_{V_c}, \qquad \forall W \in X_{0;h}(V_c). \tag{16}$$

The proof follows the same lines as the one of Lemma 2 and uses (15).

In terms of the discrete harmonic extension and the operator Π_h, we formulate our new discrete variational problem: find $(T_h, \Omega_h) \in X_{0;h}(V_c) \times S_{0;H}(V)$ such that

$$\begin{aligned} a(\Omega_h, v) + b_h(T_h, v) &= \textstyle\int_{V \setminus V_c} f \,\nabla\, v, \quad && \forall\, v \in S_{0;H}(V)\,, \\ a_c(T_h, W) + b_h(W, \Omega_h) &= \textstyle\int_{V_c} f_c \, W, && \forall\, W \in X_{0;h}(V_c)\,, \end{aligned} \tag{17}$$

where the discrete bilinear form $b_h(\cdot,\cdot)$ is defined by

$$b_h(W, v) := -\int_{V_c} W \nabla \mathcal{H}_h \Pi_h v, \quad \forall\, v \in S_{0;H}(V), \quad \forall\, W \in X_{0;h}(V_c).$$

Then, Lemma 4 and Lemma 5 yield the ellipticity under the assumption that h/H is small enough. Moreover, the unique solvability of (17) is given. The following lemma shows that optimal a priori estimates can be obtained.

Lemma 6. *For h/H small enough, problem (17) has a unique solution and there exists a constant C independent of the meshsize and a real $1 < \beta \leq 2$ such that, for $T \in (H^1(V_c))^3$ with $\nabla \times T \in (H^1(V_c))^3$ and $\Omega \in H^\beta(V)$, we have*

$$|||T - T_h|||_{V_c} + \|\Omega - \Omega_H\|_{1;V} \leq C\Big(h(||T||_{1;V_c} + ||\nabla \times T||_{1;V_c}) + H^{\beta-1}||\Omega||_{\beta;V}\Big). \tag{18}$$

Proof. The proof is carried out in two dimensions. The same proof is still valid in three dimensions, provided that the interpolation operator is replaced by an operator of Clement type, that asks for less regularity on Ω.

Three steps have to be worked out in the proof. The discretization can be bounded in terms of the sum of the best approximation error and the consistency error. Although we work with H^1- and $H(\text{curl})$-conforming discrete spaces, a consistency error term enters. This is due to the fact that the harmonic extension operator has to be replaced in the discrete approach by $\mathcal{H}_h \Pi_h$. The best approximation error is standard, see, e.g., [9],

$$\begin{aligned} &\|\Omega - I_H \Omega\|_{1;V} \leq C H^{\beta-1} ||\Omega||_{\beta;V}, \\ &|||T - R_h T|||_{V_c} \leq Ch\,(||T||_{1;V_c} + ||\nabla \times T||_{1;V_c}) \end{aligned} \tag{19}$$

where I_H denotes the Lagrange interpolation operator onto $S_{0;H}(V)$. The notation R_h stands for the natural interpolation operator onto $X_{0;h}(V_c)$ defined by

$$\int_e t_e \cdot (R_h T) \;:=\; \int_e t_e \cdot T\;,$$

where e are the edges of the triangulation $\mathcal{T}_h$. Moreover, we have to consider the two contributions for the consistency error. Let us start by considering the quantity $b_c(W, \Omega) - b_h(W, \Omega)$, $W \in X_{0;h}(V_c)$ more in detail:

$$\begin{aligned} b_c(W,\Omega) - b_h(W,\Omega) &= \textstyle\int_{V_c} W(\nabla \mathcal{H}\Omega - \nabla \mathcal{H}_h \Pi_h \Omega) \\ &= \textstyle\int_{V_c} W(\nabla \mathcal{H}\Omega - \nabla \mathcal{H}_h I_h \Omega + \nabla \mathcal{H}_h I_h \Omega - \nabla \mathcal{H}_h \Pi_h \Omega) \\ &\leq \|W\|_{0;V_c}\, \|\nabla \mathcal{H}\Omega - \nabla \mathcal{H}_h I_h \Omega\|_{0;V_c} \\ &\qquad + \|W\|_{0;V_c}\, \|\nabla \mathcal{H}_h I_h \Omega - \nabla \mathcal{H}_h \Pi_h \Omega\|_{0;V_c} \\ &\leq C|||W|||_{V_c}(h^{\beta-1}||\Omega||_{\beta;V} + |I_h\Omega - \Pi_h\Omega|_{\frac{1}{2};\partial V_c}) \\ &\leq Ch^{\beta-1}|||W|||_{V_c}(||\Omega||_{\beta;V} + |\Omega|_{\beta-\frac{1}{2};\partial V_c}) \\ &\leq Ch^{\beta-1}|||W|||_{V_c}\; ||\Omega||_{\beta;V}. \end{aligned}$$

