Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
J. Differential Equations 195 (2003) 380–397
Journal of Differential Equations
http://www.elsevier.com/locate/jde

Uniqueness and nondegeneracy for some nonlinear elliptic problems in a ball

Amandine Aftalion[a,*] and Filomena Pacella[b,1]

[a] *Laboratoire Jacques-Louis Lions, B.C. 187, Université Paris 6, 175 rue du Chevaleret, 75013 Paris, France*
[b] *Dipartimento di Matematica, Università di Roma "La Sapienza", P.le A. Moro 2, 00185 Roma, Italy*

Received October 31, 2002

Abstract

In this paper we study the uniqueness and nondegeneracy of positive solutions of nonlinear problems of the type $\Delta_p u + f(r,u) = 0$ in the unit ball B, $u = 0$ on ∂B. Here Δ_p denotes the p Laplace operator $\Delta_p = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $p>1$. The main ideas rely on the Maximum Principle and an implicit function theorem that we derive in a suitable weighted space. This space is essential to deal with the case $p \neq 2$.

1. Introduction

In this paper, we study the uniqueness and nondegeneracy of solutions of nonlinear problems of the following type:

$$\Delta_p u + f(r,u) = 0 \quad \text{in } B, \quad u>0 \text{ in } B, \quad u = 0 \quad \text{on } \partial B, \tag{1.1}$$

where B is the unit ball in $\mathbb{R}^n$ centered at the origin, $n \geqslant 2$, $r = |x|$ and Δ_p denotes the p Laplace operator

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u), \quad p>1.$$

*Corresponding author. Fax: +33-1-4427-7200.
E-mail addresses: aftalion@ann.jussieu.fr (A. Aftalion), pacella@mat.uniroma1.it (F. Pacella).
[1] Research supported by MURST, project "Variational methods and Nonlinear Differential Equations".

0022-0396/03/$ - see front matter
doi:10.1016/S0022-0396(02)00194-8

For the nonlinearity f, we will make different sets of hypotheses. First, we consider the case where f does not depend on r and we assume that f satisfies

(H1) $f \in C^0[0,\infty) \cap C^1(0,\infty)$, $f(0)=0$, $f<0$ in $(0,\theta)$ and $f>0$ in (θ,∞), for some $\theta>0$,
(H2) $K(u) = uf'(u)/f(u)$ is nonincreasing in (θ,∞),
(H3) $uf'(u) - (p-1)f(u) > 0$ for $u>0$.

The model nonlinearity for these hypotheses is $f(u) = u^q - u^{p-1}$ with $q>p-1$. This is the same class of nonlinearities as in the paper of Serrin and Tang [23] for the study of problems in $\mathbb{R}^n$. In the case $p=2$, we will also assume that f is differentiable at 0. In the case $p \neq 2$, since we want the Hopf lemma to hold, we will make some regularity hypothesis for f at 0.

To make the presentation of the results clearer, we will also distinguish between the case $p=2$, i.e. when Δ_p reduces to the ordinary Laplace operator and $p\neq 2$.

In the first case, $p=2$, we will prove the following result.

Theorem 1.1. *If* $p=2, f\in C^1[0,\infty)$ *and* (H1)–(H3) *hold, then any classical solution of* (1.1) *has Morse index less than or equal to* 1 *and is nondegenerate.*

We recall that the Morse index of a solution u is the number of negative eigenvalues of the linearized operator $-L$ where $L = \Delta + f_u(r,u)$ with zero Dirichlet boundary condition. The solution u is said to be nondegenerate if zero is not an eigenvalue for L.

Let us observe that under the hypotheses of Theorem 1.1, any solution of (1.1) is radially symmetric and radially decreasing by the theorem of Gidas et al. [11]. Then using the nondegeneracy result of Theorem 1.1 and the implicit function theorem as in other papers (see for instance [18] or [25]), we can get the following uniqueness result.

Theorem 1.2. *If* $p=2, f\in C^1[0,\infty)$ *and* (H1)–(H3) *hold, then problem* (1.1) *has at most one classical solution.*

In the case $p\neq 2$, several difficulties arise due to the degenerate or singular nature of the p Laplace operator. First of all, the solutions of (1.1) have to be considered in a weak sense because they are only of class $C^{1,\alpha}$. The analogous of the Gidas–Ni–Nirenberg symmetry result only holds in the case $p<2$ (see [7,8]) or for $p>2$ and f positive or monotone (see [3,13]). Thus in these cases, we will get a complete uniqueness result. From now on, we will prove results only for radial solutions which satisfy

$$\begin{cases} r^{1-n}(r^{n-1}|u'|^{p-2}u')' + f(r,u) = 0, \\ u>0 \text{ in } (0,1),\ u'(0)=0,\ u(1)=0. \end{cases} \tag{1.2}$$

We also have $u' < 0$ in $(0,1)$ by Franchi et al. [10]. To get uniqueness of solutions of (1.2), arguing as in the case $p = 2$, we first prove a nondegeneracy result and then an implicit function theorem which allows to carry out the bifurcation analysis. It turns out that for $p \neq 2$, it can be done in a suitable weighted space. More precisely, let us define for $\gamma \geqslant 0$ the Banach spaces

$$\mathscr{E}_\gamma^n = \{v \in C^n(0,1] \text{ s.t.} |v^{(k)}(r)| \leqslant c r^{\gamma-k}, \ \forall k, \ 0 \leqslant k \leqslant n\},$$

$$\mathscr{C}_\gamma^n = \{v \text{ s.t. } \exists w \in \mathscr{E}_\gamma^n, \ \lambda \in \mathbb{R} \text{ with } v = \lambda + w\},$$

$$\mathscr{C}_{\gamma,0}^n = \{v \in \mathscr{C}_\gamma^n \text{ with } v(1) = 0\}.$$

One may refer to [19] for the introduction of these spaces and the norms which make them Banach spaces. In the following, we will call

$$X = \mathscr{C}^2_{p/(p-1),0}.$$

By regularity results and a simple proof which uses the L'Hopital rule, it is possible to show that solutions of (1.2) are indeed in X (see Section 4). The linearized operator at a solution u of (1.2) along functions v in X is

$$Lv = (p-1)r^{1-n}(r^{n-1}|u'|^{p-2}v')' + f_u(r,u)v.$$

Finally, since we want to have that the Hopf Lemma holds on the boundary for solutions of (1.1), we also require that f satisfies some growth condition near 0 as indicated by a result of Vazquez [26], where a more general condition is assumed. We assume

(R) there exists $s_0 > 0$ and $c_1 > 0$ such that $-f(s) \leqslant c_1 s^{p-1}$ and the positive part of $f'(s)$ is bounded for s in $(0, s_0)$.

