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On the recovery of a surface with prescribed first and second fundamental forms

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Dedicated to Klaus Kirchgässner on the occasion of his seventieth birthday

Abstract

The fundamental theorem of surface theory asserts that, if a field of positive definite symmetric matrices of order two and a field of symmetric matrices of order two together satisfy the Gauss and Codazzi–Mainardi equations in a connected and simply connected open subset of \mathbb{R}^2 , then there exists a surface in \mathbb{R}^3 with these fields as its first and second fundamental forms (global existence theorem) and this surface is unique up to isometries in \mathbb{R}^3 (rigidity theorem).

The aim of this paper is to provide a self-contained and essentially elementary proof of this theorem by showing how it can be established as a simple corollary of another well-known theorem of differential geometry, which asserts that, if the Riemann–Christoffel tensor associated with a field of positive definite symmetric matrices of order three vanishes in a connected and simply connected open subset of \mathbb{R}^3 , then this field is the metric tensor field of an open set that can be isometrically imbedded in \mathbb{R}^3 (global existence theorem) and this open set is unique up to isometries in \mathbb{R}^3 (rigidity theorem). For convenience, we also give a self-contained proof of this theorem, as such a proof does not seem to be easy to locate in the existing literature.

In addition to the simplicity of its principle, this approach has the merit to shed light on the analogies existing between these two fundamental theorems of differential geometry. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Résumé

Le théorème fondamental de la théorie des surfaces affirme que, si un champ de matrices symétriques définies positives d'ordre deux et un champ de matrices symétriques d'ordre deux vérifient ensemble les équations de Gauss et de Codazzi–Mainardi dans un ouvert connexe et simplement connexe de \mathbb{R}^2 , alors il existe une surface dans \mathbb{R}^3 dont ces champs sont les première et deuxième formes fondamentales (théorème d'existence globale) et cette surface est unique aux isométries de \mathbb{R}^3 près (théorème de rigidité).

Le but de cet article est de donner une preuve autosuffisante et essentiellement élémentaire de ce théorème en montrant comment il peut être obtenu comme un simple corollaire d'un autre théorème bien connu de géométrie différentielle, qui affirme que, si le tenseur de Riemann–Christoffel associé à un champ de matrices symétriques définies positives d'ordre trois s'annule sur un ouvert connexe et simplement connexe de \mathbb{R}^3 , alors ce champ est le tenseur métrique d'un ouvert qui peut être plongé isométriquement dans \mathbb{R}^3 (théorème d'existence globale) et cet ouvert est unique aux isométries de \mathbb{R}^3 près (théorème de rigidité). Par commodité, on donne également une preuve autosuffisante de ce théorème, car une telle preuve ne semble pas aisée à localiser dans la littérature.

Outre la simplicité de son principe, cette approche a le mérite d'illustrer les analogies existant entre ces deux théorèmes fondamentaux de la géométrie différentielle. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

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Introduction

The two-dimensional equations proposed by Koiter [12] for modeling a *nonlinearly elastic shell* made with a homogeneous and isotropic material are derived from three-dimensional elasticity on the basis of two *a priori* assumptions: One assumption, of a *geometrical* nature, is the *Kirchhoff–Love assumption*; it asserts that any point situated on a normal to the middle surface remains on the normal to the deformed middle surface after the deformation has taken place and that, in addition, the distance between such a point and the middle surface remains constant. The other assumption, of a *mechanical* nature, asserts that the state of stress inside the shell is planar and parallel to the middle surface (this second assumption is itself based on delicate *a priori* estimates due to John [11]).

Taking these *a priori* assumptions into account, W.T. Koiter then reaches the conclusion that the strain energy V per unit area of the undeformed middle surface of the shell takes the form (cf. [12, equations (4.2), (8.1), and (8.3)]):

$$V = \frac{\varepsilon}{2} a^{\alpha\beta\sigma\tau} (\tilde{a}_{\sigma\tau} - a_{\sigma\tau})(\tilde{a}_{\alpha\beta} - a_{\alpha\beta}) + \frac{\varepsilon^3}{6} a^{\alpha\beta\sigma\tau} (\tilde{b}_{\sigma\tau} - b_{\sigma\tau})(\tilde{b}_{\alpha\beta} - b_{\alpha\beta}),$$

where 2ε is the thickness of the shell,

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu (a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}),$$

$\lambda > 0$ and $\mu > 0$ are the two *Lamé constants* of the constituting material, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the given undeformed middle surface, $(a^{\alpha\beta}) = (a_{\alpha\beta})^{-1}$ (details about these notions are provided in Section 3), and finally $\tilde{a}_{\alpha\beta}$ and $\tilde{b}_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the *unknown deformed middle surface* under the action of given applied forces (*ad hoc* boundary conditions are also specified along the boundary of the middle surface).

In order that they actually define a surface imbedded in a three-dimensional Euclidean space, the unknown components $\tilde{a}_{\alpha\beta}$ and $\tilde{b}_{\alpha\beta}$ must therefore satisfy *ad hoc* sufficient compatibility conditions, which take the form of the classical *Gauss and Codazzi–Mainardi equations*. This crucial observation thus explains why “a system of fully consistent equations of compatibility is considered essential in a discussion of the general nonlinear theory of shells” (cf. [12, p. 21]).

Viewed as *equality constraints* (in the sense of optimization theory) imposed on the unknown functions $\tilde{a}_{\alpha\beta}$ and $\tilde{b}_{\alpha\beta}$, these compatibility conditions also play a key rôle for finding the expression of the stress resultants and stress couples, by means of *ad hoc* Lagrange multipliers (cf. [12, equations (7.3)–(7.4)]).

It is this type of consideration that motivated the present work, whose objective is to provide a self-contained and essentially elementary proof (Theorem 5) of the sufficiency of the Gauss and Codazzi–Mainardi equations for the existence of a surface with given first and second fundamental forms in a three-dimensional Euclidean space.

The novelty lies in the proof itself, where this sufficiency is obtained as a simple corollary of another well-known theorem of differential geometry, which asserts that the vanishing of the Riemann–Christoffel tensor is a sufficient condition for the existence of a three-dimensional open set with a given metric tensor in a three-dimensional Euclidean space. For completeness, we also give a self-contained proof of this result (Theorem 2), as such a proof does not seem to be easy to locate in the existing literature.

For completeness again, we also give a self-contained proof of the associated “*rigidity theorems*”, which assert that, when these sufficient conditions are satisfied, the surface or the three-dimensional open set are unique up to isometries in a three-dimensional Euclidean space (Theorems 3 and 6).

1. The metric tensor of a three-dimensional open set

To begin with, we list some notations and conventions that will be consistently used throughout the article.

All spaces, matrices, etc., considered are real. For $n = 2$ and 3 , \mathbb{M}^n , \mathbb{S}^n , and $\mathbb{S}_>^n$ respectively designate the sets of all square matrices of order n , of all symmetric matrices of order n , and of all symmetric, positive definite matrices of order n .

Latin indices and exponents vary in the set $\{1, 2, 3\}$, except when they are used for indexing sequences, and the summation convention with respect to repeated indices or exponents is systematically used in conjunction with this rule. Kronecker’s symbols are designated by δ_i^j , δ_{ij} , or δ^{ij} according to the context.

Let \mathbf{E}^3 denote a *three-dimensional Euclidean space*, let $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ denote the Euclidean inner product and exterior product of $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$, and let $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$ denote the Euclidean norm of $\mathbf{a} \in \mathbf{E}^3$. In addition, let there be given a *three-dimensional vector space*, identified with \mathbb{R}^3 . Let x_i denote the coordinates of a point $x \in \mathbb{R}^3$ and let $\partial_i := \partial/\partial x_i$ and $\partial_{ij} := \partial^2/\partial x_i \partial x_j$.

Let Ω be an open subset of \mathbb{R}^3 and let $\Theta \in C^1(\Omega; \mathbf{E}^3)$ be an *immersion*, i.e., a mapping such that the three vectors $\partial_i \Theta(x)$ are linearly independent at all points $x \in \Omega$. Hence the set $\Theta(\Omega)$ is *open* in \mathbf{E}^3 (for a proof, see, e.g., [16, Theorem 3.8.10]).

The *metric tensor of the set* $\Theta(\Omega)$ is defined by means of its *covariant components*

$$g_{ij}(x) := \partial_i \Theta(x) \cdot \partial_j \Theta(x), \quad x \in \Omega,$$

which are used in particular for computing *lengths of curves inside the set* $\Theta(\Omega)$, considered as being *isometrically imbedded in* \mathbf{E}^3 . This means that their length is precisely that induced by the Euclidean metric of the Euclidean space \mathbf{E}^3 .

It is also well-known that the matrix field $(g_{ij}) : \Omega \rightarrow \mathbb{S}_>^3$ defined in this fashion cannot be arbitrary. More specifically, its components and some of their partial derivatives must satisfy *necessary conditions* taking the form of relations (1.3) below (according to our rule governing Latin indices and exponents, relations (1.3) are meant to hold for all $i, j, k, q \in \{1, 2, 3\}$).

Theorem 1. *Let Ω be an open subset of \mathbb{R}^3 , let $\Theta \in C^3(\Omega; \mathbf{E}^3)$ be an immersion, and let*

$$g_{ij} := \partial_i \Theta \cdot \partial_j \Theta \tag{1.1}$$

denote the covariant components of the metric tensor of the set $\Theta(\Omega)$. *Let the functions* $\Gamma_{ijq} \in C^1(\Omega)$ *and* $\Gamma_{ij}^p \in C^1(\Omega)$ *be defined by*

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \quad \text{and} \quad \Gamma_{ij}^p := g^{pq} \Gamma_{ijq}, \quad \text{where } (g^{pq}) := (g_{ij})^{-1}. \tag{1.2}$$

Then, necessarily,

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp} = 0 \quad \text{in } \Omega. \tag{1.3}$$

Proof (provided for completeness). Let $\mathbf{g}_i := \partial_i \Theta$. It is then immediately verified that the functions Γ_{ijq} are also given by

$$\Gamma_{ijq} = \partial_i \mathbf{g}_j \cdot \mathbf{g}_q. \tag{1.4}$$

For each $x \in \Omega$, let the three vectors $\mathbf{g}^j(x)$ be defined by the relations $\mathbf{g}^j(x) \cdot \mathbf{g}_i(x) = \delta_i^j$. Since we also have $\mathbf{g}^j = g^{ij} \mathbf{g}_i$, relations (1.4) imply that $\Gamma_{ij}^p = \partial_i \mathbf{g}_j \cdot \mathbf{g}^p$. Therefore,

$$\partial_i \mathbf{g}_j = \Gamma_{ij}^p \mathbf{g}_p \tag{1.5}$$

since $\partial_i \mathbf{g}_j = (\partial_i \mathbf{g}_j \cdot \mathbf{g}^p) \mathbf{g}_p$. Differentiating relations (1.4) yields

$$\partial_k \Gamma_{ijq} = \partial_{ik} \mathbf{g}_j \cdot \mathbf{g}_q + \partial_i \mathbf{g}_j \cdot \partial_k \mathbf{g}_q,$$

so that relations (1.4) and (1.5) together give

$$\partial_i \mathbf{g}_j \cdot \partial_k \mathbf{g}_q = \Gamma_{ij}^p \mathbf{g}_p \cdot \partial_k \mathbf{g}_q = \Gamma_{ij}^p \Gamma_{kqp}.$$

Consequently,

$$\partial_{ik} \mathbf{g}_j \cdot \mathbf{g}_q = \partial_k \Gamma_{ijq} - \Gamma_{ij}^p \Gamma_{kqp}. \tag{1.6}$$

Since $\partial_{ik} \mathbf{g}_j = \partial_{ij} \mathbf{g}_k$, we also have

$$\partial_{ik} \mathbf{g}_j \cdot \mathbf{g}_q = \partial_j \Gamma_{ikq} - \Gamma_{ik}^p \Gamma_{jqp}, \tag{1.7}$$

so that relations (1.3) are simply obtained by subtracting (1.6) from (1.7). \square

Remarks. (1) The vectors \mathbf{g}_i and \mathbf{g}^j introduced above form the *covariant* and *contravariant bases*, the function g^{ij} are the *contravariant* components of the metric tensor, the functions Γ_{ij}^p and Γ_{ijq} are the *Christoffel symbols of the first*, and *second*, *kind* and finally, the functions

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp}$$

are the covariant components of the *Riemann–Christoffel curvature tensor*, of the set $\Theta(\Omega)$. The relations $R_{qijk} = 0$ found in Theorem 1 thus express that the *Riemann–Christoffel tensor of the set* $\Theta(\Omega)$ (equipped with the metric tensor with covariant components g_{ij}) *vanishes*. For details, see, e.g., [5, p. 303].