Here, we have used standard a priori estimates for a Dirichlet problem and the approximation property of I_h and Π_h in the $H^{1/2}$-norm on ∂V_c. Using this bound, we find that

$$\sup_{W \in X_{0;h}(V_c)} \frac{b_c(W,\Omega) - b_h(W,\Omega)}{|||W|||_{V_c}} \leq CH||\Omega||_{\beta;V}. \tag{20}$$

In an analogous way, we can bound the term $b_c(T,v) - b_h(T,v)$, $v \in S_{0;H}(V)$. We start by applying integration by parts and obtain

$$\begin{aligned} b_c(T,v) - b_h(T,v) &= \int_{V_c} T(\nabla \mathcal{H} v - \nabla \mathcal{H}_h \Pi_h v) \\ &= \int_{V_c} -\nabla(T)(\mathcal{H} v - \mathcal{H}_h \Pi_h v) \\ &\quad + \int_{\partial V_c} T \cdot n_c (\mathcal{H} v - \mathcal{H}_h \Pi_h v) \ . \end{aligned}$$

Now, we estimate the different terms on the right hand side separately. We note that the operator Π_h is by construction $H^{1/2}$-stable on ∂V_c, and the Lagrange interpolation operator is H^1-stable on V_c. Using an Aubin–Nitsche argument, we find for the volume term

$$\begin{aligned} \|\mathcal{H} v - \mathcal{H}_h \Pi_h v\|_{0;V_c} &\leq Ch(|\mathcal{H} v - \mathcal{H}_h \Pi_h v|_{1;V_c}) \\ &\leq Ch(|v|_{1;V_c} + |\Pi_h v|_{\frac{1}{2};\partial V_c}) \\ &\leq Ch|v|_{1;V_c}. \end{aligned}$$

We observe that the boundary term can be associated with the $H^{1/2}$-duality, and we have to consider $\|\mathcal{H} v - \mathcal{H}_h \Pi_h v\|_{-1/2;\partial V_c}$ more in detail. Then, the approximation property of the operator Π_h yields

$$\|\mathcal{H} v - \mathcal{H}_h \Pi_h v\|_{-1/2;\partial V_c} = \|v - \Pi_h v\|_{-\frac{1}{2};\partial V_c} \leq Ch\|v\|_{\frac{1}{2};\partial V_c} \leq Ch\|v\|_{1;V_c}.$$

Combing the last two upper bounds and observing that $\|T \cdot n_c\|_{1/2;\partial V_c}$ is bounded by $\|T\|_{1;V_c}$, we find that

$$\sup_{v \in S_{0;H}(V)} \frac{b_c(T,v) - b_h(T,v)}{\|v\|_{1;V}} \leq Ch\|T\|_{1;V}. \tag{21}$$

Now, the a priori estimates are an easy consequence of (19), (20) and (21).

4 Stability in time

In this section, we consider the stability of our variational formulation. We start with the time continuous setting. For convenience, we recall the general variational form: Find $(T, \Omega) \in H_0(\mathrm{curl}; V_c) \times H_0^1(V)$ such that

$$\begin{aligned} &\int_{V_c} \frac{1}{\sigma\,\mu} \nabla \times T \, \nabla \times W + \int_{V_c} (\partial_t (T - \nabla \mathcal{H} \Omega)) W = 0, \quad \forall\, W \in H_0(\mathrm{curl}; V_c), \\ &\int_V \nabla \Omega \nabla v - \int_{V_c} T \nabla \mathcal{H} v = \int_{V \setminus V_c} f \nabla\, v, \quad \forall\, v \in H_0^1(V). \end{aligned} \tag{22}$$