Note that with this hypothesis, any solution of (1.1) is such that $u'(1) < 0$. Moreover, (R) and (H3) yield that the behaviour of f near 0 is similar to $-s^{p-1}$ since because of (H3), $f(u) < -c_0 u^{p-1}$ for u close to 0, for some c_0. Moreover, (H3) and (R) also imply $f'(s) \geqslant -c_1(p-1)s^{p-2}$. Then we get

Theorem 1.3. *If $p > 1$, f satisfies* (R) *and* (H1)–(H3), *then any weak radial solution of* (1.1) *is nondegenerate in X. More generally, the linearized operator $-L$ does not have any radial sign-changing eigenfunction in X corresponding to an eigenvalue $\mu \leqslant 0$.*

Using Theorem 1.3, we get that L is an isomorphism from X into $\mathscr{E}_0^0$ and we deduce uniqueness from the implicit function theorem.

Theorem 1.4. *If $p > 1$, f satisfies* (R) *and* (H1)–(H3), *then problem* (1.1) *has at most one weak radial solution, which is the only solution if $p \leqslant 2$.*

Let us mention that when f does not depend on r, the uniqueness question for the type of nonlinearities we consider has been addressed by many authors, first in the case of the Laplacian in $\mathbb{R}^n$: Coffman [4], McLeod and Serrin [17], and Kwong [14]. Then Kwong and Zhang [15] used Coffman's method and its improvement to get uniqueness in the case of the ball. The proof relies on the study of the zero of the function $\partial u/\partial \alpha$ where $\alpha = u(0)$ is the shooting parameter and uses a Sturm comparison principle.

A different approach to study uniqueness in $\mathbb{R}^n$ was initiated by Peletier and Serrin [20], which relies on the monotone separation theorem. This method was continued by Franchi et al. [10], Pucci and Serrin [21], and Serrin and Tang [23] to treat the case of quasilinear operators which include the p Laplacian. In particular, Serrin and Tang [23] studied the same type of nonlinearities satisfying (H1)–(H3), but in the case of $\mathbb{R}^n$ or of compactly supported solutions, and left as an open conjecture the case of the ball.

In a recent paper, Ouyang and Shi [18] provide uniqueness proofs for a large class of nonlinearities in the case $p = 2$. Their idea is to study the nondegeneracy problem instead of the shooting problem: they use Crandall–Rabinowitz bifurcation theorem [5] to derive uniqueness and exact multiplicity in a number of cases.

Our uniqueness result of Theorem 1.2 ($p = 2$) is already contained in [15] or [18]. However our proof is different and Theorem 1.1 relies only on the Maximum Principle without the use of ODE techniques. Hence it could be hopefully used to get some results in other domains.

For simplicity, let us explain our ideas in the case $p = 2$. To prove Theorem 1.1, we show that the second eigenvalue μ_2 of the linearized operator $-L$ is positive. This proves at the same time that the Morse index of the solution is less than 1 and u is nondegenerate since it is easy to see that zero cannot be the first eigenvalue because the corresponding eigenfunction must change sign.

To prove that μ_2 is positive, we argue by contradiction assuming that $\mu_2 \leqslant 0$ and we denote by ϕ_2 the corresponding eigenfunction. Then, by a result in [6] (see also [16]), we know that ϕ_2 is radial and has two nodal regions which are a ball $B_1 \subset B$ and an annulus $B \backslash \bar{B}_1$. Thus, in $B_1 \subset B$ and $B \backslash \bar{B}_1$, the first eigenvalue of L is $\mu_2 \leqslant 0$. The main idea is to prove that in whatever way one divides B into two radial regions $B_1 \subset B$ and $B \backslash \bar{B}_1$, at least in one of these two domains, the Maximum Principle holds and hence the first eigenvalue must be positive, against the fact that $\mu_2 \leqslant 0$.

To check the validity of the Maximum Principle in $B_1 \subset B$ or $B \backslash \bar{B}_1$, we use the following sufficient condition:

if there exists $g \in W^{2,N}(D) \cap C(\bar{D})$ such that $g > 0$ in D with $Lg \leqslant 0$ in D and g is not identically 0 on ∂D, then the Maximum Principle holds in D for the elliptic operator L.

As a test function for this criterion, we use the function $g = x \cdot \nabla u + \beta u$ with β suitably chosen depending on ϕ_2. Let us remark that this function g appears in many other papers but in a different way.

In the case $p \neq 2$, the scheme of the proof of Theorem 1.3 is the same, but we need to prove directly the validity of the Maximum Principle once we have constructed the function g since a sufficient condition similar to the case of the Laplacian is not known. To get Theorem 1.4, we perform an implicit function theorem in X. This theorem seems to be new and we think that it could be useful for other related questions.

The result of Theorem 1.4 is, to our knowledge, the first uniqueness result for p Laplace problems in a ball with nonlinearities of our type. Some previous theorems deal with the case $f>0$ and $p\leqslant n$ [1,9]. However, we will see that our method easily applies to the case $f(u)=u^q$ and any p (also when $p>n$) to get nondegeneracy.

More generally, we will consider nonlinearities which depend on r, satisfying

(H3′) $uf_u(r,u)-(p-1)f(r,u)>0$ for $u>0$,
(H4) $f(r,0)=0\ \forall r$, $f_r(r,u)$ is nonpositive and

$$\alpha(r)=\frac{2f(r,u(r))+rf_r(r,u(r))}{u(r)f_u(r,u(r))-(p-1)f(r,u(r))}$$

is nonincreasing as a function of r, for r in $(0,1)$.
(H5) $G(r)=nF(r,u(r))-(n-p)uf(r,u(r))/p+rF_r(r,u(r))$ is either nonnegative in $(0,1)$ or positive in $(0,r_0)$ and negative in $(r_0,1)$ for some r_0, where $F(r,u)=\int_0^u f(r,s)\,ds$.

As soon as $f_r(r,u)$ is nonpositive and $p\leqslant 2$, it follows from [11] or [8] that any solution is radial. Note that hypotheses (H4) and (H5) are a natural extension of the hypotheses we made when f does not depend on r. Indeed, if f does not depend on r and satisfies (H1) and (H2), then $\alpha(r)$ is nonincreasing when u is bigger than θ and we will prove that (H5) holds. We will check at the end of Section 2 that a particular set of functions satisfying (H3′), (H4), and (H5) are functions of the type

(H4′) $f(r,u)=u^q-a(r)u^{p-1}$, $a\in C^1([0,\infty),[0,\infty))$, $\max(1,p-1)<q<(np-n+p)/(n-p)$ if $p<n$, any $q>\max(1,p-1)$ if $p\geqslant n$,
(H5′) a is nondecreasing and $ra'+a$ is nondecreasing.