(2) The necessary conditions $R_{qijk} = 0$ of Theorem 1 thus simply constitute a re-writing of the relations $\partial_{ik} \mathbf{g}_j = \partial_{ij} \mathbf{g}_k$ in the form of the equivalent relations $\partial_{ik} \mathbf{g}_j \cdot \mathbf{g}_q = \partial_{ij} \mathbf{g}_k \cdot \mathbf{g}_q$.

2. Existence and uniqueness of a mapping defined on an open set in \mathbb{R}^3 that gives rise to a prescribed metric tensor

We now turn to the reciprocal questions:

Given an open subset Ω of \mathbb{R}^3 and a smooth enough matrix field $(g_{ij}) : \Omega \rightarrow \mathbb{S}_{>}^3$, when is it the metric tensor field of an open set $\Theta(\Omega) \subset \mathbb{E}^3$, i.e., when does there exist an immersion $\Theta : \Omega \rightarrow \mathbb{E}^3$ such that $g_{ij} = \partial_i \Theta \cdot \partial_j \Theta$ in Ω ?

If such an immersion exists, to what extent is it unique?

The answers turn out to be remarkably simple: *Under the additional assumptions that Ω is connected and simply connected, the necessary conditions (1.3) of Theorem 1 are also sufficient for the existence of such an immersion, and this immersion is unique up to isometries in \mathbb{R}^3 .*

As it does not seem easy to locate a self-contained, elementary, and complete proof of this well-known result of differential geometry in the existing literature, we provide one here for completeness. Its outline follows, with some modifications and simplifications, that of Blume [1]. In addition, we have included and adapted to our present purposes the proof of a crucial global existence theorem for systems of partial differential equations due to Cartan [4]. “Local” versions of the next theorem, based on the theory of locally integrable Pfaff systems and on the Frobenius theorem, are found by Choquet-Bruhat, Dewitt-Morette, Dillard-Bleick [5, p. 303] and Malliavin [14, p. 133].

This result comprises two essentially distinct parts, a *global existence result* (Theorem 2) and a *uniqueness result* (Theorem 3), the latter being called the *rigidity theorem*. Note that these two results are established under *different assumptions* on the set Ω and on the smoothness of the field (g_{ij}) .

Theorem 2 (global existence theorem). *Let Ω be a connected and simply connected open subset of \mathbb{R}^3 and $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}_{>}^3)$ be a matrix field that satisfies*

$$\partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp} = 0 \quad \text{in } \Omega, \quad (2.1)$$

where

$$\Gamma_{ijq} := \frac{1}{2}(\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \quad \text{and} \quad \Gamma_{ij}^p := g^{pq} \Gamma_{ijq}, \quad \text{where } (g^{pq}) := (g_{ij})^{-1}. \quad (2.2)$$

Then there exists an immersion $\Theta \in \mathcal{C}^3(\Omega; \mathbb{E}^3)$ such that

$$g_{ij} = \partial_i \Theta \cdot \partial_j \Theta \quad \text{in } \Omega. \quad (2.3)$$

The proof of Theorem 2 relies on a simple, yet crucial, observation. When the mapping $\Theta = (\Theta_\ell)$ is *a priori* given (as in Section 1), its components Θ_ℓ satisfy the relations $\partial_j \Theta_\ell = \Gamma_{ij}^p \partial_p \Theta_\ell$, which are nothing but another way of writing the relations $\partial_i g_j = \Gamma_{ij}^p g_p$ (see the proof of Theorem 1). This observation thus suggests to begin by solving (Lemma 2) the linear system of partial differential equations

$$\partial_i F_{\ell j} = \Gamma_{ij}^p F_{\ell p} \quad \text{in } \Omega, \quad (2.4)$$

whose solutions $F_{\ell j} : \Omega \rightarrow \mathbb{R}$ then constitute natural candidates for the partial derivatives $\partial_j \Theta_\ell$ of the unknown mapping $\Theta = (\Theta_\ell)$ (Lemma 3).

To begin with, we prove the following lemma, which will in turn allow us to re-write relations (2.1) in a slightly different form (cf. equations (2.8)), more appropriate for the existence result of Lemma 2.

Lemma 1. *Let Ω be an open subset of \mathbb{R}^3 and let there be given a field $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^3)$ of symmetric invertible matrices. The functions Γ_{ijq} , Γ_{ij}^p , and g^{pq} being defined as in (2.2), define the functions*

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp}, \quad (2.5)$$

$$R_{.ijk}^p := \partial_j \Gamma_{ik}^p - \partial_k \Gamma_{ij}^p + \Gamma_{ik}^\ell \Gamma_{j\ell}^p - \Gamma_{ij}^\ell \Gamma_{k\ell}^p. \quad (2.6)$$

Then

$$R_{.ijk}^p = g^{pq} R_{qijk} \quad \text{and} \quad R_{qijk} = g_{pq} R_{.ijk}^p. \quad (2.7)$$

Proof. Let us establish relations (2.7)_a. Using the relations

$$\Gamma_{jq\ell} + \Gamma_{\ell jq} = \partial_j g_{q\ell} \quad \text{and} \quad \Gamma_{ikq} = g_{q\ell} \Gamma_{ik}^\ell,$$

which themselves follow from definitions (2.2)_a and (2.2)_b, and noting that

$$(g^{pq} \partial_j g_{q\ell} + g_{q\ell} \partial_j g^{pq}) = \partial_j (g^{pq} g_{q\ell}) = 0,$$

we obtain

$$\begin{aligned} g^{pq} (\partial_j \Gamma_{ikq} - \Gamma_{ik}^r \Gamma_{jqr}) &= \partial_j \Gamma_{ik}^p - \Gamma_{ikq} \partial_j g^{pq} - \Gamma_{ik}^\ell g^{pq} (\partial_j g_{q\ell} - \Gamma_{\ell jq}) \\ &= \partial_j \Gamma_{ik}^p + \Gamma_{ik}^\ell \Gamma_{j\ell}^p - \Gamma_{ik}^\ell (g^{pq} \partial_j g_{q\ell} + g_{q\ell} \partial_j g^{pq}) = \partial_j \Gamma_{ik}^p + \Gamma_{ik}^\ell \Gamma_{j\ell}^p. \end{aligned}$$

Likewise,

$$g^{pq} (\partial_k \Gamma_{ijq} - \Gamma_{ij}^r \Gamma_{kqr}) = \partial_k \Gamma_{ij}^p + \Gamma_{ij}^\ell \Gamma_{kl}^p,$$

and thus relations (2.7)_a are established. Relations (2.7)_a and (2.7)_b are clearly equivalent. \square

Following Cartan [4], we now establish the existence of solutions to the linear system of partial differential equations (2.4). Note that Cartan’s approach was later adapted to *nonlinear* systems of partial differential equations by Thomas [19].

Lemma 2. *Let Ω be a connected and simply connected open subset of \mathbb{R}^3 and let there be given functions $\Gamma_{ij}^p = \Gamma_{ji}^p \in C^1(\Omega)$ satisfying the relations*

$$\partial_j \Gamma_{ik}^p - \partial_k \Gamma_{ij}^p + \Gamma_{ik}^\ell \Gamma_{j\ell}^p - \Gamma_{ij}^\ell \Gamma_{k\ell}^p = 0 \quad \text{in } \Omega, \tag{2.8}$$

which, by Lemma 1, are equivalent to relations (2.1) when the functions Γ_{ij}^p are defined as in Theorem 2. Let a point $x^0 \in \Omega$ and a matrix $(F_{\ell j}^0) \in \mathbb{M}^3$ be given. Then there exists one, and only one, field $(F_{\ell j}) \in C^2(\Omega; \mathbb{M}^3)$ that satisfies

$$\partial_i F_{\ell j}(x) = \Gamma_{ij}^p(x) F_{\ell p}(x), \quad x \in \Omega, \tag{2.9}$$

$$F_{\ell j}(x^0) = F_{\ell j}^0. \tag{2.10}$$

Proof. For clarity, the proof is broken into three parts, numbered (i) to (iii).

(i) Let x^1 be an arbitrary point in the set Ω , distinct from x^0 . Since Ω is *connected*, there exists a path $\gamma = (\gamma^i) \in C^1([0, 1]; \mathbb{R}^3)$ joining x^0 to x^1 in Ω ; this means that

$$\gamma(0) = x^0, \quad \gamma(1) = x^1, \quad \text{and } \gamma(t) \in \Omega \quad \text{for all } 0 \leq t \leq 1.$$

Letting $x = \gamma(t)$, $0 \leq t \leq 1$, in equations (2.9), we conclude that, if a matrix field $(F_{\ell j}) \in C^1(\Omega; \mathbb{M}^3)$ satisfies equations (2.9), then, for each integer $\ell \in \{1, 2, 3\}$, the three functions $\zeta_j \in C^1([0, 1])$ defined by (for simplicity, the dependence on ℓ is dropped in what follows)

$$\zeta_j(t) := F_{\ell j}(\gamma(t)), \quad 0 \leq t \leq 1, \tag{2.11}$$

satisfy the following *Cauchy problem* for a linear system of three ordinary differential equations with respect to three unknowns:

$$\frac{d\zeta_j}{dt}(t) = \Gamma_{ij}^p(\gamma(t)) \frac{d\gamma^i}{dt}(t) \zeta_p(t), \quad 0 \leq t \leq 1, \tag{2.12}$$

$$\zeta_j(0) = \zeta_j^0, \tag{2.13}$$

where the “initial” values appearing in the Cauchy conditions (2.13) are given by

$$\zeta_j^0 := F_{\ell j}^0. \tag{2.14}$$

Note in passing that the three Cauchy problems (2.12)–(2.13) obtained by letting $\ell = 1, 2$, or 3 only differ by their initial values ζ_j^0 .

It is well known that a Cauchy problem of the form (with self-explanatory notations)

$$\frac{d\zeta}{dt}(t) = \mathbf{A}(t)\zeta(t), \quad 0 \leq t \leq 1, \quad \zeta(0) = \zeta^0,$$

has *one and only one* solution $\zeta \in C^1([0, 1]; \mathbb{R}^3)$ if $\mathbf{A} \in C^0([0, 1]; \mathbb{M}^3)$ (see, e.g., [16, Theorem 4.3.1, p. 388]). Hence each Cauchy problem (2.12)–(2.13) has one and only one solution.

Incidentally, this result already shows that, *if the system (2.9)–(2.10) has a solution, it is unique.*

(ii) In order that the three values $\zeta_j(1)$ found by solving (2.12)–(2.13) for a given integer $\ell \in \{1, 2, 3\}$ be acceptable candidates for the three unknown values $F_{\ell j}(x^1)$, they must be of course *independent of the path chosen for joining x^0 to x^1* .