Lemma 7. *There exists a constant, not depending on the time t, such that*

$$\|T(t)\|_{0;V_c}^2 + |\Omega(t)|_{1;V}^2 + \int_{t_0}^{t} \frac{1}{\sigma\,\mu} \|\nabla \times T\|_{0;V_c}^2 \ \leq C(\|T(t_0)\|_{0;V_c}^2 + |\Omega(t_0)|_{1;V}^2).$$

Proof. Here, we restrict ourselves to the case of zero source terms. Replacing the second equation in (22) by

$$\int_V \nabla(\partial_t \Omega)\nabla v \;-\int_V (\partial_t T)\nabla\mathcal{H}v \;=0$$

and setting $v=\Omega$ and $W=T$ in the first equation of (22), we find

$$\frac{1}{\sigma\,\mu}\|\nabla\times T\|^2_{0;V_c}+\frac{1}{2}\partial_t\|T\|^2_{0;V_c}-\partial_t\int_{V_c} T\nabla\mathcal{H}\Omega \;+\frac{1}{2}\partial_t|\Omega|^2_{1;V}=0.$$

In the next step, we integrate from t_0 to t, and we obtain

$$\begin{aligned}&\int_{t_0}^{t}\tfrac{1}{\sigma\,\mu}\|\nabla\times T\|^2_{0;V_c}\;+\tfrac{1}{2}(\|T(t)\|^2_{0;V_c}-\|T(t_0)\|^2_{0;V_c})\\ &-(\int_{V_c} T(t)\nabla\mathcal{H}\Omega(t)\;-\int_{V_c} T(t_0)\nabla\mathcal{H}\Omega(t_0)\;)+\tfrac{1}{2}(|\Omega(t)|^2_{1;V}-|\Omega(t_0)|^2_{1;V})=0.\end{aligned}$$

From this equality, we obtain the following upper bound

$$\begin{aligned}&\int_{t_0}^{t}\tfrac{2}{\sigma\,\mu}\|\nabla\times T\|^2_{0;V_c}\;+\|T(t)\|^2_{0;V_c}-2\int_{V_c} T(t)\nabla\mathcal{H}\Omega(t)\;+|\Omega(t)|^2_{1;V}\\ &\le\;\|T(t_0)\|^2_{0;V_c}-2\int_{V_c} T(t_0)\nabla\mathcal{H}\Omega(t_0)\;+|\Omega(t_0)|^2_{1;V}.\end{aligned}$$

An upper bound for the right hand side is given in terms of the continuity and a lower bound for the left hand side in terms of the ellipticity. Here, we have to use the form of the ellipticity estimate which does not involve the curl-term, i.e., we have to use $\|T\|_{0;V_c}$. Since $\gamma_1<1$ still yields the ellipticity, we get

$$c(\|T(t)\|^2_{0;V_c}+|\Omega(t)|^2_{1;V})\le\|T(t)\|^2_{0;V_c}-2\int_{V_c} T(t)\nabla\mathcal{H}\Omega(t)\;+|\Omega(t)|^2_{1;V}$$

and this ends the proof.

If the continuous variational problem (22) is discretized in time by an implicit Euler scheme, then we have to solve in each time step $t_n:=t_0+n\,\Delta t$ a variational problem of the form (11):

$$\begin{aligned}&\int_{V_c}\tfrac{1}{\sigma\,\mu}\nabla\times T^n\,\nabla\times W\;+\tfrac{1}{\Delta t}\int_{V_c}((T^n-T^{n-1})\;-\nabla\mathcal{H}(\Omega^n-\Omega^{n-1}))W\\ &\qquad\qquad\qquad=\int_{V_c} f_c\,W\;,\quad\forall\,W\in H_0(\mathrm{curl};V_c),\\ &\int_V\nabla\Omega^n\nabla v\;-\int_V T^n\nabla\mathcal{H}v\;=\int_{V\setminus V_c} f\,\nabla\,v\;,\qquad\forall\,v\in H^1_0(V).\end{aligned}\tag{23}$$

Here, we use the upper index ^{n} to denote the solution at the time step t_n. The discrete version of Lemma 7 reads as follows.

