A linearly semi-implicit AMR method for 1D gas-liquid flows

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Abstract

We consider a drift-flux model for the numerical solution of 1D gas-liquid flows in pipes. The model is given by a set of conservation laws, consisting of two continuity equations for the two phases, and a single momentum equation for the mixture. The model is closed by providing a slip law, i.e., an empirical relation between the velocities of two phases. There are two principal issues concerning the numerical solution of the model. Firstly, for certain slip laws, the model is not hyperbolic. Secondly, there are two very different time scales in the problem: the slow one, related to the transport, and the fast one, related to propagation of acoustic waves. In practical applications, the precise resolution of the fast scale is not important. We employ a relaxation method to deal with potential non-hyperbolicity of the model. In order to account for different time scales of the model, we use a linearly semi-implicit version of the scheme. Furthermore, we apply the resulting finite volume scheme on a dynamically adaptive grid. This allows us to obtain a significant acceleration of the computation.

Keywords: finite volumes, adaptive mesh refinement, relaxation, drift-flux model, slug flow.

1 Introduction

We consider a two-phase, two-component flow of oil and gas in the pipelines during the offshore oil production. Since the pipeline is typically laid on the sea bottom, the pipeline slope is in general not constant. The change of the slope can lead to the formation of the slug flow, which is characterized by abrupt changes in the flow velocity, pressure, etc. The flow becomes highly unsteady, which is undesirable for the oil production. Given the pipeline geometry, the task of the
mathematical modeling is to predict the resulting flow regime.

To this goal, we employ a one-dimensional drift-flux model, which consists of two continuity equations for the two phases, and a momentum equation for the mixture. The gravity and friction are represented by the source terms. The system of equations is closed by an empirical slip law, which gives the difference of phase velocities as a function of the state variables.

It appears to be difficult to solve the resulting system of equations directly: due to complexity of the slip law one cannot provide its characteristic analysis. Also, at the practical implementation, each call to the slip law function appears to be very expensive computationally. To avoid these problems, we employ a relaxation method of Baudin et al. \[1\].

In our applications it is necessary to compute the two-phase transport accurately, whereas the acoustic waves propagation is less important. Since the typical flow velocity is several orders of magnitude slower than the speed of propagation of the acoustic waves, we have to deal with very different time scales of the problem. To address this issue numerically, we employ a semi-implicit version of the relaxation method.

In the slug flow simulation, one is typically interested in the accurate representation of the material front. Therefore, it is necessary to use a fine computational mesh in the neighborhood of the front. However, using the same fine mesh everywhere in the domain becomes prohibitively expensive computationally. To deal with this dilemma, we use an adaptive mesh refinement (AMR) technique of Cohen et al. \[2\]. With its help, we obtain up to three-fold acceleration of the calculation, compared to that on the fine mesh everywhere in the domain.

The paper is organized as follows. In Section 2 we present the drift-flux model and discuss its mathematical properties. In Section 3 we present the relaxation method. In Section 4 we provide some details about the AMR algorithm. Finally, Section 5 contains numerical results.

2 The drift-flux model

Denote the phases by the subscripts \(l\) (liquid) and \(g\) (gas). For \(k = l, g\), denote the phase densities by \(\rho_k\), the fraction surfaces (the part of the pipeline cross-section, occupied by the phase \(k\)) by \(R_k\), and the velocities by \(v_k\). We suppose the pressure equilibrium between the two phases, \(p_l = p_g =: p\). Consider the mixture density \(\rho := \rho_l R_l + \rho_g R_g\), the mass fractions \(X := \rho_l R_l / \rho, Y := \rho_g R_g / \rho\), the mixture velocity \(v := X v_l + Y v_g\), the liquid mass flow \(q_l := \rho_l R_l v_l\), and the gas mass flow \(q_g := \rho_g R_g v_g\). We have obviously \(R_l + R_g = 1, X + Y = 1, R_l, g, X, Y \in [0, 1]\).