So, let $\gamma_0 \in C^1([0, 1]; \mathbb{R}^3)$ and $\gamma_1 \in C^1([0, 1]; \mathbb{R}^3)$ be two paths joining x^0 to x^1 in Ω . The open set Ω being *simply connected*, there exists a *homotopy* $\mathbf{G} = (G^i) : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^3$ joining γ_0 to γ_1 in Ω , i.e., such that

$$\begin{aligned} \mathbf{G}(\cdot, 0) &= \gamma_0, & \mathbf{G}(\cdot, 1) &= \gamma_1, & \mathbf{G}(t, \lambda) &\in \Omega \quad \text{for all } 0 \leq t \leq 1, 0 \leq \lambda \leq 1, \\ \mathbf{G}(0, \lambda) &= x^0 \quad \text{and} \quad \mathbf{G}(1, \lambda) &= x^1 \quad \text{for all } 0 \leq \lambda \leq 1, \end{aligned}$$

and that is smooth enough, in the sense that

$$\mathbf{G} \in C^1([0, 1] \times [0, 1]; \mathbb{R}^3) \quad \text{and} \quad \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{G}}{\partial \lambda} \right) = \frac{\partial}{\partial \lambda} \left(\frac{\partial \mathbf{G}}{\partial t} \right) \in C^0([0, 1] \times [0, 1]; \mathbb{R}^3).$$

Let $\zeta(\cdot, \lambda) = (\zeta_j(\cdot, \lambda)) \in C^1([0, 1]; \mathbb{R}^3)$ denote for each $0 \leq \lambda \leq 1$ the solution of the Cauchy problem (2.12)–(2.13) corresponding to the path $\mathbf{G}(\cdot, \lambda)$ joining x^0 to x^1 . We thus have

$$\frac{\partial \zeta_j}{\partial t}(t, \lambda) = \Gamma_{ij}^p(\mathbf{G}(t, \lambda)) \frac{\partial G^i}{\partial t}(t, \lambda) \zeta_p(t, \lambda) \quad \text{for all } 0 \leq t \leq 1, 0 \leq \lambda \leq 1, \tag{2.15}$$

$$\zeta_j(0, \lambda) = \zeta_j^0 \quad \text{for all } 0 \leq \lambda \leq 1. \tag{2.16}$$

Our objective is to show that

$$\frac{\partial \zeta_j}{\partial \lambda}(1, \lambda) = 0 \quad \text{for all } 0 \leq \lambda \leq 1, \tag{2.17}$$

as this relation will imply that $\zeta_j(1, 0) = \zeta_j(1, 1)$ as desired. For this purpose, a direct differentiation of (2.15) shows that, for all $0 \leq t \leq 1, 0 \leq \lambda \leq 1$,

$$\frac{\partial}{\partial \lambda} \left(\frac{\partial \zeta_j}{\partial t} \right) = \{ \Gamma_{ij}^q \Gamma_{kq}^p + \partial_k \Gamma_{ij}^p \} \zeta_p \frac{\partial G^k}{\partial \lambda} \frac{\partial G^i}{\partial t} + \Gamma_{ij}^p \zeta_p \frac{\partial}{\partial \lambda} \left(\frac{\partial G^i}{\partial t} \right) + \sigma_q \Gamma_{ij}^q \frac{\partial G^i}{\partial t}, \tag{2.18}$$

where

$$\sigma_j := \frac{\partial \zeta_j}{\partial \lambda} - \Gamma_{kj}^p \zeta_p \frac{\partial G^k}{\partial \lambda}, \tag{2.19}$$

on the one hand (in (2.18), (2.19), and in (2.20) below, $\Gamma_{ij}^q, \partial_k \Gamma_{ij}^p$, etc., stand for $\Gamma_{ij}^q(\mathbf{G}(\cdot, \cdot)), \partial_k \Gamma_{ij}^p(\mathbf{G}(\cdot, \cdot))$, etc.). On the other, a direct differentiation of (2.19) shows that, for all $0 \leq t \leq 1, 0 \leq \lambda \leq 1$,

$$\frac{\partial}{\partial t} \left(\frac{\partial \zeta_j}{\partial \lambda} \right) = \frac{\partial \sigma_j}{\partial t} + \left\{ \partial_i \Gamma_{kj}^p \frac{\partial G^i}{\partial t} \zeta_p + \Gamma_{kj}^q \frac{\partial \zeta_q}{\partial t} \right\} \frac{\partial G^k}{\partial \lambda} + \Gamma_{ij}^p \zeta_p \frac{\partial}{\partial t} \left(\frac{\partial G^i}{\partial \lambda} \right),$$

so that, by (2.15),

$$\frac{\partial}{\partial t} \left(\frac{\partial \zeta_j}{\partial \lambda} \right) = \frac{\partial \sigma_j}{\partial t} + \{ \partial_i \Gamma_{kj}^p + \Gamma_{kj}^q \Gamma_{iq}^p \} \zeta_p \frac{\partial G^i}{\partial t} \frac{\partial G^k}{\partial \lambda} + \Gamma_{ij}^p \zeta_p \frac{\partial}{\partial t} \left(\frac{\partial G^i}{\partial \lambda} \right). \tag{2.20}$$

Since

$$\frac{\partial}{\partial \lambda} \left(\frac{\partial \zeta_j}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial \zeta_j}{\partial \lambda} \right) \quad \text{and} \quad \frac{\partial}{\partial \lambda} \left(\frac{\partial G^i}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial G^i}{\partial \lambda} \right)$$

by assumption, subtracting (2.20) from (2.18) yields

$$\frac{\partial \sigma_j}{\partial t} + \{ \partial_i \Gamma_{kj}^p - \partial_k \Gamma_{ij}^p + \Gamma_{kj}^q \Gamma_{iq}^p - \Gamma_{ij}^q \Gamma_{kq}^p \} \zeta_p \frac{\partial G^k}{\partial \lambda} \frac{\partial G^i}{\partial t} - \Gamma_{ij}^q \frac{\partial G^i}{\partial t} \sigma_q = 0.$$

The assumed symmetries $\Gamma_{ij}^p = \Gamma_{ji}^p$ combined with the assumed relations (2.8) show that

$$\partial_i \Gamma_{kj}^p - \partial_k \Gamma_{ij}^p + \Gamma_{kj}^q \Gamma_{iq}^p - \Gamma_{ij}^q \Gamma_{kq}^p = 0,$$

on the one hand. On the other,

$$\sigma_j(0, \lambda) = \frac{\partial \zeta_j}{\partial \lambda}(0, \lambda) - \Gamma_{kj}^p(\mathbf{G}(0, \lambda)) \zeta_p(0, \lambda) \frac{\partial G^k}{\partial \lambda}(0, \lambda) = 0,$$

since $\zeta_j^0(0, \lambda) = \zeta_j^0$ (cf. (2.16)) and $\mathbf{G}(0, \lambda) = x^0$ for all $0 \leq \lambda \leq 1$. Therefore, for any fixed value of the parameter $\lambda \in [0, 1]$, each function $\sigma_j(\cdot, \lambda)$ defined by (2.19) satisfies a Cauchy problem for an ordinary differential equation, viz.,

$$\frac{d\sigma_j}{dt}(t, \lambda) = \Gamma_{ij}^q(\mathbf{G}(t, \lambda)) \frac{\partial G^i}{\partial t}(t, \lambda) \sigma_q(t, \lambda), \quad 0 \leq t \leq 1, \tag{2.21}$$

$$\sigma_j(0, \lambda) = 0. \tag{2.22}$$

But the solution of (2.21)–(2.22) is unique; consequently $\sigma_j(t, \lambda) = 0$ for all $0 \leq t \leq 1$. In particular then,

$$\sigma_j(1, \lambda) = \frac{\partial \zeta_j}{\partial \lambda}(1, \lambda) - \Gamma_{kj}^p(\mathbf{G}(1, \lambda)) \zeta_p(1, \lambda) \frac{\partial G^k}{\partial \lambda}(1, \lambda) = 0 \quad \text{for all } 0 \leq \lambda \leq 1,$$

and thus relations (2.17) hold since $\mathbf{G}(1, \lambda) = x^1$ for all $0 \leq \lambda \leq 1$.

(iii) For each integer ℓ , we may thus unambiguously define a vector field $(F_{\ell j}) : \Omega \rightarrow \mathbb{R}^3$ by letting

$$F_{\ell j}(x^1) := \zeta_j(1) \quad \text{for any } x^1 \in \Omega,$$

where $\boldsymbol{\gamma} \in \mathcal{C}^1([0, 1]; \mathbb{R}^3)$ is any path joining x^0 to x^1 in Ω and the vector field $(\zeta_j) \in \mathcal{C}^1([0, 1])$ is the solution of the Cauchy problem (2.12)–(2.13) corresponding to such a path.

To establish that such a vector field is indeed the ℓ th row-vector field of the unknown matrix field we are seeking, we need to show that $(F_{\ell j})_{j=1}^3 \in \mathcal{C}^1(\Omega; \mathbb{R}^3)$ and that this field does satisfy the partial differential equations (2.9) corresponding to the fixed integer ℓ used in the Cauchy problem (2.12)–(2.13).

Let x be an arbitrary point in Ω and let the integer $i \in \{1, 2, 3\}$ be fixed in what follows. Then there exist $x^1 \in \Omega$, a path $\boldsymbol{\gamma} \in \mathcal{C}^1([0, 1]; \mathbb{R}^3)$ joining x^0 to x^1 in Ω , $\tau \in]0, 1[$, and an open interval $I \subset [0, 1]$ containing τ such that

$$\boldsymbol{\gamma}(t) = x + (t - \tau)\mathbf{e}_i \quad \text{for } t \in I,$$

where \mathbf{e}_i is the i th basis vector in \mathbb{R}^3 . Since each function ζ_j is continuously differentiable in $[0, 1]$ and satisfies the differential equation (2.12),

$$\zeta_j(t) = \zeta_j(\tau) + (t - \tau) \frac{d\zeta_j}{dt}(\tau) + o(t - \tau) = \zeta_j(\tau) + (t - \tau) \Gamma_{ij}^p(\boldsymbol{\gamma}(\tau)) \zeta_p(\tau) + o(t - \tau)$$

for all $t \in I$. Equivalently,

$$F_{\ell j}(x + (t - \tau)\mathbf{e}_i) = F_{\ell j}(x) + (t - \tau) \Gamma_{ij}^p(x) F_{\ell p}(x) + o(t - x).$$

This relation shows that each function $F_{\ell j}$ possesses partial derivatives in the set Ω , given at each $x \in \Omega$ by

$$\partial_i F_{\ell p}(x) = \Gamma_{ij}^p(x) F_{\ell p}(x).$$

Consequently, the matrix field $(F_{\ell j})$ is of class \mathcal{C}^1 in Ω (its partial derivatives are continuous in Ω) and it satisfies the partial differential equations (2.9), as desired. Differentiating equations (2.9) shows that the matrix field $(F_{\ell j})$ is in fact of class \mathcal{C}^2 in Ω . \square

Remark. The assumptions (2.8) made in Lemma 2 on the functions $\Gamma_{ij}^p = \Gamma_{ji}^p$ are thus sufficient conditions for equations (2.9) to have solutions. Conversely, a simple computation shows that they may be also viewed as necessary conditions, simply expressing that, if equations (2.9) have a solution, then necessarily $\partial_{ik} F_{\ell j} = \partial_{ki} F_{\ell j}$ in Ω .

It is no surprise that these necessary conditions are of the same nature as those of Theorem 1, viz., $\partial_{ik} \mathbf{g}_j = \partial_{ij} \mathbf{g}_k$ in Ω .

In order to conclude the proof of Theorem 1, it remains to adequately choose the “initial” conditions $F_{\ell j}^0$ appearing in equations (2.10):

Lemma 3. Let Ω be a connected and simply connected subset of \mathbb{R}^3 and let $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}_>^3)$ be a matrix field satisfying equations (2.1), the functions $\Gamma_{ijq}^p, \Gamma_{ij}^p$, and g^{pq} being defined as in (2.2).

Given an arbitrary point $x^0 \in \Omega$, let $(F_{\ell j}^0) \in \mathbb{S}_>^3$ denote the square root of the matrix $(g_{ij}^0) := (g_{ij}(x^0)) \in \mathbb{S}_>^3$.