Lemma 8. *There exists a constant C not depending on the time t such that*

$$\sum_{n=0}^{N} \frac{\Delta t}{\sigma\,\mu} \|\nabla \times T^n\|^2_{0;V_c} + \|T^N\|^2_{0;V_c} + |\Omega^N|^2_{1;V} \le C(\|T^0\|^2_{0;V_c} + |\Omega^0|^2_{1;V}).$$

Proof. The proof follows basically the same lines as the one in the continuous setting. We take $W = T^n$, $v = \Omega^n$ and subtract the variational equality

$$\int_V \nabla \Omega^{n-1} \nabla v \;-\int_{V_c} T^{n-1} \nabla \mathcal{H} v \;= \int_{V \setminus V_c} f\,\nabla\,v\;, \qquad \forall\, v \in H^1_0(V)$$

on the previous time step form the second equation in (23). Adding the two equations in (23) yields

$$\begin{aligned}
&\tfrac{1}{\sigma\,\mu}\|\nabla \times T^n\|^2_{0;V_c} + \tfrac{1}{\Delta t}(\|T^n\|^2_{0;V_c} - (T^n, T^{n-1})_{0;V_c} - (T^n, \nabla\mathcal{H}(\Omega^n - \Omega^{n-1}))_{0;V_c})\\
&\quad + \tfrac{1}{\Delta t}(\|\Omega^n\|^2_{1;V} - (\Omega^n, \Omega^{n-1})_{1;V} - (T^n - T^{n-1}, \nabla\mathcal{H}\Omega^n)_{0;V_c})\\
&= \tfrac{1}{\sigma\,\mu}\|\nabla \times T^n\|^2_{0;V_c} + \tfrac{1}{2\Delta t}(\|T^n\|^2_{0;V_c} - \|T^{n-1}\|^2_{0;V_c} + \|T^n - T^{n-1}\|^2_{0;V_c})\\
&\quad + \tfrac{1}{2\Delta t}(\|\Omega^n\|^2_{1;V} - \|\Omega^{n-1}\|^2_{1;V} + \|\Omega^n - \Omega^{n-1}\|^2_{1;V}) - \tfrac{1}{\Delta t}((T^n, \nabla\mathcal{H}\Omega^{n-1})_{0;V_c}\\
&\quad - (T^{n-1}, \nabla\mathcal{H}\Omega^{n-1})_{0;V_c} + (T^n - T^{n-1}, \nabla\mathcal{H}(\Omega^n - \Omega^{n-1}))_{0;V_c}) = 0.
\end{aligned}$$

We note that $\|T^n - T^{n-1}\|^2_{0;V_c} + \|\Omega^n - \Omega^{n-1}\|^2_{1;V} - 2(T^n - T^{n-1}, \nabla\mathcal{H}(\Omega^n - \Omega^{n-1}))_{0;V_c}$ is non negative. Thanks to the telescopic cancellation, we find

$$\begin{aligned}
\tfrac{1}{\sigma\,\mu}\sum_{n=0}^{N} \|\nabla \times T^n\|^2_{0;V_c} + \tfrac{1}{2\Delta t}(\|T^N\|^2_{0;V_c} + \|\Omega^N\|^2_{1;V} - (T^N, \nabla\mathcal{H}\Omega^N)_{0;V_c})\\
\le \tfrac{1}{2\Delta t}(\|T^0\|^2_{0;V_c} + \|\Omega^0\|^2_{1;V} - (T^0, \nabla\mathcal{H}\Omega^0)_{0;V_c}).
\end{aligned}$$

by summing over n. Now, we conclude as before and use the continuity to bound the right-hand side and the ellipticity to bound the left-hand side.

5 An application and an iterative procedure

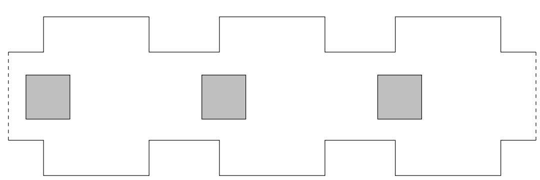

Fig. 3. Geometry of a linear motor in 2D

In this section, we consider a simple 2D example of a linear motor. Figure 3 illustrates the geometry. It is sufficient to consider a part of the domain, see Figure 4. Due to the symmetry and periodicity of the problem, we can reduce the geometry to a quarter. The outer boundary conditions are given by $H \times n = 0$. Having in mind that T is zero outside of V_c, we find that $\nabla \Omega \times n = 0$ and thus we can assume homogeneous Dirichlet boundary conditions for Ω on Γ_D. Thus the constant γ_1 does not degenerate. Moreover, the symmetry yields homogeneous Neumann boundary conditions for T on γ_N and for Ω on Γ_N, see also Figure 4 for the notation.