The pipeline is characterized by its length \(L\) and diameter \(D\). In this work, we only consider the plane pipelines, i.e., the geometry of the pipeline is determined by the angle \(\theta = \theta(x)\), where the \(x\)-axis is oriented along the pipeline. Assuming that \(D \ll L\), we can consider the flow as one-dimensional.
The system of governing equations reads

\begin{align}
\rho_t + (\rho v)_x &= 0 \\
(\rho Y)_t + (\rho Y v - \sigma)_x &= 0 \\
(pv)_t + (pv^2 + P)_x &= -\rho g \sin \theta - C \rho v|v|.
\end{align}

Here \(C = 2C_f/D\), \(C_f\) is the friction coefficient,

\[\sigma = \rho Y(1 - Y)\phi, \quad P = p + \rho Y(1 - Y)\phi^2,\]

and the function \(\phi = \phi(\rho, Y, v) := v_l - v_g\) is given by the slip law (see below).

We assume that the flow is isothermal. We use the ideal gas equation of state for the gas,

\[\frac{p}{\rho g} = a_g^2 = \text{const},\]

where \(a_g\) is the constant gas sound speed. The equation of state for the liquid is

\[\frac{p - p^0}{\rho_l - \rho_0^l} = a_l^2 = \text{const},\]

where \(p^0\) is the atmospheric pressure, \(\rho_0^l\) is the liquid density at \(p = p^0\), and \(a_l\) is the constant liquid sound speed. If the liquid is incompressible, \(\rho_l = \rho_0^l\) and \(a_l \to \infty\).

The hydrodynamic closure is given by the slip law. In this work, we use the law of Zuber and Findlay [3]

\[\phi = -\frac{(1 - c_0)v - c_1}{c_0b + 1 - Y},\]

with \(c_0 = 1.07\), \(c_1 = 0.2162\) and \(b = (1 - Y)(\rho/\rho_l - 1)\). One can show that the system (1), closed with the law (5), is hyperbolic with the characteristic speeds of the form

\[c^- = \hat{v} - a_m, \quad c^0 = v_g, \quad c^+ = \hat{v} + a_m,\]

where \(\hat{v} = (1 - \delta)v_l + \delta v_g\) and \(a_m\) is the mixture sound speed. Physically, \(a_m \gg \hat{v}\) so that \(c^- < 0, c^+ > 0\). Similar to the case of the Euler equations, the fast characteristic speeds \(c^-\) and \(c^+\) are associated with the acoustic waves, whereas the slow speed \(c^0\) is associated with the transport. In our applications we are mainly interested in the accurate resolution of the slow waves, since they correspond to the movement of material interfaces.

Note that the Zuber–Findlay law (5) is not valid in case of pure phases \(Y = 0\) and \(Y = 1\). In these cases we will use the modified Zuber–Findlay law

\[\phi = \frac{\mu v + \nu}{1 + \mu b},\]

with \(\mu = 0.2 \sin \theta, \nu = 0.35\sqrt{gD} \sin \theta\). Due to relative complexity of the law (7), it is impossible to provide the characteristic analysis of the system (1), closed with the law (7).
3 The relaxation method

We approach the solutions of the original system (1) by the solutions of the relaxation system

\[ u_t + f(u_x) = \frac{1}{\epsilon} s_{\text{rel}}(u) + s(u), \]

where

\[ u = \begin{bmatrix} \rho \\ \rho Y \\ \rho v \\ \rho \Pi \\ \rho \Sigma \end{bmatrix}, \quad f(u) = \begin{bmatrix} \rho v \\ \rho Y v - \Sigma \\ \rho v^2 + \Pi \\ \rho \Pi v + a^2 v \\ \rho \Sigma v - b^2 Y \end{bmatrix}, \]

\[ s_{\text{rel}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ P(u) - \Pi \\ \sigma(u) - \Sigma \end{bmatrix}, \quad s(u) = \begin{bmatrix} 0 \\ 0 \\ -\rho g \sin \theta - C \rho v|v| \\ 0 \\ 0 \end{bmatrix}. \]

Here \( P = P(\rho, Y, v), \sigma = \sigma(\rho, Y, v) \) are given by eqns (2), \( \epsilon \) is the relaxation parameter, and \( a, b \) are the relaxation coefficients. Under certain conditions on \( a \) and \( b \), we have \( \Pi \rightarrow P \) and \( \Sigma \rightarrow \sigma \) as \( \epsilon \rightarrow 0 \), so the relaxation system (8) reduces to the original system (1), see Baudin et al. [1].