Let $(F_{\ell j}) \in \mathcal{C}^2(\Omega; \mathbb{M}^3)$ denote the solution of the corresponding system (2.9)–(2.10); this solution exists and is unique by Lemmas 1 and 2. Then there exists an immersion $\boldsymbol{\Theta} = (\Theta_\ell) \in \mathcal{C}^3(\Omega; \mathbf{E}^3)$ such that

$$\partial_j \Theta_\ell = F_{\ell j} \quad \text{and} \quad g_{ij} = \partial_i \boldsymbol{\Theta} \cdot \partial_j \boldsymbol{\Theta} \quad \text{in } \Omega. \tag{2.23}$$

Proof. (i) We first show that *the three vector fields defined by*

$$\mathbf{g}_j := (F_{\ell j})_{\ell=1}^3 \in C^2(\Omega; \mathbb{R}^3) \tag{2.24}$$

satisfy

$$\mathbf{g}_i \cdot \mathbf{g}_j = g_{ij} \quad \text{in } \Omega. \tag{2.25}$$

To this end, we note that, by construction, these fields satisfy

$$\partial_i \mathbf{g}_j = \Gamma_{ij}^p \mathbf{g}_p \quad \text{in } \Omega, \tag{2.26}$$

$$\mathbf{g}_j(x^0) = \mathbf{g}_j^0, \tag{2.27}$$

where \mathbf{g}_j^0 is the j th column vector of the matrix $\sqrt{g_{ij}(x^0)} \in \mathbb{S}_{>}^3$. Hence the matrix field $(\mathbf{g}_i \cdot \mathbf{g}_j) \in C^2(\Omega; \mathbb{M}^3)$ satisfies

$$\partial_k (\mathbf{g}_i \cdot \mathbf{g}_j) = \Gamma_{kj}^m (\mathbf{g}_m \cdot \mathbf{g}_i) + \Gamma_{ki}^m (\mathbf{g}_m \cdot \mathbf{g}_j) \quad \text{in } \Omega, \tag{2.28}$$

$$(\mathbf{g}_i \cdot \mathbf{g}_j)(x^0) = g_{ij}^0. \tag{2.29}$$

By (2.2)_a, $\partial_k g_{ij} = \Gamma_{ikj} + \Gamma_{jki}$, and by (2.2)_b–(2.2)_c, $\Gamma_{ijq} = g_{pq} \Gamma_{ij}^p$. Hence the matrix field $(g_{ij}) \in C^2(\Omega; \mathbb{S}_{>}^3)$ satisfies

$$\partial_k g_{ij} = \Gamma_{kj}^m g_{mi} + \Gamma_{ki}^m g_{mj} \quad \text{in } \Omega, \tag{2.30}$$

$$g_{ij}(x^0) = g_{ij}^0. \tag{2.31}$$

Viewed as a system of partial differential equations, together with “initial” conditions, with respect to the matrix field $(g_{ij}) : \Omega \rightarrow \mathbb{M}^3$, the system (2.30)–(2.31) can have *at most one solution* in the space $C^2(\Omega; \mathbb{M}^3)$. To see this, let $x^1 \in \Omega$ be distinct from x^0 and let $\boldsymbol{\gamma} \in C^1([0, 1]; \mathbb{R}^3)$ be any path joining x^0 to x^1 in Ω (as in part (i) of the proof of Lemma 2). Then the nine functions $g_{ij}(\boldsymbol{\gamma}(t))$, $0 \leq t \leq 1$, satisfy a Cauchy problem for a linear system of nine ordinary differential equations and this Cauchy problem has *at most one* solution. By inspecting (2.28)–(2.29) and (2.30)–(2.31), we thus conclude that the fields $\mathbf{g}_i \in C^2(\Omega; \mathbb{R}^3)$ defined in (2.24) satisfy (2.25).

(ii) It thus remains to show that *there exists an immersion $\Theta \in C^3(\Omega; \mathbf{E}^3)$ such that*

$$\partial_i \Theta = \mathbf{g}_i \quad \text{in } \Omega, \tag{2.32}$$

where \mathbf{g}_i are the vector fields defined in (2.24).

By (2.2), $\Gamma_{ij}^p = \Gamma_{ji}^p$; hence any solution $(F_{\ell j}) \in C^2(\Omega; \mathbb{M}^3)$ of the system (2.9)–(2.10) satisfies

$$\partial_i F_{\ell j} = \partial_j F_{\ell i} \quad \text{in } \Omega.$$

The open set Ω being *simply connected*, Poincaré’s theorem shows that, for each integer ℓ , there exists a function $\Theta_\ell \in C^3(\Omega)$ such that

$$\partial_i \Theta_\ell = F_{\ell i} \quad \text{in } \Omega,$$

or, equivalently, such that the mapping $\Theta := (\Theta_\ell) \in C^3(\Omega; \mathbf{E}^3)$ satisfies equations (2.32). That Θ is an immersion follows from the assumed invertibility of the matrices (g_{ij}) . The proof of Lemma 3, and consequently that of Theorem 2, are thus complete. \square

Remark. The assumed positive definiteness of the matrices (g_{ij}) is used only in Lemma 3, for defining an *ad hoc* “initial” vector \mathbf{g}_i^0 in the system (2.26)–(2.27).

We now turn to the question of *uniqueness*. The proof given here is adapted from Ciarlet [6, Theorems 1.8–1 and 1.8–2] and Blume [1, Theorem 2.1].

Theorem 3 (rigidity theorem). *Let Ω be a connected open subset of \mathbb{R}^3 and let $\Theta \in C^1(\Omega; \mathbf{E}^3)$ and $\tilde{\Theta} \in C^1(\Omega; \mathbf{E}^3)$ be two immersions whose associated metric tensors satisfy (with self-explanatory notations)*

$$g_{ij} = \tilde{g}_{ij} \quad \text{in } \Omega. \tag{2.33}$$

Then there exist a vector $\mathbf{c} \in \mathbf{E}^3$ and an orthogonal matrix \mathbf{Q} of order three such that

$$\Theta(x) = \mathbf{c} + \mathbf{Q}\tilde{\Theta}(x) \quad \text{for all } x \in \Omega. \tag{2.34}$$

Proof. For convenience, the three-dimensional vector space \mathbb{R}^3 is identified throughout this proof with the Euclidean space \mathbf{E}^3 . In particular then, \mathbb{R}^3 inherits the inner product and norm of \mathbf{E}^3 . We also use the following notation: The matrix representing the Fréchet derivative at $x \in \Omega$ of a differentiable mapping $\Theta = (\Theta_\ell) : \Omega \rightarrow \mathbf{E}^3$ is denoted

$$\nabla \Theta(x) := (\partial_j \Theta_\ell(x)) \in \mathbb{M}^3,$$

where ℓ is the row index and j the column index (equivalently, $\nabla \Theta(x)$ is the matrix of order three whose j th column vector is $\partial_j \Theta$), and the spectral norm of a matrix $\mathbf{A} \in \mathbb{M}^3$ is denoted

$$|\mathbf{A}| := \sup\{|\mathbf{A}\mathbf{v}|; \mathbf{v} \in \mathbb{R}^3, |\mathbf{v}| = 1\}.$$

To begin with, we consider the special case where $\tilde{\Theta} : \Omega \rightarrow \mathbf{E}^3 = \mathbb{R}^3$ is the identity mapping. Solving equations (2.33) reduces in this case to finding $\Theta \in C^1(\Omega; \mathbf{E}^3)$ such that

$$\nabla \Theta(x)^T \nabla \Theta(x) = \mathbf{I} \quad \text{for all } x \in \Omega. \tag{2.35}$$

How to solve equations (2.35) is the object of parts (i) to (iii).

(i) We first establish that a mapping $\Theta \in C^1(\Omega; \mathbf{E}^3)$ that satisfies (2.35) is locally an isometry: Given any point $x^0 \in \Omega$, there exists an open neighborhood V of x^0 contained in Ω such that

$$|\Theta(y) - \Theta(x)| = |y - x| \quad \text{for all } x, y \in V. \tag{2.36}$$

Let B be an open ball centered at x^0 and contained in Ω . Since the set B is convex, the mean-value theorem shows that

$$|\Theta(y) - \Theta(x)| \leq \sup_{z \in]x, y[} |\nabla \Theta(z)| |y - x| \quad \text{for all } x, y \in B.$$

Since the spectral norm of an orthogonal matrix is one, we thus have

$$|\Theta(y) - \Theta(x)| \leq |y - x| \quad \text{for all } x, y \in B. \tag{2.37}$$

Since the matrix $\nabla \Theta(x^0)$ is invertible, the local inversion theorem shows that there exist an open neighborhood V of x^0 contained in Ω and an open neighborhood \widehat{V} of $\Theta(x^0)$ in \mathbf{E}^3 such that the restriction of Θ to V is a C^1 -diffeomorphism from V onto \widehat{V} . Besides, there is no loss of generality in assuming that V is contained in B and that \widehat{V} is convex (to see this, apply the local inversion theorem first to the restriction of Θ to B , thus producing a “first” neighborhood V' of x^0 contained in B , then to the restriction of the inverse mapping obtained in this fashion to an open ball \widehat{V} centered at $\Theta(x^0)$ and contained in $\Theta(V')$).

Let $\Theta^{-1} : \widehat{V} \rightarrow V$ denote the inverse mapping of $\Theta : V \rightarrow \widehat{V}$. The chain rule applied to the relation $\Theta^{-1}(\Theta(x)) = x$ for all $x \in V$ then shows that

$$\widehat{\nabla} \Theta^{-1}(\widehat{x}) = \nabla \Theta(x)^{-1} \quad \text{for all } \widehat{x} = \Theta(x), x \in V.$$

The matrix $\widehat{\nabla} \Theta^{-1}(\widehat{x})$ being thus orthogonal for all $\widehat{x} \in \widehat{V}$, the mean-value theorem applied in the convex set \widehat{V} shows that

$$|\Theta^{-1}(\widehat{y}) - \Theta^{-1}(\widehat{x})| \leq |\widehat{y} - \widehat{x}| \quad \text{for all } \widehat{x}, \widehat{y} \in \widehat{V},$$

or equivalently, that

$$|y - x| \leq |\Theta(y) - \Theta(x)| \quad \text{for all } x, y \in V. \tag{2.38}$$

Since $V \subset B$, inequalities (2.37) and (2.38) together yield the desired relation (2.36).

(ii) We next establish that, if a mapping $\Theta \in C^1(\Omega; \mathbf{E}^3)$ is locally an isometry, in the sense that, given any $x^0 \in \Omega$, there exists an open neighborhood V of x^0 contained in Ω such that relation (2.36) is satisfied, then its derivative is locally constant, in the sense that

$$\nabla \Theta(x) = \nabla \Theta(x^0) \quad \text{for all } x \in V. \tag{2.39}$$

The set V being that found in (i), let the differentiable function $F : V \times V \rightarrow \mathbb{R}$ be defined for all $x = (x_p) \in V$ and all $y = (y_p) \in V$ by

$$F(x, y) := (\Theta_\ell(y) - \Theta_\ell(x))(\Theta_\ell(y) - \Theta_\ell(x)) - (y_\ell - x_\ell)(y_\ell - x_\ell).$$

Then $F(x, y) = 0$ for all $x, y \in V$ by (i). Hence

$$G_i(x, y) := \frac{1}{2} \frac{\partial F}{\partial y_i}(x, y) = \frac{\partial \Theta_\ell}{\partial y_i}(y)(\Theta_\ell(y) - \Theta_\ell(x)) - \delta_{i\ell}(y_\ell - x_\ell) = 0$$

for all $x, y \in V$. For a fixed $y \in V$, each function $G_i(\cdot, y) : V \rightarrow \mathbb{R}$ is differentiable and its derivative vanishes. Consequently,

$$\frac{\partial G_i}{\partial x_j}(x, y) = -\frac{\partial \Theta_\ell}{\partial y_i}(y) \frac{\partial \Theta_\ell}{\partial x_j}(x) + \delta_{ij} = 0 \quad \text{for all } x, y \in V,$$

or equivalently, in matrix form,

$$\nabla \Theta(y)^T \nabla \Theta(x) = \mathbf{I} \quad \text{for all } x, y \in V.$$

Letting $y = x^0$ in this relation yields relation (2.39).

(iii) By (ii), the mapping $\nabla \Theta : \Omega \rightarrow \mathbb{M}^3$ is differentiable and its derivative vanishes in Ω . Therefore *the mapping $\Theta : \Omega \rightarrow \mathbb{E}^3$ is twice differentiable and its second Fréchet derivative vanishes in Ω* . The open set Ω being *connected*, a classical result from differential calculus (see, e.g., [16, Theorem 3.7.10]) shows that the mapping Θ is *affine* in Ω , i.e., there exists a vector $\mathbf{c} \in \mathbb{E}^3$ and a matrix $\mathbf{Q} \in \mathbb{M}^3$ such that

$$\Theta(x) = \mathbf{c} + \mathbf{Q} \mathbf{o} x \quad \text{for all } x \in \Omega.$$

Besides, $\mathbf{Q} = \nabla \Theta(x^0)$ is an orthogonal matrix by (2.35).

(iv) We now consider the “general” equations (2.33), noting that they equivalently read

$$\nabla \Theta(x)^T \nabla \Theta(x) = \nabla \tilde{\Theta}(x)^T \nabla \tilde{\Theta}(x) \quad \text{for all } x \in \Omega. \tag{2.40}$$

Given any point $x^0 \in \Omega$, let the neighborhoods V of x^0 and \widehat{V} of $\Theta(x^0)$ and the mapping $\Theta^{-1} : \widehat{V} \rightarrow V$ be defined as in part (i) (by assumption, the mapping Θ is an immersion; hence the matrix $\nabla \Theta(x^0)$ is invertible).