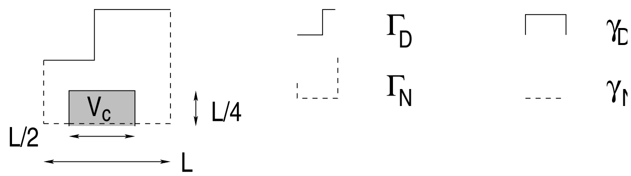

Fig. 4. Quarter of a linear motor in 2D

Applying our approach to the special situation here, we obtain the following variational problem. In contrast to (11), the harmonic extension has to be adapted to the special boundary structure of V_c. We use an extension operator having zero normal derivative on γ_N, i.e.,

$$\mathcal{H}v_{|\gamma_D} := v_{|\gamma_D}, \quad \int_{V_c} \nabla \mathcal{H}v \; \nabla w \, d\sigma = 0, \qquad \forall \, w \in H^1_{0;\gamma_D}(V_c).$$

Now, all results presented in the previous sections hold here, and we can work with the following variational problem: Find $(T, \Omega) \in H_{0;\gamma_D}(\mathrm{curl}; V_c) \times H^1_{0;\Gamma_D}(V)$ such that

$$\begin{aligned} a_c(T, H) + b_h(H, \Omega) &= \textstyle\int_{V_c} f_c \, H, \qquad && H \in H_{0;\gamma_D}(\mathrm{curl}; V_c) \\ b_h(T, v) + a(\Omega, v) &= \textstyle\int_{V \setminus V_c} f \, \nabla \, v, && v \in H^1_{0;\Gamma_D}(V). \end{aligned} \tag{24}$$

Let us denote by A_c and A the standard stiffness matrices associated with the bilinear forms $a_c(\cdot, \cdot)$ and $a(\cdot, \cdot)$ respectively. Let Q be the matrix associated with the mortar projection Π_h from $S_{0;H}(V)$ to $W_h(\partial V_c)$ and

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{II} & \tilde{A}_{IR} \\ \tilde{A}_{RI} & \tilde{A}_{RR} \end{pmatrix}$$

the stiffness matrix associated with the Laplacian operator restricted to V_c but computed on S_H. Then, the harmonic extension $\mathcal{H}_h$ from $W_h(\partial V_c)$ to $S_h(V_c)$ has the following algebraic representation

$$S : v_R \to \begin{pmatrix} v_R \\ v_I \end{pmatrix} \qquad \text{with} \qquad \tilde{A}_{IR} v_R + \tilde{A}_{II} v_I = 0$$

and hence

$$Sv_R = \begin{pmatrix} v_R \\ -\tilde{A}_{II}^{-1}\tilde{A}_{IR}v_R \end{pmatrix} = \begin{pmatrix} Id \\ -\tilde{A}_{II}^{-1}\tilde{A}_{IR} \end{pmatrix} v_R\,,$$

i.e., to obtain v_I we have to solve one mixed Dirichlet boundary problem:

$$\tilde{A}_{II}v_I = -\tilde{A}_{IR}v_R\,.$$

The operator

$$S^t : \begin{pmatrix} v_R \\ v_I \end{pmatrix} \to \tilde{v}_R$$

is defined as follows

$$S^t \begin{pmatrix} v_R \\ v_I \end{pmatrix} = (Id\, -\, \tilde{A}_{RI}\tilde{A}_{II}^{-1}) \begin{pmatrix} v_R \\ v_I \end{pmatrix} = v_R - \tilde{A}_{RI}\tilde{A}_{II}^{-1}v_I,$$

i.e., to obtain $\tilde{v}_R$, we have to solve one homogeneous Dirichlet boundary value problem:

$$\tilde{A}_{II}\hat{v}_I = v_I\,, \qquad \tilde{v}_R = v_R - \tilde{A}_{RI}\hat{v}_I\,.$$

The algebraic form of the discrete problem (24) reads [1]: find two vectors T_h and Ω_H solution of the linear system

$$\begin{aligned} A_cT_h + BSQ\Omega_H &= F_c\,, \\ A\Omega_H + Q^tS^tB^tT_h &= F\,, \end{aligned} \tag{25}$$

where B is a rectangular stiffness matrix associated with the bilinear form $b_h(T_h, \Omega_H)$ and ^{t} denotes the transpose operator. As a linear iterative solver for (25), we suggest the use of a block Gauss-Seidel method, i.e. starting from Ω^n, we compute T^{n+1} through

$$A_cT^{n+1} + A_bSQ\Omega^n = F_c \tag{26}$$

and then Ω^{n+1} by means of

$$A\Omega^{n+1} + Q^tS^tA_b^tT^{n+1} = F\,. \tag{27}$$

This algorithm converges thanks to the following lemma, whose proof relies on Lemma 1, Lemma 2 and on the continuity and coerciveness of the bilinear forms $a(\cdot,\cdot)$ and $a_c(\cdot,\cdot)$.