A typical function a in this case is a_0+r^γ, $a_0\geqslant 0$, $\gamma\geqslant 1$. Among the set of functions satisfying (H3′), (H4), and (H5) are also functions of the type

(H4″) $f(r,u)=a(r)u^q$, $a\in C^1([0,\infty),[0,\infty))$, $\max(1,p-1)<q<(np-n+p)/(n-p)$ if $p<n$, any $q>\max(1,p-1)$ if $p\geqslant n$,
(H5″) a is nonincreasing and ra'/a is nonincreasing.

These include for instance $f(r,u)=u^q/(1+r^2)^\alpha$ for $\alpha\geqslant 0$. Note that in this case, we need a nonincreasing only to know that the solution is radial. But the nondegeneracy holds for radial solutions without this hypothesis. Next, we also assume the equivalent of (R):

(R′) $f\in C^0([0,\infty)\times[0,\infty))\cap C^1([0,\infty)\times(0,\infty))$ and the positive part of f_u is bounded for u close to 0 independently of r. Moreover, either f is nonnegative or there exists $s_0>0$ such that $-f(r,s)\leqslant c_0 s^{p-1}$ for s in $(0,s_0)$ and r in a neighborhood of 1.

Note that with this hypothesis, any solution of (1.1) is such that $u'(1) < 0$. We are able to show nondegeneracy:

Theorem 1.5. *If* $p > 1$, f *satisfies* (R′) *and* (H3′), (H4), *and* (H5), *then any weak solution of* (1.1) *is nondegenerate in* X.

We can deduce uniqueness from the nondegeneracy in X:

Theorem 1.6. *If* $p > 1$ *and* (H4′) *and* (H5′) *hold, then problem* (1.1) *has at most one weak radial solution.*

Note that our uniqueness result holds for a class of nonlinearities which is smaller than the one for nondegeneracy. This is due to the fact that in order to perform the implicit function theorem, we add a parameter λ into the equation and describe the bifurcation diagram of $u_\lambda(0)$ as a function of λ. When f does not depend on r, we just add λ in front of the nonlinearity, but when f depends on r, we need to know some asymptotic behaviour of the bifurcation diagram when λ is small. Thus, it is easier to consider a special kind of nonlinearities.

The outline of the paper is the following. In Section 2, we present a preliminary lemma. Theorems 1.1, 1.2 and 1.5, 1.6 in the case $p = 2$ are proved in Section 3. Section 4 is devoted to the case $p \neq 2$.

2. Preliminaries

Let u be a C^1 radial solution of (1.1) with f satisfying either (R), (H1)–(H3) or (R′), (H3′), (H4), and (H5). By Proposition 1.2.6 of [10], we have

$$u'(r) < 0 \quad \text{in } (0,1). \tag{2.1}$$

and hence u belongs to $C^2(0,1] \cap C^1[0,1]$ and satisfies

$$\begin{cases} \left((p-1)u'' + \dfrac{n-1}{r}u'\right)|u'|^{p-2} + f(r,u) = 0 \text{ in } (0,1), \\ u > 0 \text{ in } (0,1),\ u'(0) = u(1) = 0. \end{cases} \tag{2.2}$$

Finally, let $F(r,u) = \int_0^u f(r,s)\,ds$. The following Pohozaev identities hold:

$$\int_0^s r^{n-1}\left(nF(u) - \frac{(n-p)}{p}uf(u)\right)dr = s^{n-1}H(s) \tag{2.3}$$

if f does not depend on r, where H is given by

$$H(r) = \frac{n-p}{p}uu'|u'|^{(p-2)} + \frac{(p-1)}{p}r|u'|^p + rF(u) \tag{2.4}$$

or

$$\int_0^s r^{n-1}\left(nF(u) - \frac{(n-p)}{p}uf(u) + r\frac{\partial F}{\partial r}\right)dr = s^{n-1}H(s) \tag{2.5}$$

if f depends on r, with the same $H(r)$ as in (2.4).

We start with a technical lemma that will be used later to study the sign of the function $g = ru' + \beta u$ for some suitable β.

Lemma 2.1. *The function $h(r) = -ru'/u$ is increasing.*

Proof. The idea of the proof comes from [18]. We start by assuming that f does not depend on r and satisfies (R), (H1)–(H3). We compute h' and get by (2.2) and (2.4)

$$h'(r) = \frac{p}{(p-1)u^2|u'|^{p-2}}\left(H(r) + \frac{ruf(u)}{p} - rF(u)\right). \tag{2.6}$$

It follows from (H3) that $\frac{uf(u)}{p} - F(u) \geqslant 0$. Thus, if we prove that $H \geqslant 0$, it will imply that $h' \geqslant 0$. Let $J(r) = r^{n-1}H(r)$. We have $J(0) = 0$ and $J(1) > 0$ since $u(1) = 0$ and the Hopf Lemma holds at the boundary. Let us study the sign of J'. From (2.3), we get

$$J'(r) = r^{n-1}G(r),$$

where $G(r) = nF(u(r)) - (n-p)/pu(r)f(u(r))$. So we have to study the sign of G. If $p = n$, it follows easily that G changes sign exactly once, so J increases and then decreases to a positive number, thus J remains positive and h is increasing. If $p \neq n$, we have

$$G'(r) = (pf(u(r)) + [(p-1)f(u(r)) - uf'(u(r))](n-p)/p)u'.$$

Recall that $(p-1)f(u) - uf'(u) < 0$ by (H3).

Case $p > n$: In the region where $f \leqslant 0$, that is for r close to the boundary, we have that $G \leqslant 0$ and in the region where $f \geqslant 0$, $G' \leqslant 0$. Hence G is positive in $(0, \bar{r})$ and negative in $(\bar{r}, \infty)$ for some $\bar{r}$ less than one. This means that J increases from $J(0) = 0$ and decreases to $J(1)$ which is positive. Hence J and H remain positive.