Consider the composite mapping

$$\widehat{\Phi} := \tilde{\Theta} \circ \Theta^{-1} : \widehat{V} \rightarrow \mathbb{E}^3.$$

Clearly, $\widehat{\Phi} \in \mathcal{C}^1(\widehat{V}; \mathbb{E}^3)$ and

$$\widehat{\nabla} \widehat{\Phi}(\widehat{x}) = \nabla \tilde{\Theta}(x) \widehat{\nabla} \Theta^{-1}(\widehat{x}) = \nabla \tilde{\Theta}(x) \nabla \Theta(x)^{-1} \quad \text{for all } \widehat{x} = \Theta(x), x \in V,$$

so that, by relations (2.40),

$$\widehat{\nabla} \widehat{\Phi}(\widehat{x})^T \widehat{\nabla} \widehat{\Phi}(\widehat{x}) = \mathbf{I} \quad \text{for all } \widehat{x} \in \widehat{V}.$$

By parts (i) to (iii), there thus exist a vector $\mathbf{c} \in \mathbb{R}^3$ and an orthogonal matrix \mathbf{Q} of order three such that

$$\widehat{\Phi}(\widehat{x}) = \tilde{\Theta}(x) = \mathbf{c} + \mathbf{Q} \Theta(x) \quad \text{for all } \widehat{x} = \Theta(x), x \in V,$$

hence such that

$$\mathcal{E}(x) := \nabla \tilde{\Theta}(x) \nabla \Theta(x)^{-1} = \mathbf{Q} \quad \text{for all } x \in V.$$

The *continuous* mapping $\mathcal{E} : V \rightarrow \mathbb{M}^3$ defined in this fashion is thus *locally constant* in Ω . As in part (iii), we conclude from the assumed *connectedness* of Ω that the mapping \mathcal{E} is *constant* in Ω . Thus the proof is complete. \square

Remarks. (1) In terms of metric tensors, parts (i) to (iii) of the above proof provide the solution to the equations $g_{ij} = \delta_{ij}$ in Ω , while part (iv) provides the solution to the equations $g_{ij} = \partial_i \tilde{\Theta} \cdot \partial_j \tilde{\Theta}$ in Ω , where $\tilde{\Theta} \in \mathcal{C}^1(\Omega; \mathbb{E}^3)$ is a given immersion.

(2) The classical *Mazur–Ulam theorem* asserts the following: Let Ω be a connected subset in \mathbb{R}^n , and let $\Theta : \Omega \rightarrow \mathbb{R}^n$ be a mapping that satisfies

$$|\Theta(y) - \Theta(x)| = |y - x| \quad \text{for all } x, y \in \Omega.$$

Then there exist a vector $\mathbf{c} \in \mathbb{R}^n$ and an orthogonal matrix \mathbf{Q} of order n such that

$$\Theta(x) = \mathbf{c} + \mathbf{Q} \mathbf{o} x \quad \text{for all } x \in \Omega.$$

Parts (ii) and (iii) of the above proof thus provide a proof of this theorem under the additional assumption that the mapping Θ is of class \mathcal{C}^1 .

(3) When \mathbb{R}^3 is identified with \mathbb{E}^3 as in the proof of Theorem 3, immersions such as $\Theta \in \mathcal{C}^1(\Omega; \mathbb{E}^3)$ may be thought of as *deformations* in the sense of “geometrically exact” three-dimensional elasticity (although they should then be in addition injective and orientation-preserving in order to qualify for this definition; for details, see, e.g., [6, Section 1.4]).

3. The first and second fundamental forms of a surface

In addition to the rules governing Latin indices and exponents that we set in Section 1, we henceforth require that *Greek* indices and exponents vary in the set $\{1, 2\}$. The summation convention with respect to repeated indices and exponents is extended to these indices and exponents.

Let there be given a *two-dimensional vector space*, identified with \mathbb{R}^2 . Let y_α denote the coordinates of a point $y \in \mathbb{R}^2$ and let $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$.

Let ω be an open subset of \mathbb{R}^2 and let $\theta \in C^2(\omega; \mathbf{E}^3)$ be an *immersion*, i.e., a mapping such that the two vectors $\partial_\alpha \theta(y)$ are linearly independent at all points $y \in \omega$. The image $\theta(\omega)$ is a *surface* in \mathbf{E}^3 .

The *first fundamental form of the surface* $\theta(\omega)$ is defined by means of its *covariant components*

$$a_{\alpha\beta}(y) := \partial_\alpha \theta(y) \cdot \partial_\beta \theta(y), \quad y \in \omega,$$

which are used in particular for computing *lengths of curves on the surface* $\theta(\omega)$, considered as being *isometrically imbedded* in \mathbf{E}^3 .

The *second fundamental form of the surface* $\theta(\omega)$ is defined by means of its *covariant components*

$$b_{\alpha\beta}(y) := \partial_{\alpha\beta} \theta(y) \cdot \left\{ \frac{\partial_1 \theta(y) \wedge \partial_2 \theta(y)}{|\partial_1 \theta(y) \wedge \partial_2 \theta(y)|} \right\}, \quad y \in \omega,$$

which, together with those of the first fundamental form, are used for computing *curvatures of curves on the surface* $\theta(\omega)$.

It is also well known that the matrix fields $(a_{\alpha\beta}) : \omega \rightarrow \mathbb{S}_>^2$ and $(b_{\alpha\beta}) : \omega \rightarrow \mathbb{S}^2$ defined in this fashion cannot be arbitrary. More specifically, their components and some of their partial derivatives must satisfy *necessary conditions* taking the form of relations (3.3)–(3.4) below (meant to hold for all $\alpha, \beta, \sigma, \tau \in \{1, 2\}$), which respectively constitute the *Gauss*, and *Codazzi–Mainardi*, equations.

Theorem 4. *Let ω be an open subset of \mathbb{R}^2 , let $\theta \in C^3(\omega; \mathbf{E}^3)$ be an immersion, and let*

$$a_{\alpha\beta} := \partial_\alpha \theta \cdot \partial_\beta \theta \quad \text{and} \quad b_{\alpha\beta} := \partial_{\alpha\beta} \theta \cdot \left\{ \frac{\partial_1 \theta \wedge \partial_2 \theta}{|\partial_1 \theta \wedge \partial_2 \theta|} \right\} \tag{3.1}$$

denote the first and second fundamental forms of the surface $\theta(\omega)$. Let the functions $C_{\alpha\beta\tau} \in C^1(\omega)$ and $C_{\alpha\beta}^\sigma \in C^1(\omega)$ be defined by

$$C_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}) \quad \text{and} \quad C_{\alpha\beta}^\sigma := a^{\sigma\tau} C_{\alpha\beta\tau}, \quad \text{where } (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}. \tag{3.2}$$

Then, necessarily,

$$\partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C_{\alpha\beta}^\mu C_{\sigma\tau\mu} - C_{\alpha\sigma}^\mu C_{\beta\tau\mu} = b_{\alpha\sigma} b_{\beta\tau} - b_{\alpha\beta} b_{\sigma\tau} \quad \text{in } \omega, \tag{3.3}$$

$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + C_{\alpha\sigma}^\mu b_{\beta\mu} - C_{\alpha\beta}^\mu b_{\sigma\mu} = 0 \quad \text{in } \omega. \tag{3.4}$$

Proof (provided for completeness). Let $\mathbf{a}_\alpha := \partial_\alpha \theta$. It is then immediately verified that the functions $C_{\alpha\beta\tau}$ are also given by

$$C_{\alpha\beta\tau} = \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}_\tau. \tag{3.5}$$

Let $\mathbf{a}_3 := (\mathbf{a}_1 \wedge \mathbf{a}_2)/|\mathbf{a}_1 \wedge \mathbf{a}_2|$ and, for each $y \in \omega$, let the three vectors $\mathbf{a}^j(y)$ be defined by the relations $\mathbf{a}^j(y) \cdot \mathbf{a}_i(y) = \delta_i^j$. Since we also have $\mathbf{a}^\beta = a^{\alpha\beta} \mathbf{a}_\alpha$ and $\mathbf{a}^3 = \mathbf{a}_3$, relations (3.5) imply that $C_{\alpha\beta}^\sigma = \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}^\sigma$, hence that

$$\partial_\alpha \mathbf{a}_\beta = C_{\alpha\beta}^\sigma \mathbf{a}^\sigma + b_{\alpha\beta} \mathbf{a}_3, \tag{3.6}$$

since $\partial_\alpha \mathbf{a}_\beta = (\partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}^\sigma) \mathbf{a}_\sigma + (\partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}^3) \mathbf{a}_3$. Differentiating relations (3.5) yields

$$\partial_\sigma C_{\alpha\beta\tau} = \partial_{\alpha\sigma} \mathbf{a}_\beta \cdot \mathbf{a}_\tau + \partial_\alpha \mathbf{a}_\beta \cdot \partial_\sigma \mathbf{a}_\tau,$$

so that relations (3.5) and (3.6) together give

$$\partial_\alpha \mathbf{a}_\beta \cdot \partial_\sigma \mathbf{a}_\tau = C_{\alpha\beta}^\mu \mathbf{a}_\mu \cdot \partial_\sigma \mathbf{a}_\tau + b_{\alpha\beta} \mathbf{a}_3 \cdot \partial_\sigma \mathbf{a}_\tau = C_{\alpha\beta}^\mu C_{\sigma\tau\mu} + b_{\alpha\beta} b_{\sigma\tau}.$$

Consequently,

$$\partial_{\alpha\sigma} \mathbf{a}_\beta \cdot \mathbf{a}_\tau = \partial_\sigma C_{\alpha\beta\tau} - C_{\alpha\beta}^\mu C_{\sigma\tau\mu} - b_{\alpha\beta} b_{\sigma\tau}. \tag{3.7}$$

Since $\partial_{\alpha\sigma} \mathbf{a}_\beta = \partial_{\alpha\beta} \mathbf{a}_\sigma$, we also have

$$\partial_{\alpha\sigma} \mathbf{a}_\beta \cdot \mathbf{a}_\tau = \partial_\beta C_{\alpha\sigma\tau} - C_{\alpha\sigma}^\mu C_{\beta\tau\mu} - b_{\alpha\sigma} b_{\beta\tau}, \tag{3.8}$$

so that the Gauss equations (3.3) are simply obtained by subtracting (3.7) from (3.8).

Since $\partial_\alpha \mathbf{a}_3 = (\partial_\alpha \mathbf{a}_3 \cdot \mathbf{a}_\sigma) \mathbf{a}^\sigma + (\partial_\alpha \mathbf{a}_3 \cdot \mathbf{a}_3) \mathbf{a}^3$ and $\partial_\alpha \mathbf{a}_3 \cdot \mathbf{a}_\sigma = -\partial_\alpha \mathbf{a}_\sigma \cdot \mathbf{a}_3 = -b_{\alpha\sigma}$, we have

$$\partial_\alpha \mathbf{a}_3 = -b_{\alpha\sigma} \mathbf{a}^\sigma. \tag{3.9}$$

Differentiating the relations $b_{\alpha\beta} = \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}_3$, we obtain

$$\partial_\sigma b_{\alpha\beta} = \partial_{\alpha\sigma} \mathbf{a}_\beta \cdot \mathbf{a}_3 + \partial_\alpha \mathbf{a}_\beta \cdot \partial_\sigma \mathbf{a}_3,$$

and from (3.6) and (3.9), we obtain

$$\partial_\alpha \mathbf{a}_\beta \cdot \partial_\sigma \mathbf{a}_3 = -C_{\alpha\beta}^\mu b_{\sigma\mu}.$$

Consequently,

$$\partial_{\alpha\sigma} \mathbf{a}_\beta \cdot \mathbf{a}_3 = \partial_\sigma b_{\alpha\beta} + C_{\alpha\beta}^\mu b_{\sigma\mu}. \tag{3.10}$$

Since $\partial_{\alpha\sigma} \mathbf{a}_\beta = \partial_{\alpha\beta} \mathbf{a}_\sigma$, we also have

$$\partial_{\alpha\sigma} \mathbf{a}_\beta \cdot \mathbf{a}_3 = \partial_\beta b_{\alpha\sigma} + C_{\alpha\sigma}^\mu b_{\beta\mu}, \tag{3.11}$$

so that the Codazzi–Mainardi equations (3.4) are simply obtained by subtracting (3.10) from (3.11). \square

Remarks. (1) The vectors \mathbf{a}_α and \mathbf{a}^β introduced above respectively form the *covariant* and *contravariant bases of the tangent plane* to the surface $\theta(\omega)$, while the unit vector $\mathbf{a}_3 = \mathbf{a}^3$ is *normal* to the surface. The functions $a^{\alpha\beta}$ are the *contravariant* components of the metric tensor, the functions $C_{\alpha\beta}^\sigma$ and $C_{\alpha\beta\tau}$ are the *Christoffel symbols of the first, and second, kind*, and finally, the functions

$$S_{\tau\alpha\beta\sigma} := \partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C_{\alpha\beta}^\mu C_{\sigma\tau\mu} - C_{\alpha\sigma}^\mu C_{\beta\tau\mu}$$

are the covariant components of the *Riemann–Christoffel curvature tensor* of the surface $\theta(\omega)$.