The advantage of considering the system (8) instead of (1) is that the mathematical structure of (8) appears to be quite simple. In particular, the system (8) is strictly hyperbolic. Analogous to (6), we can distinguish between the slow transport waves and the fast acoustic waves of (8). For the homogeneous part of (8) (i.e., without the source terms (10)), we can also solve the Riemann problem for (8) (i.e. the initial-value problem with piecewise constant initial data) exactly, see Baudin et al. [1]. Therefore, we can use Godunov’s method for the numerical solution of (8).

Consider a computational mesh in the domain \([0, L]\), consisting of cells \( j \) with mesh sizes \( \Delta x_j, j = 1, \ldots, N \). In general, the mesh sizes \( \Delta x_j \) can be different. We will describe the construction of the mesh in Section 4. Denote by \( u^n_j \) the average values at the mesh cell \( j \) at time \( t^n, j = 1, \ldots, N \). The values at the left and right of the cell interface \( j + 1/2 \) are denoted by \( u^n_{j+1/2} \). They are obtained from the average values \( u^n_j \) using a modified MUSCL approach, so that the resulting scheme will be of second order. The time step \( \Delta t \) is the same for all cells.
The homogeneous part of (8) is solved as follows. We consider an implicit formulation of the Godunov method for the homogeneous part of (8),

\[ u_j^{n+1} := u^n_j - \frac{\Delta t}{\Delta x_j} \left[ f(u^*(u_j^{n+1,-}, u_j^{n+1,+})) - f(u^*(u_{j-1/2}^{n+1,-}, u_{j-1/2}^{n+1,+})) \right], \]  

for \( j = 1, \ldots, N \). Here \( f(u) \) is given by (9), and \( u^*(u_{j+1/2}^{n,-}, u_{j+1/2}^{n,+}), j = 0, \ldots, N \), represents the exact solution to the Riemann problem with initial data \( u_{j+1/2}^{n,-} \) et \( u_{j+1/2}^{n,+} \). The Taylor expansion for the Godunov flux \( f(u^*(u_{j+1/2}^{n,-}, u_{j+1/2}^{n,+})) \) gives

\[ f(u^*(u_{j+1/2}^{n,-}, u_{j+1/2}^{n,+})) = f(u_{j+1/2}^{n,-}, u_{j+1/2}^{n,+}) + \nabla f_{j+1/2} \delta u_{j+1/2}^{n,+}, \quad j = 0, \ldots, N, \]  

with \( \delta u_{j+1/2}^{n,+} := u_{j+1/2}^{n,+} - u_{j+1/2}^{n,-} \). In order to calculate the Jacobians in (12), we interpret the Godunov flux as a Roe flux,

\[ f(u^*(u_{j+1/2}^{n,-}, u_{j+1/2}^{n,+})) = F(u_{j+1/2}^{n,-}, u_{j+1/2}^{n,+}) = \frac{1}{2} \left[ f(u_{j+1/2}^{n,-}) + f(u_{j+1/2}^{n,+}) - |A_{j+1/2}|(u_{j+1/2}^{n,+} - u_{j+1/2}^{n,-}) \right], \]

where \( A_{j+1/2} = R_{j+1/2} A_{j+1/2} R_{j+1/2}^{-1} \) is the Roe matrix, \( R_{j+1/2} \) is the matrix of its right eigenvectors, and \( A_{j+1/2} \) contains the corresponding eigenvalues on the main diagonal and zeros elsewhere. Using (13), we calculate the Jacobians in (12) as

\[ \frac{\partial F(u_{j+1/2}^{n,-}, u_{j+1/2}^{n,+})}{\partial u_{j+1/2}^{n,-}} = \frac{1}{2} \left[ \nabla f(u_{j+1/2}^{n,-}) + |A_{j+1/2}| \right], \]

\[ \frac{\partial F(u_{j+1/2}^{n,-}, u_{j+1/2}^{n,+})}{\partial u_{j+1/2}^{n,+}} = \frac{1}{2} \left[ \nabla f(u_{j+1/2}^{n,+}) - |A_{j+1/2}| \right]. \]  