Case $p < n$: In the region where $f \leqslant 0$, that is for r close to the boundary, we have $G'(r) \geqslant 0$. Since $G(1) = 0$, it means that $G \leqslant 0$ near 1. Because $J(0) = 0$ and $J(1) > 0$, it cannot be that $G \leqslant 0$ in the whole interval $(0, 1)$: G has to be positive somewhere, so G' has to change sign. In the region where $f(u) > 0$, we have

$$G'(r) = \left(\frac{n-p}{p}f(u)\left(\frac{np-n+p}{n-p} - \frac{uf'(u)}{f(u)}\right)\right)u'. \tag{2.7}$$

Then using (H2) and (2.1), we have that G' changes sign exactly once in the region where f is positive and G is positive in $(0, \bar{r})$ and negative in $(\bar{r}, \infty)$ for some $\bar{r}$ less than one. This means that J increases from $J(0) = 0$ and decreases to $J(1)$ which is positive. Hence J and H remain positive.

In the case f depends on r, and satisfies (H3′), (H4), and (H5), we get from (2.5) that $G(r) = nF(r,u) - (n-p)/puf(r,u) + rF_r(r,u)$. It follows directly from (H5) that either $G \geqslant 0$ in the whole interval $(0,1)$ or $G \geqslant 0$ near 0 and then $\leqslant 0$. Hence, J remains positive and h is increasing. □

Remark 2.1. If f satisfies (H4′)–(H5′) or (H4″)–(H5″), then it satisfies (H5). Indeed, G is, respectively, equal to

$$G(r) = u^p \left(u^{q+1-p} \left(\frac{n}{q+1} - \frac{n-p}{p} \right) - a - \frac{ra'}{p} \right),$$

$$G(r) = au^{q+1} \left(\frac{n}{q+1} - \frac{n-p}{p} + \frac{ra'}{(q+1)a} s \right).$$

With our hypotheses, the term inside the parentheses is decreasing. But it cannot be negative for all r in $(0,1)$, otherwise this would imply that J is decreasing and contradict $J(0) = 0$ and $J(1) \geqslant 0$. This yields that f satisfies (H5). Moreover (H4) follows from straightforward computation using (H5′) or (H5″).

Remark 2.2. Note that our proof in the case $p < n$ and f does not depend on r, implies that G' changes sign in the region where $f > 0$, so that from (2.7), we deduce that for u sufficiently large $uf'(u)/f(u) < \frac{np-n+p}{n-p}$. Conversely, this condition and $\lim_{u\to\infty} uf'(u)/f(u) > p-1$ imply the existence of a solution of (1.1) using the Mountain Pass Lemma [2]. Recall that our hypotheses yield $\lim_{u\to\infty} uf'(u)/f(u) \geqslant p-1$.

Remark 2.3. We conclude this section recalling a well-known condition for the Maximum Principle to hold for uniformly elliptic operators M in a domain D. Let $M = a_{i,j}\partial_{i,j} + b_i\partial_i$ be a uniformly elliptic operator and c be such that the positive part of c is in $L^n(D)$. Assume that there is a function g in $W^{2,n} \cap C(\bar{D})$ such that $Mg + c(x)g \leqslant 0$ and $g > 0$ in D, but not identically zero on ∂D. Then the Maximum Principle holds for $M + c$ in D.

3. The case $p = 2$

Now we restrict our attention to the case $p = 2$. Let u be a solution of (1.1) and denote by L the linearized operator at u, i.e.

$$Lv = \Delta v + f_u(r,u)v. \tag{3.1}$$

Lemma 3.1. *If u is a solution of* (1.1), $p = 2$, *and if* $\partial f/\partial r \leqslant 0$, *then any solution of* $-Lv = \mu v$ *with* $\mu \leqslant 0$ *and* $v = 0$ *on* ∂B *is radial.*

Proof. We adapt the proof contained in [6]. We let $D = B \cap \{x_1 > 0\}$ and we define $w = v^+ - v$ in D where $v^+(x_1, x') = v(-x_1, x')$. Then $-Lw = \mu w$ in D and $w = 0$ on ∂D. By the result of [11], we know that u is radial and $\partial u/\partial x_1 < 0$ in D. Moreover

$$L\left(\frac{\partial u}{\partial x_1}\right) = -u\frac{\partial f}{\partial x_1} \geqslant 0,$$

and $\partial u/\partial x_1 < 0$ on $\bar{D} \cap \{x_1 > 0\}$ by the Hopf Lemma. Thus by Remark 2.3, the Maximum Principle holds for L in D which implies that $w \equiv 0$. This is true in any direction so that v is radial. □

Proof of Theorems 1.1 and 1.5 in the case $p = 2$**.** We will show that the second eigenvalue μ_2 of $-L$ in B with zero Dirichlet boundary condition is positive. Arguing by contradiction, let us assume that $\mu_2 \leqslant 0$ and denote by v the corresponding eigenfunction. Of course, v must change sign and has only two nodal regions.

By Lemma 3.1, we know that v is radial, hence its two nodal regions are a ball $B_1 \subset B$ of radius r_1 and an annulus $B \backslash \bar{B}_1$. In each of these two regions, the first eigenvalue of $-L$ is μ_2. We denote by u_1 the value of u at $r = r_1$.

Step 1: *Let L be the linearized operator defined in* (3.1). *Then, there exists a radial function g such that $g(1) < 0$ and either*

$$Lg \leqslant 0,\ g > 0 \ \text{in}\ B_1 \quad \text{or} \quad Lg \geqslant 0,\ g < 0\ \text{in}\ B \backslash \bar{B}_1. \tag{3.2}$$

Let us prove this claim. We let $g(r) = ru' + \beta u$ with $\beta \geqslant 0$. We have $g(1) < 0$ since $u'(1) < 0$ by the Hopf Lemma, and the function g is negative for r near 1, positive near $r = 0$ when $\beta > 0$. Thus, if $\beta = 0$, $g < 0$ in $(0, 1)$ and if $\beta > 0$, by Lemma 2.1, g has a unique zero in the interval $(0, 1)$. Moreover,

$$Lg = (uf_u(r,u) - f(r,u))\left(\beta - \frac{2f(r,u) + rf_r(r,u)}{uf_u(r,u) - f(r,u)}\right). \tag{3.3}$$

Recall from (H3) or (H3′) that $uf_u(r,u) - f(r,u) > 0$.

If (H4) holds, we let

$$\beta = \frac{2f(r_1,u_1) + r_1 f_r(r_1,u_1)}{u_1 f_u(r_1,u_1) - f(r_1,u_1)}, \tag{3.4}$$

so that $Lg = 0$ for $r = r_1$. It follows directly from (H4) that $Lg \leqslant 0$ in B_1 and $Lg \geqslant 0$ in $B \backslash \bar{B}_1$. Moreover, since g has a unique zero, it is either in B_1 or in $B \backslash \bar{B}_1$, hence (3.2) is satisfied.