(2) The notations $C_{\alpha\beta\tau}$ and $C_{\alpha\beta}^\sigma$ are intended to avoid confusions with the “three-dimensional” Christoffel symbols Γ_{ijq} and Γ_{ij}^p introduced in Section 1.

(3) Relations (3.6) and (3.9) constitute the well-known *formulas of Gauss and Weingarten*.

(4) The necessary conditions (3.3) and (3.4) of Theorem 4 thus simply constitute a re-writing of the relations $\partial_{\alpha\sigma} \mathbf{a}_\beta = \partial_{\alpha\beta} \mathbf{a}_\sigma$ in the form of the equivalent relations $\partial_{\alpha\sigma} \mathbf{a}_\beta \cdot \mathbf{a}_\tau = \partial_{\alpha\beta} \mathbf{a}_\sigma \cdot \mathbf{a}_\tau$ and $\partial_{\alpha\sigma} \mathbf{a}_\beta \cdot \mathbf{a}_3 = \partial_{\alpha\beta} \mathbf{a}_\sigma \cdot \mathbf{a}_3$.

4. Existence and uniqueness of a surface with prescribed first and second fundamental forms

We now turn to the reciprocal questions:

Given an open subset ω of \mathbb{R}^2 and two smooth enough matrix fields $(a_{\alpha\beta}) : \omega \rightarrow \mathbb{S}_>^2$ and $(b_{\alpha\beta}) : \omega \rightarrow \mathbb{S}^2$, when are they the first and second fundamental forms of a surface $\theta(\omega) \subset \mathbf{E}^3$, i.e., when does there exist an immersion $\theta : \omega \rightarrow \mathbf{E}^3$ such that

$$a_{\alpha\beta} := \partial_\alpha \theta \cdot \partial_\beta \theta \quad \text{and} \quad b_{\alpha\beta} := \partial_\alpha \theta \cdot \left\{ \frac{\partial_1 \theta \wedge \partial_2 \theta}{|\partial_1 \theta \wedge \partial_2 \theta|} \right\} \quad \text{in } \omega?$$

If such an immersion exists, to what extent is it unique?

As in Section 2, the answers turn out to be remarkably simple: *Under the additional assumptions that ω is connected and simply connected, the necessary conditions (3.3) and (3.4) of Theorem 4, i.e., the Gauss and Codazzi–Mainardi equations, are also sufficient for the existence of such an immersion, and this immersion is unique up to isometries in \mathbb{R}^3 .*

We now give a self-contained, complete, and essentially elementary, proof of this well-known result, sometimes referred to as the “*fundamental theorem of surface theory*”. This proof, announced in [7, Section 2.8] and briefly outlined in [8], amounts to showing that it can be established as a simple *corollary* to Theorems 2 and 3. In addition to its simplicity, it has the merit to shed light on the analogies (which cannot remain unnoticed!) between the assumptions and conclusions of both *existence* results (compare Theorems 2 and 5) and both *uniqueness* results (compare Theorems 3 and 6).

A direct proof of the fundamental theorem of surface theory is given by Klingenberg [13, Theorem 3.8.8], where the global existence of the mapping θ is based on an existence theorem for ordinary differential equations, analogous to that used in Lemma 2. The “local” version of this theorem, which constitutes *Bonnet’s theorem*, is proved by, e.g., do Carmo [2].

This result is a special case of the *fundamental theorem of Riemannian geometry*. This theorem asserts that a simply connected Riemannian manifold can be isometrically immersed into a Euclidean space if and only if there exist tensors satisfying the *Gauss–Codazzi equations* and that the isometric immersions are unique up to rigid motions. A substantial literature has been devoted to this theorem and its various proofs, which usually rely on basic notions of Riemannian geometry such as connections or normal bundles and on the theory of differential forms. See in particular the earlier papers of Janet [10] and Cartan [3] and the more recent references of Szczarba [17], Tenenblat [18], and Jacobowitz [9].

Like the “three-dimensional” result established in Section 2, this theorem comprises two essentially distinct parts, a *global existence result* (Theorem 5) and a *uniqueness result* (Theorem 6), the latter being called the *rigidity theorem*. Note that these two results are established under *different assumptions* on the set ω and on the smoothness of the fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$.

Theorem 5 (global existence theorem). *Let ω be a connected and simply connected open subset of \mathbb{R}^2 and let $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}_{>}^2)$ and $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2)$ be two matrix fields that satisfy the Gauss and Codazzi–Mainardi equations, viz.,*

$$\partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C_{\alpha\beta}^\mu C_{\sigma\tau\mu} - C_{\alpha\sigma}^\mu C_{\beta\tau\mu} = b_{\alpha\sigma} b_{\beta\tau} - b_{\alpha\beta} b_{\sigma\tau} \quad \text{in } \omega, \tag{4.1}$$

$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + C_{\alpha\sigma}^\mu b_{\beta\mu} - C_{\alpha\beta}^\mu b_{\sigma\mu} = 0 \quad \text{in } \omega, \tag{4.2}$$

where

$$C_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}) \quad \text{and} \quad C_{\alpha\beta}^\sigma := a^{\sigma\tau} C_{\alpha\beta\tau}, \quad \text{where } (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}. \tag{4.3}$$

Then there exists an immersion $\theta \in \mathcal{C}^3(\omega; \mathbf{E}^3)$ such that

$$a_{\alpha\beta} = \partial_\alpha \theta \cdot \partial_\beta \theta \quad \text{and} \quad b_{\alpha\beta} = \partial_\alpha \beta \theta \cdot \left\{ \frac{\partial_1 \theta \wedge \partial_2 \theta}{|\partial_1 \theta \wedge \partial_2 \theta|} \right\} \quad \text{in } \omega. \tag{4.4}$$

Proof. The proof of Theorem 5 as a corollary to Theorem 2 relies on the following elementary observation: Given a smooth enough immersion $\theta : \omega \rightarrow \mathbf{E}^3$ and $\varepsilon > 0$, let the mapping $\Theta : \omega \times]-\varepsilon, \varepsilon[\rightarrow \mathbf{E}^3$ be defined by

$$\Theta(y, x_3) := \theta(y) + x_3 \mathbf{a}_3(y) \quad \text{for all } (y, x_3) \in \omega \times]-\varepsilon, \varepsilon[,$$

where $\mathbf{a}_3 := (\partial_1 \theta \wedge \partial_2 \theta) / |\partial_1 \theta \wedge \partial_2 \theta|$, and let

$$g_{ij} := \partial_i \Theta \cdot \partial_j \Theta.$$

Then an immediate computation shows that

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \quad \text{and} \quad g_{i3} = \delta_{i3} \quad \text{in } \omega \times]-\varepsilon, \varepsilon[,$$

where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant components of the first and second fundamental forms of the surface $\theta(\omega)$ and $c_{\alpha\beta} := a^{\sigma\tau} b_{\alpha\sigma} b_{\beta\tau}$.

Assume that the matrices (g_{ij}) constructed in this fashion are *invertible*, hence positive definite, over the set $\omega \times]-\varepsilon, \varepsilon[$ (they are not necessarily invertible; but the resulting difficulty is easily circumvented; see parts (i) and (viii) below). Then the field $(g_{ij}) : \omega \times]-\varepsilon, \varepsilon[\rightarrow \mathbb{S}_{>}^3$ becomes a natural candidate for applying the “three-dimensional” existence result of Theorem 2, provided of course that the “three-dimensional” sufficient conditions (2.1) of Theorem 2 can be shown to be hold, as consequences of the “two-dimensional” sufficient conditions (4.1)–(4.2): That this is indeed the case is the essence of the present proof (see parts (i) to (vii)).

By Theorem 2, there then exists an immersion $\Theta : \omega \times]-\varepsilon, \varepsilon[\rightarrow \mathbf{E}^3$ that satisfies $g_{ij} = \partial_i \Theta \cdot \partial_j \Theta$ in $\omega \times]-\varepsilon, \varepsilon[$. It thus remains to check that $\theta := \Theta(\cdot, 0)$ indeed satisfies the announced relations (4.4) (see part (ix)).

The actual implementation of this “program” essentially involves elementary, but sometimes lengthy, computations, which accordingly will be omitted for their most part; only the main intermediate results will be recorded.

For clarity, the proof is broken into nine parts, numbered (i) to (ix).

(i) Given two matrix fields $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}_{>}^2)$ and $(b_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2)$, let the matrix field $(g_{ij}) \in \mathcal{C}^2(\omega \times \mathbb{R}; \mathbb{S}^3)$ be defined by

$$g_{\alpha\beta} := a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \quad \text{and} \quad g_{i3} := \delta_{i3} \quad \text{in } \omega \times \mathbb{R} \tag{4.5}$$

(the variable $y \in \omega$ is omitted; x_3 designates the variable in \mathbb{R}), where

$$c_{\alpha\beta} := b_\alpha^\tau b_{\beta\tau} \quad \text{and} \quad b_\alpha^\tau := a^{\sigma\tau} b_{\alpha\sigma} \quad \text{in } \omega. \tag{4.6}$$

Let ω_0 be an open subset of \mathbb{R}^2 such that $\bar{\omega}_0$ is a compact subset of ω . Then there exists $\varepsilon_0 = \varepsilon_0(\omega_0) > 0$ such that the symmetric matrices (g_{ij}) are positive definite at all points in $\bar{\Omega}_0$, where

$$\Omega_0 := \omega_0 \times]-\varepsilon_0, \varepsilon_0[. \tag{4.7}$$

Besides, the inverse matrix (g^{pq}) is given at any point in $\bar{\Omega}_0$ by

$$g^{\alpha\beta} = \sum_{n \geq 0} (n+1)x_3^n a^{\alpha\sigma} (\mathbf{B}^n)_\sigma^\beta \quad \text{and} \quad g^{i3} = \delta^{i3}, \tag{4.8}$$

where

$$(\mathbf{B})_\sigma^\beta := b_\sigma^\beta \quad \text{and} \quad (\mathbf{B}^n)_\sigma^\beta := b_{\sigma_{n-1}}^{\sigma_1} \cdots b_{\sigma_{n-1}}^\beta \quad \text{for } n \geq 2, \tag{4.9}$$

i.e., $(\mathbf{B}^n)_\sigma^\beta$ designates for any $n \geq 0$ the element at the σ th row and β th column of the matrix \mathbf{B}^n . Each series in (4.8) is absolutely convergent in the space $C^2(\bar{\Omega}_0)$.

Let a priori $g^{\alpha\beta} = \sum_{n \geq 0} x_3^n h_n^{\alpha\beta}$ where $h_n^{\alpha\beta}$ are functions of $y \in \bar{\omega}_0$ only, so that the relations $g^{\alpha\beta} g_{\beta\tau} = \delta_\tau^\alpha$ read

$$h_0^{\alpha\beta} a_{\beta\tau} + x_3 (h_1^{\alpha\beta} a_{\beta\tau} - 2h_0^{\alpha\beta} b_{\beta\tau}) + \sum_{n \geq 2} x_3^n (h_n^{\alpha\beta} a_{\beta\tau} - 2h_{n-1}^{\alpha\beta} b_{\beta\tau} + h_{n-2}^{\alpha\beta} c_{\beta\tau}) = \delta_\tau^\alpha.$$

It is then easily verified that the functions $h_n^{\alpha\beta}$ are given by

$$h_n^{\alpha\beta} = (n+1)a^{\alpha\sigma} (\mathbf{B}^n)_\sigma^\beta, \quad n \geq 0,$$

so that

$$g^{\alpha\beta} = \sum_{n \geq 0} (n+1)x_3^n a^{\alpha\sigma} b_{\sigma_{n-1}}^{\sigma_1} \cdots b_{\sigma_{n-1}}^\beta.$$

It is clear that such a series is absolutely convergent in the space $C^2(\bar{\omega}_0 \times]-\varepsilon_0, \varepsilon_0[)$ if $\varepsilon_0 > 0$ is small enough.