For the semi-implicit version of the relaxation method, we rewrite the Jacobians in (14) as

\[ \nabla f(u_{j+1/2}^{n,\pm}) = R_{j+1/2}^\pm A_{j+1/2}^\pm \left( R_{j+1/2}^\pm \right)^{-1}, \]  

where \( A_{j+1/2}^\pm \) and \( R_{j+1/2}^\pm \) are the matrices of right eigenvectors and corresponding eigenvalues of \( f'(u_j^{n,\pm}) \). The semi-implicit version of the scheme consists in replacing the eigenvalues in \( A_{j+1/2}^\pm \) and \( A_{j+1/2}^\pm \), corresponding to the transport waves, by zeros. This amounts in solving the slow transport waves explicitly, and the fast acoustic waves — implicitly. The motivation for this approach comes from the linear case, see [1] for details.
Using (13), we substitute (14) in (12). Assuming \( \delta u_{j-1/2}^+ := \delta u_j \), \( \delta u_{j+1/2}^- := \delta u_j \), for \( j = 1, \ldots, N \), we obtain by (11) the linear system for the values of \( \delta u_j \), which is three-diagonal by blocks \( 5 \times 5 \). We solve it using the LU-factorization.

The time step is subject to the two CFL constraints: the explicit and the implicit one. We demand that the slow characteristic speeds satisfy the explicit CFL condition with \( \text{CFL}_{\text{exp}} := 0.5 \). The fast characteristic speeds must satisfy the implicit CFL condition with \( \text{CFL}_{\text{imp}} := 20 \), for instance.

The source terms \( s_{\text{rel}}(u) \) and \( s(u) \), given by (10), are treated with the splitting technique. Assuming that the relaxation take place instantaneously, the splitting for the relaxation term \( s_{\text{rel}}(u) \) reads

\[
P_j^{n+1} := P(p_j^{n+1}, v_j^{n+1}, v_j^{n+1}) \tag{16a}
\]

\[
\Sigma_j^{n+1} := \sigma(p_j^{n+1}, v_j^{n+1}, v_j^{n+1}). \tag{16b}
\]

In practical implementation, it is necessary to solve (16a) implicitly, see [1].

4 Adaptive mesh refinement

On the interval \([0, L]\) we introduce the ensemble of uniform grids, which are characterized by the level \( k \), \( k = K_c, \ldots, K_f \), with the cell size \( \Delta x^k \). On the level \( k \), the total number of cells is taken equal to \( 2^k \), so that \( \Delta x^k = L/2^k \). The cell \( i \) on the level \( k \) is defined as \( M^k_i := \Delta x^k[i-1, i], i = 1, \ldots, 2^k \). We have therefore an ensemble of \( K_f - K_c + 1 \) imbricated grids. The grid of the level \( K_f \) is the finest, and the grid of the level \( K_c \) is the coarsest one.

Consider the solution \( u(x) := u(x, t^n) \) to the relaxation system (8) at the time instant \( t^n \). We denote by \( \bar{u}^k_i \) the average value of the solution \( u(x) \) for \( x \in M^k_i \). Given the initial condition \( u^k_i, i = 1, \ldots, 2^k \) on the fine grid, we compute the solution \( u^{k-1}_i, i = 1, \ldots, 2^{k-1} \) on the coarser grid by

\[
u^{k-1}_i := \frac{1}{2}(u^k_{2i-1} + u^k_{2i}), \quad i = 1, \ldots, 2^{k-1}. \tag{17}\]

The inverse operation consists in predicting the values \( \bar{u}^k_{2i-1}, \bar{u}^k_{2i} \) on the fine grid by using the values on the coarser grid. Here we use the reconstruction operator, based on the second order polynomials,

\[
u^k_{2i-1} := u^{k-1}_i - \frac{1}{8}(u^k_{i+1} - u^k_{i-1}), \quad \nu^k_{2i} := u^{k-1}_i + \frac{1}{8}(u^k_{i+1} - u^k_{i-1}), \tag{18}\]

for \( i = 2, \ldots, 2^{k-1} - 1 \). For the boundary cells, we use the first order polynomials.