If, (H2) holds, that is f does not depend on r, we have to argue according to the position of u_1 with respect to θ, the value where f changes sign (see (H1)). If $u_1 > \theta$, we let β as in (3.4) so that $Lg = 0$ for $r = r_1$. We use (H2) and the monotonicity of u to deduce as before that $Lg \leqslant 0$ in B_1 and $Lg \geqslant 0$ in $B \backslash \bar{B}_1$. Since g has a unique zero, (3.2) holds.

If $u_1 \leqslant \theta$, then $f(u) < 0$ for $u \leqslant u_1$. We choose $\beta = 0$ so that $Lg \geqslant 0$ for $u < u_1$ that is in $B \backslash \bar{B}_1$, while $g < 0$ in $\bar{B} \backslash \{0\}$ because u is radially decreasing, hence the property of (3.2) is satisfied in $B \backslash \bar{B}_1$.

Step 2: It follows from Step 1 and Remark 2.3 that the Maximum Principle holds either in B_1 or $B \backslash \bar{B}_1$. This is equivalent to say that in one of the two regions, the first eigenvalue of $-L$ with zero Dirichlet condition is positive against the assumption that $\mu_2 \leqslant 0$.

Having proved that $\mu_2 > 0$, if $\mu = 0$ is an eigenvalue for L in B, the only possibility is that it is the first eigenvalue. Thus the corresponding eigenfunction should not change sign and this is not possible. Indeed v would satisfy

$$\Delta v + f_u(r, u)v = 0. \tag{3.5}$$

Multiplying (3.5) by u and (1.1) by v, we get

$$\int_B (uf_u(r, u) - f(r, u))uv = 0.$$

This implies that v changes sign in B and completes the proof. □

Proof of Theorem 1.2. Uniqueness follows from nondegeneracy, analyzing the bifurcation diagram. We are going to take the radius of the ball B_R as a bifurcation parameter. Note that by rescaling the solution to the unit ball, this is the same as adding a parameter $\lambda = R^2$ in front of the nonlinearity in (1.1) so that it becomes $\lambda f(u)$ and the solution is u_λ. We study the bifurcation diagram of $u_\lambda(0)$ versus λ. By classical bifurcation results ([5] or [22]), we know that there is a branch of solutions bifurcating from $\lambda = \lambda_*$, where $\lambda_* = \lambda_1 / f'(\infty)$, for which $u_\lambda(0)$ goes to ∞. Note that because of our hypotheses, $uf'(u)/f(u)$ has a limit at infinity. If this limit is bigger than 1, then $\lambda_* = 0$.

Then using the uniqueness theorem for initial value problem for ODEs and the nondegeneracy result of Theorem 1.1, we can say that the branch continues for all values of $\lambda \in (\lambda_*, \infty)$. The nondegeneracy result excludes secondary bifurcations or turning points, hence the branch is monotone decreasing. Since this branch is unbounded, if there was another branch, the uniqueness theorem for initial value problem for ODEs would provide that this other branch is such that $u_\lambda(0)$ is bounded. But then, on such a branch, $u(0)$ bounded and λ bounded from below would imply the existence of a turning point which is impossible since we have the nondegeneracy result. Hence all solutions of our problem are on the branch we have constructed and this provides uniqueness. □

Proof of Theorem 1.6 in the case $p = 2$. Let us introduce a parameter λ in the nonlinearity f: $f_\lambda(r,u) = u^q - \lambda a(r)u$. We claim that

$$\exists \lambda_0 \text{ such that there is only one solution for } \lambda \in (0, \lambda_0). \tag{3.6}$$

To prove the claim, let us first observe that near $\lambda = 0$, the solutions are uniformly bounded. Indeed, if u_n is a solution with $\lambda = \lambda_n$ and λ_n tends to 0, we are going to prove that $M_n = \sup u_n$ is bounded. We rescale the function u_n to

$$\tilde{u}_n(r) = \frac{1}{M_n} u_n \left(\frac{r}{M_n^{(q-1)/2}} \right).$$

Then, if M_n goes to infinity, $\tilde{u}_n$ converges to a positive solution of $\Delta u + u^q = 0$ in $\mathbb{R}^n$. This is impossible by the result of Gidas and Spruck [12] since $q < (n+2)/(n-2)$. So the sequence M_n remains bounded. Next, we deduce that u_n tends to the unique solution u_0 of $\Delta u + u^q = 0$ in the ball with zero boundary data. Since this solution is nondegenerate, it implies that there is a unique branch of solutions near $(0, u_0)$. Otherwise let u_n and v_n be two solutions: the function

$$w_n = (u_n - v_n)/||u_n - v_n||_\infty$$

converges uniformly to a nontrivial solution of the linearized problem $\Delta w + q u_0^{q-1} w = 0$ in B, $w = 0$ on ∂B. This is impossible since the solution u_0 is nondegenerate [6,25]. Hence the claim is proved.

Then let us define $(0, \bar{\lambda})$ to be the maximal interval for which uniqueness holds. We claim that $\bar{\lambda}$ is infinite. In fact, if $\bar{\lambda}$ is finite, one can prove that solutions stay uniformly bounded as in the case $\lambda = 0$ using the blow-up. Then by the nondegeneracy result, we have that there is only one solution for $\lambda = \bar{\lambda}$, otherwise using the implicit function theorem at each solution, we would derive nonuniqueness for $\lambda < \bar{\lambda}$. Next, if there exists a sequence λ_n tending to $\bar{\lambda}$, $\lambda_n > \bar{\lambda}$, for which the problem has two solutions u_n and v_n, we define $w_n = (u_n - v_n)/||u_n - v_n||_\infty$. The two solutions u_n and v_n converge to the unique solution $\bar{u}$ at $\lambda = \bar{\lambda}$. Then w_n converges to a nontrivial solution of the linearized problem at $u = \bar{u}$. This is impossible since $\bar{u}$ is nondegenerate by Theorem 1.1. Thus if $\bar{\lambda}$ is finite, we have reached a contradiction so that $\bar{\lambda}$ is in finite and uniqueness holds for every λ in $(0, \infty)$, in particular for $\lambda = 1$. □

4. Case $p \neq 2$

We will be concerned with radial solutions, which are the only solutions if $p < 2$ by Damascelli and Pacella [7,8]. We consider the p-Laplace operator in the space $X = \mathscr{C}^2_{p/(p-1),0}$ defined in the introduction. With the following norm, this space is a

Banach space (see [19]).