(ii) The functions $C_{\alpha\beta}^\sigma$ being defined as in (4.3)_b, define the functions

$$b_\alpha^\tau |_\beta := \partial_\beta b_\alpha^\tau + C_{\beta\mu}^\tau b_\alpha^\mu - C_{\alpha\beta}^\mu b_\mu^\tau, \tag{4.10}$$

$$b_{\alpha\beta} |_\sigma := \partial_\sigma b_{\alpha\beta} - C_{\alpha\sigma}^\mu b_{\beta\mu} - C_{\beta\sigma}^\mu b_{\alpha\mu} = b_{\beta\alpha} |_\sigma. \tag{4.11}$$

Then

$$b_\alpha^\tau |_\beta = a^{\sigma\tau} b_{\alpha\sigma} |_\beta \quad \text{and} \quad b_{\alpha\sigma} |_\beta = a_{\sigma\tau} b_\alpha^\tau |_\beta. \tag{4.12}$$

Furthermore, the assumed relations (4.2) imply that

$$b_\alpha^\tau |_\beta = b_\beta^\tau |_\alpha \quad \text{and} \quad b_{\alpha\sigma} |_\beta = b_{\alpha\beta} |_\sigma. \tag{4.13}$$

Relations (4.12)–(4.13) follow from straightforward computations based on definitions (4.10)–(4.11). They are recorded here because they play a pervading rôle in the subsequent computations.

(iii) The functions $g_{ij} \in C^2(\bar{\Omega}_0)$ and $g^{ij} \in C^2(\bar{\Omega}_0)$ being defined as in (4.5) and (4.8), define the functions $\Gamma_{ijq} \in C^1(\bar{\Omega}_0)$ and $\Gamma_{ij}^p \in C^1(\bar{\Omega}_0)$ by

$$\Gamma_{ijq} := \frac{1}{2} (\partial_j g_{iq} + \partial_i g_{jq} - \partial_q g_{ij}) \quad \text{and} \quad \Gamma_{ij}^p := g^{pq} \Gamma_{ijq}. \tag{4.14}$$

Then the functions $\Gamma_{ijq} = \Gamma_{j iq}$ and $\Gamma_{ij}^p = \Gamma_{ji}^p$ have the following expressions:

$$\Gamma_{\alpha\beta\sigma} = C_{\alpha\beta\sigma} - x_3 (b_\alpha^\tau |_\beta a_{\tau\sigma} + 2C_{\alpha\beta}^\tau b_{\tau\sigma}) + x_3^2 (b_\alpha^\tau |_\beta b_{\tau\sigma} + C_{\alpha\beta}^\tau c_{\tau\sigma}), \tag{4.15}$$

$$\Gamma_{\alpha\beta 3} = -\Gamma_{\alpha 3\beta} = b_{\alpha\beta} - x_3 c_{\alpha\beta}, \tag{4.16}$$

$$\Gamma_{\alpha 33} = \Gamma_{3\beta 3} = \Gamma_{33q} = 0, \tag{4.17}$$

$$\Gamma_{\alpha\beta}^\sigma = C_{\alpha\beta}^\sigma - \sum_{n \geq 0} x_3^{n+1} b_\alpha^\tau |_\beta (\mathbf{B}^n)_\tau^\sigma, \tag{4.18}$$

$$\Gamma_{\alpha\beta}^3 = b_{\alpha\beta} - x_3 c_{\alpha\beta}, \tag{4.19}$$

$$\Gamma_{\alpha 3}^\beta = - \sum_{n \geq 0} x_3^n (\mathbf{B}^{n+1})_\alpha^\beta, \tag{4.20}$$

$$\Gamma_{3\beta}^3 = \Gamma_{33}^P = 0, \tag{4.21}$$

where the functions $c_{\alpha\beta}$, $(\mathbf{B}^n)_\tau^\sigma$, and $b_\alpha^\tau|_\beta$ are defined as in (4.6), (4.9), and (4.10).

All computations are straightforward. We simply point out that the assumed *Codazzi–Mainardi equations* (4.2) are needed to conclude that the factor of x_3 in the function $\Gamma_{\alpha\beta\sigma}$ is indeed that announced in (4.15). We also note that the computation of the factor of x_3^2 in $\Gamma_{\alpha\beta\sigma}$ relies in particular on the relations

$$\partial_\alpha c_{\beta\sigma} = b_\beta^\tau|_\alpha b_{\sigma\tau} + b_\sigma^\mu|_\alpha b_{\mu\beta} + C_{\alpha\beta}^\mu c_{\sigma\mu} + C_{\alpha\sigma}^\mu c_{\beta\mu}. \tag{4.22}$$

(iv) The functions $\Gamma_{ijq} \in C^1(\overline{\Omega}_0)$ and $\Gamma_{ij}^P \in C^1(\overline{\Omega}_0)$ being defined as in (4.14), define the functions $R_{qijk} \in C^0(\overline{\Omega}_0)$ by

$$R_{qijk} := \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^P \Gamma_{kqp} - \Gamma_{ik}^P \Gamma_{jqp}. \tag{4.23}$$

Then, in order that the relations

$$R_{qijk} = 0 \quad \text{in } \overline{\Omega}_0 \tag{4.24}$$

hold, it is sufficient that

$$R_{1212} = 0, \quad R_{\alpha 2\beta 3} = 0, \quad R_{\alpha 3\beta 3} = 0 \quad \text{in } \overline{\Omega}_0. \tag{4.25}$$

The definition of the functions R_{qijk} and relations (4.14) together imply that

$$R_{qijk} = R_{jkqi} = -R_{qikj} \quad \text{and} \quad R_{qijk} = 0 \quad \text{if } j = k \text{ or } q = i.$$

Consequently, equations (4.25)_a imply that $R_{\alpha\beta\sigma\tau} = 0$, equations (4.25)_b imply that $R_{qijk} = 0$ if exactly one index is equal to 3, and finally, equations (4.25)_c imply that $R_{qijk} = 0$ if exactly two indices are equal to 3.

(v) The functions

$$R_{\alpha 3\beta 3} := \partial_\beta \Gamma_{33\alpha} - \partial_3 \Gamma_{3\beta\alpha} + \Gamma_{3\beta}^P \Gamma_{3\alpha p} - \Gamma_{33}^P \Gamma_{\beta\alpha p} \tag{4.26}$$

satisfy

$$R_{\alpha 3\beta 3} = 0 \quad \text{in } \overline{\Omega}_0. \tag{4.27}$$

Relations (4.27) immediately follow from definition (4.26) and the expressions found in part (iii) for the functions Γ_{ijq} and Γ_{ij}^P . Note that neither the Gauss equations (4.1) nor the Codazzi–Mainardi equations (4.2) are needed here.

(vi) The functions

$$R_{\alpha 2\beta 3} := \partial_\beta \Gamma_{23\alpha} - \partial_3 \Gamma_{2\beta\alpha} + \Gamma_{2\beta}^P \Gamma_{3\alpha p} - \Gamma_{23}^P \Gamma_{\beta\alpha p} \tag{4.28}$$

satisfy

$$R_{\alpha 2\beta 3} = 0 \quad \text{in } \overline{\Omega}_0. \tag{4.29}$$

First, definitions (4.5)_a and (4.14)_a show that

$$\partial_\beta \Gamma_{23\alpha} - \partial_3 \Gamma_{2\beta\alpha} = (\partial_2 b_{\alpha\beta} - \partial_\alpha b_{2\beta}) + x_3 (\partial_\alpha c_{2\beta} - \partial_2 c_{\alpha\beta}).$$

Then the expressions found in part (iii) show that

$$\Gamma_{2\beta}^P \Gamma_{3\alpha p} - \Gamma_{23}^P \Gamma_{\beta\alpha p} = \Gamma_{3\alpha}^\sigma \Gamma_{2\beta\sigma} - \Gamma_{23}^\sigma \Gamma_{\alpha\beta\sigma} = C_{\alpha\beta}^\sigma b_{2\sigma} - C_{2\beta}^\sigma b_{\alpha\sigma} + x_3 (b_2^\sigma|_\beta b_{\alpha\sigma} - b_\alpha^\sigma|_\beta b_{2\sigma} + C_{2\beta}^\sigma c_{\alpha\sigma} - C_{\alpha\beta}^\sigma c_{2\sigma}),$$

and relations (4.27) follow by making use of relations (4.22) together with the relations

$$\partial_2 b_{\alpha\beta} - \partial_\alpha b_{2\beta} + C_{\alpha\beta}^\sigma b_{2\sigma} - C_{2\beta}^\sigma b_{\alpha\sigma} = 0,$$

which are special cases of the assumed *Codazzi–Mainardi equations* (4.2).

(vii) The function

$$R_{1212} := \partial_1 \Gamma_{221} - \partial_2 \Gamma_{211} + \Gamma_{21}^P \Gamma_{21p} - \Gamma_{22}^P \Gamma_{11p} \tag{4.30}$$

satisfies

$$R_{1212} = 0 \quad \text{in } \overline{\Omega}_0. \tag{4.31}$$

The computations leading to relation (4.31) are fairly lengthy and they require some care. We simply record the main intermediary steps, which consist in evaluating separately the various terms occurring in R_{1212} rewritten as

$$R_{1212} = (\partial_1 \Gamma_{221} - \partial_2 \Gamma_{211}) + (\Gamma_{12}^\sigma \Gamma_{12\sigma} - \Gamma_{11}^\sigma \Gamma_{22\sigma}) + (\Gamma_{123} \Gamma_{123} - \Gamma_{113} \Gamma_{223}).$$

First, the expressions found in (4.16) easily yield

$$\Gamma_{123} \Gamma_{123} - \Gamma_{113} \Gamma_{223} = (b_{12}^2 - b_{11} b_{22}) + x_3 (b_{11} c_{22} - 2b_{12} c_{12} + b_{22} c_{11}) + x_3^2 (c_{12}^2 - c_{11} c_{22}). \quad (4.32)$$

Second, the expressions found in (4.15) and (4.18) yield, after some manipulations:

$$\begin{aligned} \Gamma_{12}^\sigma \Gamma_{12\sigma} - \Gamma_{11}^\sigma \Gamma_{22\sigma} &= (C_{12}^\sigma C_{12}^\tau - C_{11}^\sigma C_{22}^\tau) a_{\sigma\tau} \\ &+ x_3 \{ (C_{11}^\sigma b_2^\tau |_2 - 2C_{12}^\sigma b_1^\tau |_2 + C_{22}^\sigma b_1^\tau |_1) a_{\sigma\tau} + 2(C_{11}^\sigma C_{22}^\tau - C_{12}^\sigma C_{12}^\tau) b_{\sigma\tau} \} \\ &+ x_3^2 \{ (b_1^\sigma |_1 b_2^\tau |_2 - b_1^\sigma |_2 b_1^\tau |_2) a_{\sigma\tau} + (C_{11}^\sigma b_2^\tau |_2 - 2C_{12}^\sigma b_1^\tau |_2 + C_{22}^\sigma b_1^\tau |_1) b_{\sigma\tau} \\ &+ (C_{11}^\sigma C_{22}^\tau - C_{12}^\sigma C_{12}^\tau) c_{\sigma\tau} \}. \end{aligned} \quad (4.33)$$

Third, after somewhat delicate computations, which in particular make use of relations (4.12) and (4.13) established in part (ii), it is found that

$$\begin{aligned} \partial_1 \Gamma_{221} - \partial_2 \Gamma_{211} &= \partial_1 C_{221} - \partial_2 C_{211} \\ &- x_3 \{ S_{1212} b_\alpha^\alpha + (C_{11}^\sigma b_2^\tau |_2 - 2C_{12}^\sigma b_1^\tau |_2 + C_{22}^\sigma b_1^\tau |_1) a_{\sigma\tau} + 2(C_{11}^\sigma C_{22}^\tau - C_{12}^\sigma C_{12}^\tau) b_{\sigma\tau} \} \\ &+ x_3^2 \{ S_{\sigma\tau 12} b_1^\sigma b_2^\tau + (b_1^\sigma |_1 b_2^\tau |_2 - b_1^\sigma |_2 b_1^\tau |_2) a_{\sigma\tau} + (C_{11}^\sigma b_2^\tau |_2 - 2C_{12}^\sigma b_1^\tau |_2 + C_{22}^\sigma b_1^\tau |_1) b_{\sigma\tau} \\ &+ (C_{11}^\sigma C_{22}^\tau - C_{12}^\sigma C_{12}^\tau) c_{\sigma\tau} \}, \end{aligned} \quad (4.34)$$

where the functions

$$S_{\tau\alpha\beta\sigma} := \partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C_{\alpha\beta}^\mu C_{\sigma\tau\mu} - C_{\alpha\sigma}^\mu C_{\beta\tau\mu} \quad (4.35)$$

are precisely those appearing in the left-hand sides of the Gauss equations (4.1).