We define the detail as the difference between the average and the predicted values of the solution,

\[
d^k_i := u^k_{2i} - \bar{u}^k_{2i}, \quad i = 1, \ldots, 2^{k-1}. \tag{19}\]

The absolute value of the detail depends on the local regularity of the solution \( u(x) \), see [4].
The relations (17), (18), and (19) define the algorithms of coding and decoding the solution. Starting from the decoded solution $u^{K_f}_{i}$ on the finest grid, we can calculate the coded solution, which is given by the average values $u^{K}_{i}$, $i = 1, \ldots, 2^{K}$ on the grid $K$, and the details $d^{k}_{i}$, $k = K, \ldots, K_f - 1$, $i = 1, \ldots, 2^{k}$. Iterating the coding $K_f - K_c$ times, we can get the coded solution on the coarsest level $K_c$. Inversely, starting from the coded solution on the level $K$, we can obtain the decoded solution on the finest grid $K_f$.

The construction of adaptive mesh consists in selecting the grid level, which will be used to represent locally the solution. The decision is taken according to the size of the detail $d^{k}_{i}$. If its norm is below certain threshold $\epsilon$, the detail is considered as negligible and the cell $M^{k}_{i}$ is marked for decoding the solution. Otherwise one has to use the coarser level.

During the evolution of the solution to the hyperbolic system (8) the singularities propagate with finite speed. In order to predict the position of singularities at the time $t^{n+1}$ using the solution at the time $t^{n}$, we use the idea of Harten [5]. It consists in augmenting the set of marked cells such that if a cell $M^{k}_{i}$ is marked, than also the neighbor cells $M^{k-1}_{i}, M^{k+1}_{i}, M^{k+1}_{2i-1}, M^{k+1}_{2i}$ will be marked. Finally, the set of marked cells is one again augmented in order to ensure that the set of the marked cells is gradual, i.e., the neighbor cells are situated either on the same level, or on the neighbor ones. Also, the marked cells should cover the whole domain $[0, L]$. Now the obtained set of the marked cells forms the adaptive mesh over the domain $[0, L]$. We can decode the average values of the solution $u^{k}_{i}$ on the adaptive mesh and use them to evolve the solution to the next time instant.

The adaptive mesh is characterized by the threshold parameter $\epsilon$ and the number of refinement levels. The smaller $\epsilon$ is, the more details are considered as non-negligible; in the limit case $\epsilon = 0$ all details are non-negligible and the finest mesh is used everywhere in the domain. On the contrary, if $\epsilon$ is too big, too many details are considered as negligible and consequently the accuracy of the numerical solution deteriorates. The choice of the optimal $\epsilon$ is specific for each class of problems and it has to be fitted manually.

5 Numerical results

5.1 Stationary solution in the W-pipeline

Consider the pipeline with length $L = 4000$, diameter $D = 0.146$, and the inclination $\theta(x) = -30^\circ$ for $0 \leq x \leq L$ or $L/2 < x \leq 3L/4$, and $\theta(x) = +30^\circ$ for $L/4 < x < L/2$ or $3L/4 < x \leq L$.

The gas sound speed is $a_{g} = 316.22777$. The liquid is compressible, the equation of state is given by (4) with $\rho_{l}^{0} = 1000$, $p_{l}^{0} = 10^{5}$ and $a_{l} = 500$. The friction coefficient $C_{f} = 0.005$ and the gravity acceleration is $g = 9.81$. We use the Zuber–Findlay slip law (5). Boundary conditions are given by $q_{l} := 5/A$, $q_{g} := 0.1/A$, $\bar{p} := 10^{6}$, where $A = \pi D^2/4$ is the cross-section of the pipeline.