$$||v||_X = ||v||_{C^2(\bar{B}\setminus B_{1/2})} + \sum_{j=0}^{2} \sup_{0<s<1/2} s^{j-\frac{p}{p-1}} |v^{(j)}(s)|.$$

4.1. Regularity

As we recalled at the beginning of Section 2, a solution u of (1.2) belongs to $C^2(0,1] \cap C^1[0,1]$ and $u'<0$ in $(0,1)$. One can derive a precise behaviour of u' near the origin using the L'Hopital rule (see [23]): indeed, it follows from Eq. (1.2) that $(|u'|^{p-1}r^{n-1})'/(r^n)'$ tends to a finite limit as r tends to zero, hence $(|u'|^{p-1}r^{-1})$ tends to the same limit, i.e.

$$u'(r)r^{\frac{-1}{p-1}} \to -\left(\frac{f(0,u(0))}{n}\right)^{\frac{1}{p-1}} \text{ as } r\to 0. \tag{4.1}$$

In other words, $u'(r)r^{\frac{-1}{p-1}}$ is in L^∞. Moreover, we derive from Eq. (2.2)

$$u''(r)r^{\frac{p-2}{p-1}} \to -\frac{1}{p-1}\left(\frac{f(0,u(0))}{n}\right)^{\frac{1}{p-1}} \text{ as } r\to 0. \tag{4.2}$$

This implies that $u\in X$.

The linearized operator at a solution u of (1.2) along functions v in X is

$$Lv = (p-1)r^{1-n}(r^{n-1}|u'|^{p-2}v')' + f_u(r,u)v \quad v\in X. \tag{4.3}$$

Let v be a radial solution of $-Lv = \mu v$ with $\mu \leqslant 0$, v in X. Note that as soon as v is in X, then v is continuous up to 0. We can be more precise about the behaviour of v at zero. Indeed, if $v\in X$, $|u'|^{p-2}r^{n-1}v'$ tends to 0 when r tends to 0, hence one can apply the L'Hopital rule to (4.3) in a similar way as for u and it yields that v' has the same behaviour as u', namely

$$v'(r)r^{\frac{-1}{p-1}} \to -v(0)\left(\frac{\mu + f_u(0,u(0))}{n(p-1)}\right)\left(\frac{f(0,u(0))}{n}\right)^{\frac{2-p}{p-1}} \text{ as } r\to 0. \tag{4.4}$$

Similarly as for u, one can derive the behaviour of v'' from the equation so that $v''(r)r^{\frac{p-2}{p-1}}$ has a limit when r tends to 0.

4.2. Proof of Theorems 1.3 and 1.5: nondegeneracy

We let v be a solution in X of the linearized problem

$$-Lv = \mu v \text{ in } (0,1), \quad v(1) = 0, \quad \text{with } \mu \leqslant 0, \tag{4.5}$$

such that v changes sign. Recall that if $\mu = 0$, any solution of $Lv = 0$ has to change sign because of (H3) or (H3′) and the fact that

$$\int_0^1 (uf_u(r,u) - (p-1)f(r,u))uvr^{n-1}\,dr = 0.$$

As in the case $p = 2$, we choose B_1 to be a ball of radius r_1 such that $v(r_1) = 0$. We call $u_1 = u(r_1)$. We are going to construct a function g in a similar way as in the proof of Theorem 1.1.

Step 1: *Let L be the linearized operator defined in* (4.5) *Then, there exists a radial function g such that* $g(1) < 0$ *and either*

$$Lg \leqslant 0,\ g > 0 \text{ in } B_1 \quad \text{or} \quad Lg \geqslant 0,\ g < 0 \text{ in } B \backslash \bar{B}_1. \tag{4.6}$$

Let us prove this claim. We let $g = ru' + \beta u$ with $\beta \geqslant 0$. We have $g(1) < 0$ since $u'(1) < 0$ by the Hopf Lemma which holds by (R) or (R′), and the function g is negative for r near 1, positive near $r = 0$ when $\beta > 0$. Thus, if $\beta = 0$, $g < 0$ in $(0,1)$ and if $\beta > 0$, by Lemma 2.1, g has a unique zero in the interval $(0,1)$. Hence either $g < 0$ in $(r_1, 1)$ or $g > 0$ in $(0, r_1)$. Moreover,

$$Lg = (uf_u(r,u) - (p-1)f(r,u))\left(\beta - \frac{2f(r,u) + rf_r(r,u)}{uf_u(r,u) - (p-1)f(r,u)}\right). \tag{4.7}$$

Recall from (H3) or (H3′) that $uf_u(r,u) - (p-1)f(r,u) > 0$.

If (H4) holds, we let

$$\beta = \frac{2f(r_1,u_1) + r_1 f_r(r_1,u_1)}{u_1 f_u(r_1,u_1) - f(r_1,u_1)}, \tag{4.8}$$

so that $Lg = 0$ for $r = r_1$. It follows directly from (H4) that (4.6) is satisfied.

If (H2) holds, that is f does not depend on r, we have to argue according to the position of u_1 with respect to θ as in the case $p = 2$. If $u_1 > \theta$, we let β as in (4.8) so that $Lg = 0$ for $r = r_1$. We use (H2) and the monotonicity of u to deduce as before that $Lg \leqslant 0$ in B_1 and $Lg \geqslant 0$ in $B \backslash \bar{B}_1$. Since g has a unique zero, (4.6) holds.

If $u_1 \leqslant \theta$, we choose $\beta = 0$ so that $Lg > 0$ for $u < u_1$ that is in $(r_1, 1)$.

Step 2: *The function g will provide a weak Maximum Principle either in* $(0, r_1)$ *or in* $(r_1, 1)$.

If $Lg \geqslant 0$ and $g < 0$ in $(r_1, 1)$, since the operator L is uniformly elliptic there and $g(1) < 0$, it follows from Remark 2.3 that the Maximum Principle holds for L in $(r_1, 1)$ and $v \equiv 0$. Note that since the positive part of $f_u(r,u)$ is bounded for u small by (R) or (R′), it satisfies the hypotheses of Remark 2.3.