Lemma 9. *Let us denote by e^n the error $\Omega_H - \Omega_H^n$ at the iteration n on the vector Ω_H. The mapping $e^n \to e^{n+1}$ is a strict contraction over $S_H(V)$: there exists a constant $0 < \theta < 1$ such that*

$$a(e^{n+1}, e^{n+1}) < \theta\, a(e^n, e^n)\,.$$

[1] In what follows, we use the same letters T_h and Ω_H for the discrete functions and their vectors of unknowns in the appropriated basis.

Proof. Let $E^n = T - T^n$ denote the error at the iteration n on the vector T. Then subtracting (26) and (27) from (25), we get

$$\begin{aligned} &b_h(E^{n+1}, v) + a(e^{n+1}, v) = 0\,,\ \forall\, v \in H^1_{0;\Gamma_D}(V)\,,\\ &a_c(E^{n+1}, H) + b_h(H, e^n) = 0\,,\ \forall\, H \in H_{0;\gamma_D}(\mathrm{curl}; V_c)\,. \end{aligned} \tag{28}$$

Then, taking $v = e^{n+1}$ in the first equation of (28), we get

$$\begin{aligned} |||e^{n+1}|||^2_V = a(e^{n+1}, e^{n+1}) &= -b_h(E^{n+1}, e^{n+1})\\ &= \textstyle\int_{V_c} E^{n+1}\, \nabla \mathcal{H}_h \Pi_h e^{n+1}\\ &= \textstyle\int_{V_c} [\nabla \Phi_h + \mathcal{E}_h E^{n+1}]\, \nabla \mathcal{H}_h \Pi_h e^{n+1}\\ &= \textstyle\int_{V_c} \mathcal{E}_h E^{n+1}\, \nabla \mathcal{H}_h \Pi_h e^{n+1}\\ &\le ||\mathcal{E}_h E^{n+1}||_{0;V_c} ||\nabla \mathcal{H}_h \Pi_h e^{n+1}||_{0;V_c}\\ &\le \hat{\gamma}_2\, |||E^{n+1}|||_{V_c}\, \hat{\gamma}_1\, |||e^{n+1}|||_V\,. \end{aligned}$$

After simplification we get

$$|||e^{n+1}|||_V \le \hat{\gamma}_1 \hat{\gamma}_2\, |||E^{n+1}|||_{V_c}\,.$$

Taking $H = E^{n+1}$ in the second equation of (28), and with similar steps as before, we have

$$|||E^{n+1}|||^2_{V_c} = a_c(E^{n+1}, E^{n+1}) = -b_h(E^{n+1}, e^n) \le \hat{\gamma}_2\, |||E^{n+1}|||_{V_c}\, \hat{\gamma}_1\, |||e^n|||_V\,.$$

Then, $|||E^{n+1}|||_{V_c} \le \hat{\gamma}_1 \hat{\gamma}_2\, |||e^n|||_V$; combining the estimations on $|||e^{n+1}|||_V$ and $|||E^{n+1}|||_{V_c}$, we get

$$|||e^{n+1}|||_V \le (\hat{\gamma}_1 \hat{\gamma}_2)^2\, |||e^n|||_V\,,$$

and we end the proof by setting $\theta = (\hat{\gamma}_1 \hat{\gamma}_2)^2$.

The convergence of Ω^n_H to Ω_H yields the one of T^n_h to T_h. In (25), the application of B, Q, Q^t and B^t is standard. We do not assemble S and S^t but solve two homogeneous Dirichlet boundary value problems. Thus, at each step n, we have to solve two Dirichlet boundary value problems on $S_h(V_c)$, one curl problem on V_c and one Dirichlet problem on $S_{0;H}(V)$.

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