The comparison of the results using the adaptive and uniform grids is shown on
Fig. 1. The calculations were done using 256 cells on the finest level with 5 levels of refinement for the the adaptive mesh, and 256 cells for the uniform mesh. We see that the density and pressure are maximal at the lowest points of the pipeline, and minimal at its highest points. On the contrary, the velocity is minimal at the lowest points and maximal at the highest points. The mesh is refined where the solution is less smooth. Note that the adaptive and uniform solutions are almost indistinguishable.

The speed-up factor of the AMR method with 256 cells on the finest level over the uniform one with 256 cells is shown in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$\epsilon = 0.01$</th>
<th>$\epsilon = 0.001$</th>
<th>$\epsilon = 0.0001$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 levels</td>
<td>1.82</td>
<td>1.38</td>
<td>1.22</td>
</tr>
<tr>
<td>4 levels</td>
<td>1.82</td>
<td>1.50</td>
<td>1.22</td>
</tr>
<tr>
<td>5 levels</td>
<td>1.82</td>
<td>1.51</td>
<td>1.23</td>
</tr>
</tbody>
</table>

For the same number of refinement levels, the acceleration diminishes as $\epsilon \to 0$. Indeed, in this case the more and more details are considered as non-negligible, so
the solution can be rarely represented on the coarsest grid. For the same value of $\epsilon$, the speed-up factor increases slightly with the number of levels.

### 5.2 Severe slugging

Consider a pipeline with length $L = 74$, diameter $D = 0.05$, and the inclination $\theta(x) = 0^\circ$ for $0 \leq x \leq 60$, and $\theta(x) = +90^\circ$ for $60 < x \leq L$.

The gas sound speed is $a_g = 316.22777$. The liquid is compressible, the equation of state is given by (4) with $\rho_l^0 = 1000$, $p_l^0 = 10^5$ and $a_l = 500$. The friction coefficient $C_f = 0.005$ and the gravity acceleration is $g = 9.81$. We use the modified Zuber–Findlay slip law (7). Boundary conditions are given by $q_l := 0.07854/A$, $q_g := 0.000196/A$, $\bar{p} := 10^5$.

The Fig. 2 shows the time evolution of the flow at the fixed position $x = 60$, with 3 refinement levels, $\epsilon = 0.0001$, and 128 cells on the finest level for the AMR method. The physical situation is the following. The liquid column in the riser (the vertical part of the pipeline) blocks the flow, causing the pressure in the horizontal part to augment. When the pressure exceeds a critical value, the pressure difference pushes the mixture towards the pipeline exit. The violent acceleration (over 16-fold in velocity) causes the appearance of the rarefaction zone near the corner. The liquid in the riser falls down and swings back and forth, finally again blocking the flow.

The speed-up factor of the AMR method with 128 cells on the finest level over the uniform one with 128 cells is shown in the following table:

<table>
<thead>
<tr>
<th>$\epsilon$ (time scale)</th>
<th>3 levels</th>
<th>4 levels</th>
<th>5 levels</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon = 0.01$</td>
<td>2.37</td>
<td>2.46</td>
<td>1.58</td>
</tr>
<tr>
<td>$\epsilon = 0.001$</td>
<td>2.27</td>
<td>2.28</td>
<td>1.51</td>
</tr>
<tr>
<td>$\epsilon = 0.0001$</td>
<td>2.18</td>
<td>2.07</td>
<td>1.49</td>
</tr>
</tbody>
</table>

As for the previous test case, the speed-up factor diminishes as $\epsilon$ goes to zero. Observe that for the same value of $\epsilon$, the speed-up factor varies non-monotonically with number of levels. The reason is that with more refinement levels, the coding-decoding operations become increasingly expensive. Note that this test is favorable for the AMR method: there are few irregularities in the horizontal part of the pipeline, and therefore the coarsest grid can be used in the most of the domain.

### 6 Conclusions and outlook

We have demonstrated that the relaxation method can be successfully used for simulating several flow phenomena in pipelines, including the slug formation. Our experience shows that the adaptive mesh refinement technique accelerates the computation by the factor of up to 3. We expect that using more realistic slip laws will increase this factor even more significantly. We are also considering the use of the local time stepping for the further speed-up of the computation.
Figure 2: The time evolution of the adaptive and uniform solutions (indistinguishable) for the severe slugging flow.

References