If $Lg \leqslant 0$ and $g > 0$ in $[0, r_1]$, we define w such that $v = wg$. Without loss of generality, we can assume that w is positive somewhere (otherwise we consider $-w$). Then w cannot achieve a positive maximum away from the origin since the operator is uniformly elliptic in any compact set not containing the origin. Let us assume that w reaches a positive maximum at zero, then is straightforward to see that w has the same behaviour as v near the origin, that is $w'(r) \sim -cr^{1/(p-1)}$ and $w''(r) \sim -(c/(p-1))r^{(2-p)/(p-1)}$ for some $c > 0$. Moreover, w satisfies

$$(p-1)|u'|^{p-2}\left(w'' + \frac{n-1}{r}w'\right) + (p-1)(|u'|^{p-2})'w' + 2(p-1)|u'|^{p-2}\frac{g'}{g}w' = -w\frac{Lg}{g} - \mu w. \tag{4.9}$$

Recall that Lg/g is negative at zero, hence the right hand side has a positive limit when r tends to 0. This provides a contradiction with (4.9) since the left hand side is going to $-\alpha$ near the origin with $\alpha > 0$. This completes the proof that any solution of (4.5) which changes sign is identically zero. □

Remark 4.1. It follows from our proof that an analogous of Remark 2.3 holds for the linearized operator L, even if this operator is not uniformly elliptic: that is if there exists a radial function g such that $g > 0$ in $\bar{D}$ and $Lg \leqslant 0$ in D, then the weak Maximum Principle holds for L in D. This is a new result that may be used in other settings.

4.3. Proof of Theorems 1.4 and 1.6

We have proved in Theorem 1.3 that the linearized operator L is injective as a map from X to $\mathscr{E}_0^0$. So we have to prove surjectivity in order to get that it is an isomorphism. This would allow to apply the implicit function theorem in X to get local uniqueness. We need a preliminary lemma.

Lemma 4.1. *There exists a solution ϕ in $C^2_{p/(p-1)}$ of the equation $L\phi = 0$ in $(0,1)$, such that $\phi(0) = 1$ and $\phi(1) \neq 0$.*

Proof. We want to solve

$$\phi'(r) = -\frac{1}{r^{n-1}|u'|^{p-2}}\int_0^r t^{n-1} f_u(t, u(t))\phi(t)\, dt \tag{4.10}$$

with the initial condition $\phi(0) = 1$. The existence of such a solution for r small follows from a fixed point argument as in the proof of Picard's theorem for the existence of a solution of initial value problems in ODEs. The solution exists up to

$r = 1$ since $\int_0^1 f_u(t, u(t))\, dt$ is finite by (R) or (R′). Indeed (R) or (R′) imply that the positive part of f_u is bounded and $-f_u \leqslant c_1(p-1)u^{p-2}$.

Next let us check that ϕ is in $C^2_{p/(p-1)}$. We have already seen that ϕ is bounded, hence it follows from (4.10) that ϕ' has the appropriate behaviour near 0. Then we can use Eq. (4.3) to deduce that ϕ'' also has the appropriate behaviour.

Notice that $\phi(1)$ is not zero, otherwise ϕ would provide a non zero solution to (4.5) in X and it would contradict Theorem 1.3 or 1.5. □

Proposition 4.1. *L is surjective from X into* $\mathscr{E}^0_0$.

Proof. For $g \in \mathscr{E}^0_0$, we want to solve $Lv = g$. First we are going to solve it without boundary condition at $r = 1$. We can use a variation constant formula starting from the solution ϕ constructed in Lemma 4.1 and looking for a special solution $v = c(r)\phi$. But one can also solve directly the equation

$$v'(r) = \frac{1}{r^{n-1}|u'|^{p-2}} \int_0^r t^{n-1}(-f_u(t, u(t))v(t) + g(t))\, dt, \tag{4.11}$$

with $v(0) = 1$. The existence follows from a fixed point argument as before, and v' and v'' behave like $r^{1/(p-1)}$ and $r^{(2-p)/(p-1)}$ near the origin, so that v is indeed in $\mathscr{C}^2_{p/(p-1)}$. If $v(1)$ is zero, then v provides a solution for our problem. Otherwise, we look for $w = v + \alpha\phi$, where ϕ is constructed in Lemma 4.1 and α is suitably chosen to satisfy $w(1) = 0$. This can be done since we have checked that $\phi(1)$ is not zero. □

Proof of Theorem 1.4. We consider problem (1.1) with a nonlinearity $\lambda f(u)$ instead of $f(u)$, where λ is a free parameter:

$$\begin{cases} \left((p-1)u'' + \dfrac{n-1}{r}u'\right)|u'|^{p-2} + \lambda f(r, u) = 0, \\ u > 0, \text{ in } (0,1),\ u'(0) = u(1) = 0. \end{cases} \tag{4.12}$$

We want to describe the bifurcation diagram of solutions, that is $d = u(0)$ vs λ. The fact that our linearized operator is an isomorphism allows to use the implicit function theorem and deduce that around any solution (λ, u_λ), there is a unique branch of solutions. Note that rescaling Eq. (4.12) on the ball of radius $\lambda^{1/p}$, problem 4.12 becomes like (2.2), but in a ball of radius $\lambda^{1/p}$. The uniqueness theorem for initial value problems for ODEs (see [10]) yields that for any $u(0)$, there is at most one λ for which there is a solution of (4.12).

Assume that for some λ_0, there exists a solution u_{λ_0}. Then using the implicit function theorem, there exists a unique branch of solutions around $(\lambda_0, u_{\lambda_0})$. Let us continue this branch in terms of λ. We can define $\underline{\lambda}$ to be the infimum of the λ's on this branch and $\bar{\lambda}$ the supremum.

First, we claim that as λ tends to $\underline{\lambda}$, $u_\lambda(0)$ has to go to infinity. Otherwise, if it stays bounded, then we can find a solution for $\lambda = \underline{\lambda}$: this is impossible if $\underline{\lambda} = 0$ because the only solution for $\lambda = 0$ is zero and $u_\lambda(0) > \theta$ by (H1) and if $\underline{\lambda} > 0$, using the implicit function theorem, we can find a solution for smaller λ.

Next, we have $\bar{\lambda} = \infty$. Indeed, if $\bar{\lambda}$ is finite, it implies that the norm of u tends to infinity (otherwise we could continue the branch for bigger λ by the implicit function theorem), but this contradicts the uniqueness theorem for initial value problem for ODEs. Thus, we constructed a decreasing branch Γ which exists from $\underline{\lambda}$ to ∞.

Finally, we claim that there is no other branch of solutions. If such branch exists, then it would have the same properties as Γ and in particular $u_\lambda(0)$ would tend to ∞, which contradicts the uniqueness theorem for initial value problem.