It is then easily seen that equations (4.32) to (4.34) together yield

$$R_{1212} = \{ S_{1212} - (b_{11} b_{22} - b_{12} b_{12}) \} - x_3 \{ S_{1212} - (b_{11} b_{22} - b_{12} b_{12}) b_\alpha^\alpha \} + x_3^2 \{ S_{\sigma\tau 12} b_1^\sigma b_2^\tau + (c_{12} c_{12} - c_{11} c_{22}) \}.$$

Since

$$\begin{aligned} S_{\sigma\tau 12} b_1^\sigma b_2^\tau &= S_{1212} (b_1^1 b_2^2 - b_1^2 b_2^1), \\ c_{12} c_{12} - c_{11} c_{22} &= (b_{11} b_{12} - b_{11} b_{22}) (b_1^1 b_2^2 - b_1^2 b_2^1), \end{aligned}$$

it is finally found that the function R_{1212} has the following remarkable expression:

$$R_{1212} = \{ S_{1212} - (b_{11} b_{22} - b_{12} b_{12}) \} \{ 1 - x_3 (b_1^1 + b_2^2) + x_3^2 (b_1^1 b_2^2 - b_1^2 b_2^1) \}.$$

By the assumed Gauss equations (4.1),

$$S_{1212} = b_{11} b_{22} - b_{12} b_{12},$$

hence $R_{1212} = 0$ as announced.

(viii) Let ω be a connected and simply connected open subset of \mathbb{R}^2 . Then there exist open subsets $\omega_n, n \geq 0$, of \mathbb{R}^2 such that $\bar{\omega}_n$ is a compact subset of ω for each $n \geq 0$ and

$$\omega = \bigcup_{n \geq 0} \omega_n. \quad (4.36)$$

Furthermore, for each $n \geq 0$, there exists $\varepsilon_n = \varepsilon_n(\omega_n) > 0$ such that the symmetric matrices (g_{ij}) are positive definite at all points in $\bar{\Omega}_n$, where

$$\Omega_n := \omega_n \times]-\varepsilon_n, \varepsilon_n[. \quad (4.37)$$

Finally, the open set

$$\Omega := \bigcup_{n \geq 0} \Omega_n \quad (4.38)$$

is connected and simply connected.

Let $\omega_n, n \geq 0$, be open subsets of ω with compact closures contained in ω such that relation (4.36) holds. For each n , a set Ω_n of the form given in (4.37) can then be constructed in the same way that the set Ω_0 was constructed in part (i).

It is clear that the set Ω defined in (4.38) is connected. To show that Ω is simply connected, let γ be a loop in Ω , i.e., a mapping $\gamma \in C^0([0, 1]; \mathbb{R}^3)$ that satisfies

$$\gamma(0) = \gamma(1) \quad \text{and} \quad \gamma(t) \in \Omega \quad \text{for all } 0 \leq t \leq 1.$$

Let the projection operator $\pi : \Omega \rightarrow \omega$ be defined by $\pi(y, x_3) = y$ for all $(y, x_3) \in \Omega$, and let $\varphi_0 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^3$ be a mapping defined by

$$\varphi_0(t, \lambda) := (1 - \lambda)\gamma(t) + \lambda\pi(\gamma(t)) \quad \text{for all } 0 \leq t \leq 1, 0 \leq \lambda \leq 1.$$

Then φ_0 is a continuous mapping such that $\varphi_0([0, 1] \times [0, 1]) \subset \Omega$ (by definition (4.38)); furthermore, $\varphi_0(t, 0) = \gamma(t)$ and $\varphi_0(t, 1) = \pi(\gamma(t))$ for all $t \in [0, 1]$.

The mapping

$$\tilde{\gamma} := \pi \circ \gamma \in C^0([0, 1]; \mathbb{R}^2)$$

is a loop in ω since $\tilde{\gamma}(0) = \pi(\gamma(0)) = \pi(\gamma(1)) = \tilde{\gamma}(1)$. Since ω is simply connected, there exist a point $y^0 \in \omega$ and a mapping $\varphi_1 \in C^0([0, 1] \times [0, 1]; \mathbb{R}^2)$ such that

$$\varphi_1(t, 1) = \tilde{\gamma}(x) \quad \text{and} \quad \varphi_1(t, 2) = y^0 \quad \text{for all } 0 \leq t \leq 1,$$

and

$$\varphi_1(t, \lambda) \in \omega \quad \text{for all } 0 \leq t \leq 1, 1 \leq \lambda \leq 2.$$

Then the mapping $\varphi \in C^0([0, 1] \times [0, 2]; \mathbb{R}^3)$ defined by

$$\varphi(t, \lambda) = \varphi_0(t, \lambda) \quad \text{for all } 0 \leq t \leq 1, 0 \leq \lambda \leq 1,$$

$$\varphi(t, \lambda) = \varphi_1(t, \lambda) \quad \text{for all } 0 \leq t \leq 1, 1 \leq \lambda \leq 2,$$

is a homotopy in Ω that reduces the loop γ to the point $(y^0, 0) \in \Omega$. Hence the set Ω is simply connected.

(ix) By parts (iv) to (viii), the functions $\Gamma_{ijq} \in C^1(\Omega)$ and $\Gamma_{ij}^p \in C^1(\Omega)$ constructed as in (4.14) satisfy relations (2.1) in the simply connected open set Ω . By Theorem 2, there thus exists an immersion $\Theta \in C^3(\Omega; \mathbf{E}^3)$ such that

$$g_{ij} = \partial_i \Theta \cdot \partial_j \Theta \quad \text{in } \Omega, \tag{4.39}$$

where the matrix field $(g_{ij}) \in C^2(\Omega; \mathbb{S}_>^3)$ is defined as in (4.5). Then the mapping $\theta \in C^3(\omega; \mathbf{E}^3)$ defined by

$$\theta(y) = \Theta(y, 0) \quad \text{for all } y \in \omega, \tag{4.40}$$

satisfies the required relations (4.4).

Let $\mathbf{g}_i := \partial_i \Theta$. Then $\partial_{33} \Theta = \partial_3 \mathbf{g}_3 = \Gamma_{33}^p \mathbf{g}_p = \mathbf{0}$ by (4.21). Hence there exists a mapping $\theta^1 \in C^3(\omega; \mathbf{E}^3)$ such that

$$\Theta(y, x_3) = \theta(y) + x_3 \theta^1(y) \quad \text{for all } (y, x_3) \in \Omega,$$

and consequently, $\mathbf{g}_\alpha = \partial_\alpha \theta + x_3 \partial_\alpha \theta^1$ and $\mathbf{g}_3 = \theta^1$. The relations $g_{i3} = \mathbf{g}_i \cdot \mathbf{g}_3 = \delta_{i3}$ (cf. (4.5)) then show that

$$(\partial_\alpha \theta + x_3 \partial_\alpha \theta^1) \cdot \theta^1 = 0 \quad \text{and} \quad \theta^1 \cdot \theta^1 = 1.$$

These relations imply that $\partial_\alpha \theta \cdot \theta^1 = 0$. Hence either $\theta^1 = \mathbf{a}_3$ or $\theta^1 = -\mathbf{a}_3$ in ω , where

$$\mathbf{a}_3 := \frac{\partial_1 \theta \wedge \partial_2 \theta}{|\partial_1 \theta \wedge \partial_2 \theta|}.$$

But $\theta^1 = -\mathbf{a}_3$ is ruled out since we must have

$$\{\partial_1 \theta \wedge \partial_2 \theta\} \cdot \theta^1 = \det(g_{ij})|_{x_3=0} > 0.$$

Noting that

$$\partial_\alpha \theta \cdot \mathbf{a}_3 = 0 \quad \text{implies} \quad \partial_\alpha \theta \cdot \partial_\beta \mathbf{a}_3 = -\partial_{\alpha\beta} \theta \cdot \mathbf{a}_3,$$

we obtain, on the one hand,

$$g_{\alpha\beta} = (\partial_\alpha \theta + x_3 \partial_\alpha \mathbf{a}_3) \cdot (\partial_\beta \theta + x_3 \partial_\beta \mathbf{a}_3) = \partial_\alpha \theta \cdot \partial_\beta \theta - 2x_3 \partial_{\alpha\beta} \theta \cdot \mathbf{a}_3 + x_3^2 \partial_\alpha \mathbf{a}_3 \cdot \partial_\beta \mathbf{a}_3 \quad \text{in } \Omega.$$

Since, on the other hand,

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \quad \text{in } \Omega$$

(cf. (4.5)), we conclude that

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \quad \text{and} \quad b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \mathbf{a}_3 \quad \text{in } \omega,$$

as desired. This completes the proof. \square

Remarks. (1) The functions $c_{\alpha\beta} = b_{\alpha}^{\tau} b_{\beta\tau} = \partial_{\alpha} \mathbf{a}_3 \cdot \partial_{\beta} \mathbf{a}_3$ introduced in (4.6) are the covariant components of the *third fundamental form* of the surface $\boldsymbol{\theta}(\omega)$.

(2) The series expansion (4.8) is known. See, e.g., Naghdi [15].

(3) The functions $b_{\alpha}^{\tau} |_{\beta}$ and $b_{\alpha\beta} |_{\sigma}$ introduced in (4.10) and (4.11) are *covariant derivatives* of the second fundamental form of the surface $\boldsymbol{\theta}(\omega)$.

(4) The *Gauss equations* (4.1) are used only *once* in the above proof, for showing that $R_{1212} = 0$ in part (vii).

Finally, we turn to the question of *uniqueness*, which, like that of existence, shall be settled as a corollary to its “three-dimensional counterpart” (Theorem 3).

Theorem 6 (rigidity theorem). *Let ω be a connected open subset of \mathbb{R}^2 and let $\boldsymbol{\theta} \in C^2(\omega; \mathbf{E}^3)$ and $\tilde{\boldsymbol{\theta}} \in C^2(\omega; \mathbf{E}^3)$ be two immersions such that their associated first and second fundamental forms satisfy (with self-explanatory notations)*

$$a_{\alpha\beta} = \tilde{a}_{\alpha\beta} \quad \text{and} \quad b_{\alpha\beta} = \tilde{b}_{\alpha\beta} \quad \text{in } \omega. \tag{4.41}$$

Then there exist a vector $\mathbf{c} \in \mathbf{E}^3$ and an orthogonal matrix \mathbf{Q} of order three such that

$$\boldsymbol{\theta}(y) = \mathbf{c} + \mathbf{Q}\tilde{\boldsymbol{\theta}}(y) \quad \text{for all } y \in \omega. \tag{4.42}$$

Proof. By parts (i) and (viii) of the proof of Theorem 5, there exist open subsets ω_n of ω and real numbers $\varepsilon_n > 0$ such that the symmetric matrices (g_{ij}) defined by

$$g_{\alpha\beta} := a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2 c_{\alpha\beta} \quad \text{and} \quad g_{i3} = \delta_{i3},$$

where $c_{\alpha\beta} := a^{\sigma\tau} b_{\alpha\sigma} b_{\beta\tau}$, are positive definite in the set

$$\Omega := \bigcup_{n \geq 0} \omega_n \times]-\varepsilon_n, \varepsilon_n[.$$

The two immersions $\boldsymbol{\Theta} \in C^1(\Omega; \mathbf{E}^3)$ and $\tilde{\boldsymbol{\Theta}} \in C^1(\Omega; \mathbf{E}^3)$ defined by (with self-explanatory notations)

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \mathbf{a}_3(y) \quad \text{and} \quad \tilde{\boldsymbol{\Theta}}(y, x_3) := \tilde{\boldsymbol{\theta}}(y) + x_3 \tilde{\mathbf{a}}_3(y) \quad \text{for all } (y, x_3) \in \Omega$$

therefore satisfy

$$g_{ij} = \tilde{g}_{ij} \quad \text{in } \Omega,$$

by assumption (4.41).

By Theorem 3, there exist a vector $\mathbf{c} \in \mathbf{E}^3$ and an orthogonal matrix \mathbf{Q} of order three such that

$$\boldsymbol{\Theta}(y, x_3) = \mathbf{c} + \mathbf{Q}\tilde{\boldsymbol{\Theta}}(y, x_3) \quad \text{for all } (y, x_3) \in \Omega. \tag{4.43}$$

Hence relations (4.42) simply follow by letting $x_3 = 0$ in (4.43). \square

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