Notice that if we assume additionnally that uf'/f tends to a limit bigger than $p-1$ at infinity, then the Mountain Pass Lemma implies existence for every λ. Hence all solutions of our problem are on the same branch which is decreasing. This provides uniqueness. □

Proof of Theorem 1.6. We argue similarly to the case $p = 2$, that is we let $f_\lambda(r, u) = u^q - \lambda a(r)u$. For bounded λ, we claim that the solutions are uniformly bounded. Indeed, if u_n is a solution with $\lambda = \lambda_n$ which tends to 0, let $M_n = \sup u_n = u_n(0)$. We rescale the function u_n to

$$\tilde{u}_n(r) = \frac{1}{M_n} u_n\left(\frac{r}{M_n^{(q-p+1)/2}}\right).$$

Then, if M_n tends to infinity, $\tilde{u}_n$ converges to a positive radial solution of $\Delta_p u + u^q = 0$ in $\mathbb{R}^n$. This is impossible by the Liouville result of Serrin and Zou [24]. Next let u_n and v_n be two solutions for $\lambda = \lambda_n$. When λ_n tends to 0, then u_n and v_n converge to the unique positive solution u_0 of $\Delta_p u + u^q = 0$ in B with zero Dirichlet data. We have seen in Theorem 1.5 that this solution is nondegenerate in X. The function $w_n = (u_n - v_n)/||u_n - v_n||_\infty$ converges to a nontrivial solution of the linearized problem at u_0, $Lw = 0$ in B, $w = 0$ on ∂B. Indeed, one can see taking the difference of the two equations for u_n and v_n that

$$(r^{n-1}|u_n'|^{p-2}w_n' + (p-2)r^{n-1}v_n' z_n^{p-3} w_n')' + r^{n-1}\lambda_n f_u(r, q_n) w_n = 0,$$

where q_n is in the interval (u_n, v_n) and z_n in the interval $(|u_n'|, |v_n'|)$. Passing to the limit contradicts the nondegeneracy of u_0. The rest of the proof follows as in the case $p = 2$.

So we get the uniqueness of a radial solution. By the symmetry result of [7,8], all positive solutions of (1.1) are radial if $1 < p < 2$, hence (1.1) has at most one positive solution. Note that in [7,8] only the case where f does not depend on r is considered, but the same proof applies if $f(r, .)$ is nonincreasing in r. □

Acknowledgments

The authors are very grateful to Franck Pacard for taking the time of explaining the properties of elliptic operators in weighted Hölder spaces. This research was partially supported by the TMR Programme "Nonlinear Parabolic Partial Differential Equations: Methods and Applications".

References

[1] A. Adimurthi, S.L. Yadava, An elementary proof of the uniqueness of positive radial solutions of a quasilinear Dirichlet problem, Arch. Rational Mech. Anal. 127 (3) (1994) 219–229.
[2] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical points theory and applications, J. Funct. Anal. 14 (1973) 349–381.
[3] F. Brock, Radial symmetry for nonnegative solutions of semilinear elliptic equations involving the p-Laplacian, Progress in Partial Differential Equations, Vol. 1 (Pont–Mousson, 1997), Pitman Research Notes in Mathematics Series, Vol. 383, Longman, Harlow, 1998, pp. 46–57.
[4] C.V. Coffman, Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions, Arch. Rational Mech. Anal. 46 (1972) 81–95.
[5] M.G. Crandall, P.H. Rabinowitz, Bifurcation, perturbation from simple eigenvalues and linearized stability, Arch. Rational Mech. Anal. 52 (1973) 161–180.
[6] L. Damascelli, M. Grossi, F. Pacella, Qualitative properties of positive solutions of semilinear elliptic equations in symmetric domains via the maximum principle, Ann. Inst. H. Poincaré Anal. Non Linéaire 16 (5) (1999) 631–652.
[7] L. Damascelli, F. Pacella, Monotonicity and symmetry of solutions of p-Laplace equations, $1<p<2$, via the moving plane method, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 26 (4) (1998) 689–707.
[8] L. Damascelli, F. Pacella, Monotonicity and symmetry results for p-Laplace equations and applications, Adv. Differential Equations 5 (7–9) (2000) 1179–1200.
[9] L. Erbe, M. Tang, Uniqueness theorems for positive radial solutions of quasilinear elliptic equations in a ball, J. Differential Equations 138 (2) (1997) 351–379.
[10] B. Franchi, E. Lanconelli, J. Serrin, Existence and uniqueness of nonnegative solutions of quasilinear equations in $\mathbb{R}^n$, Adv. in Math. 118 (2) (1996) 177–243.
[11] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (3) (1979) 209–243.
[12] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. 34 (4) (1981) 525–598.
[13] S. Kesavan, F. Pacella, Symmetry of positive solutions of a quasilinear elliptic equation via isoperimetric inequalities, Appl. Anal. 54 (1–2) (1994) 27–37.
[14] M.K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$, Arch. Rational Mech. Anal. 105 (3) (1989) 243–266.
[15] M.K. Kwong, L.Q. Zhang, Uniqueness of the positive solution of $\Delta u + f(u) = 0$ in an annulus, Differential Integral Equations 4 (3) (1991) 583–599.
[16] C.S. Lin, W.M. Ni, A counterexample to the nodal domain conjecture and a related semilinear equation, Proc. Amer. Math. Soc. 102 (2) (1988) 271–277.
[17] K. McLeod, J. Serrin, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^n$, Arch. Rational Mech. Anal. 99 (2) (1987) 115–145.
[18] Ouyang, J. Shi, Exact multiplicity of positive solutions for a class of semilinear problems, J. Differential Equations 146(1) (1998) 121–156.
[19] F. Pacard, T. Rivière, Linear and nonlinear aspects of vortices, the Ginzburg–Landau model, in: Progress in Nonlinear Differential Equations and their Applications, Vol. 39, Birkhauser, Boston, Inc., Boston, MA, 2000.

[20] L.A. Peletier, J. Serrin, Uniqueness of positive solutions of semilinear equations in $\mathbb{R}^n$, Arch. Rational Mech. Anal. 81 (2) (1983) 181–197.
[21] P. Pucci, J. Serrin, Uniqueness of ground states for quasilinear elliptic operators, Indiana Univ. Math. J. 47 (2) (1998) 501–528.
[22] P.H. Rabinowitz, On bifurcation from infinity, J. Differential Equations 14 (1973) 462–475.
[23] J. Serrin, M. Tang, Uniqueness of ground states for quasilinear elliptic equations, Indiana Univ. Math. J. 49 (3) (2000) 897–923.
[24] J. Serrin, H. Zou, Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations, 2002, to appear.
[25] P.N. Srikanth, Uniqueness of solutions of nonlinear Dirichlet problems, Differential Integral Equations 6 (3) (1993) 663–670.
[26] J.L. Vazquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (1984) 191–202.