AN ULTRA WEAK FINITE ELEMENT METHOD AS AN ALTERNATIVE TO A MONTE CARLO METHOD FOR AN ELASTO-PLASTIC PROBLEM WITH NOISE *

ALAIN BSENSUSSAN †, JANOS TURI ‡, LAURENT MERTZ §, AND OLIVIER PIRONNEAU ¶

Abstract. An efficient method for obtaining numerical solutions of a stochastic variational inequality modeling an elasto-plastic oscillator with noise is considered. Since Monte Carlo simulations for the underlying stochastic process are too slow to produce results, as an alternative, approximate solutions of the partial differential equation defining the invariant measure of the process are studied. The regularity of the solution of that partial differential equation is not sufficient to employ a “standard” finite element method. To overcome that difficulty, an ultra weak finite element method has been developed and successfully implemented.

Key words. Random Vibration, Ergodicity of degenerate diffusion, Numerical approximation of invariant measure

AMS subject classifications. 37M25, 65N99

1. Introduction. Modeling and simulation of elasto-plastic materials under random excitation has been studied by many authors (see e.g., [5] and the bibliography therein) providing important information on the resistance of structures to earthquakes and also on fatigue in general. To understand the problem the simplest is to consider a rod excited at one end by a random force and clamped at the other end. We are interested by the displacement \( X(t) \) in general. To understand the problem the simplest is to consider a rod excited at one end by a random force and clamped at the other end. We are interested by the displacement \( X(t) \) of the free end of the rod. The velocity at this end point is denoted by \( Y(t) \), and we have \( X(t) = Y(t) \). Newton’s law implies \( Y(t) = -(\alpha_0 Y(t) + F(X(t)) + f(t) \) where \( \alpha_0 Y(t) \) is a damping term, \( -F(X(t)) \) is a restoring force and \( f(t) \) is the force applied at the free end of the rod. We assume that \( f(t) \) is white noise. The conservation of forces written as a stochastic differential equation is

\[
dY(t) = -(\alpha_0 Y(t) + F(X(t)))dt + dW(t) \tag{1.1}
\]

where \( W(t) \) is a standard Wiener process.

Beyond a given threshold \( |X(t)| > b \) for the displacement the material goes through plastic deformation. Introducing \( X(t) \), the total plastic yielding accumulated up to time \( t \), we can define a new state variable \( Z(t) \) as \( Z(t) = X(t) - X(t) \). It follows that in the plastic regime, \( Z(t) = \) 0, while \( Y(t) \) satisfies (1.1) where now the restoring force, \( F(X(t)) \), is written as \( F(X(t)) = kZ(t) \) for some constant \( k \) and \( |Z(t)| < b \). This is the nonlinear single degree of freedom model of [5]; its mechanical analogy is a system containing a linear mass, dashpot and spring and Coulomb friction-slip joint studied in [8].

Engineers are interested in the asymptotic regime at large time i.e. the probability density \( m \) such that \( m = \lim_{t \to \infty} p_t \) where \( p_t \) is the probability density function of the process \( (Z(t), Y(t)) \) i.e. \( \mathbb{P}((Z(t), Y(t)) \in [z, z + dz] \times [y, y + dy]) = p_t(z, y)dzdy. \)

*This research was partially supported by a grant from CEA, Commissariat à l’énergie atomique and by the National Science Foundation under grant DMS-0705247
†International Center for Decision and Risk Analysis, ICDRiA, School of Management, University of Texas at Dallas, Richardson, TX 75083 (Alain.Bensoussan@utdallas.edu)
‡Department of Mathematical Sciences EC35, University of Texas at Dallas, Richardson, TX 75083 (turi@utdallas.edu)
§Université Pierre et Marie Curie-Paris 6, Laboratoire Jacques Louis Lions, 175 rue du Chevaleret 75013 Paris (mertz@ann.jussieu.fr)
¶Université Pierre et Marie Curie-Paris 6, Laboratoire Jacques Louis Lions, 175 rue du Chevaleret 75013 Paris (pironneau@ann.jussieu.fr)
The numerical simulation of the system by a Monte-Carlo method is straightforward to implement but it is slow. The paper is an attempt to find a better numerical method, by solving a partial differential equation (PDE) for $m$. The Kolmogorov forward equation on $p$ is given by

$$\frac{\partial}{\partial t} p - y \frac{\partial}{\partial z} p + \frac{1}{2} \frac{\partial^2}{\partial y^2} p = 0$$

(1.2)

It must be solved forward in time in the domain $(-b, b) \times \mathcal{R}$ with appropriate boundary conditions. Finding the solution for large times is expensive, so we may prefer to compute the invariant measure $m$ of the process which is known to satisfy

$$-y \frac{\partial}{\partial z} m + \frac{1}{2} \frac{\partial^2}{\partial y^2} m = 0$$

(1.3)

However the solution of this equation with the boundary conditions is not regular and as a consequence of that classical numerical methods fail. To overcome that difficulty we have employed an unusual, so called ultra-weak variational method to solve (1.3).

Ultra-weak methods are used to establish existence of some difficult partial differential equations but rarely numerically, even though one of the authors had explored this strategy in an earlier paper [4]. To explain the method consider the Dirichlet problem

$$-\Delta u = f, \text{ in } \Omega \quad u|_{\partial \Omega} = 0$$

The usual variational formulation searches $u \in H^1_0(\Omega)$, (the space of functions with square integrable first derivatives and with zero trace on the boundary), such that, for all $w \in H^1_0(\Omega)$,

$$\int_{\Omega} \nabla u \cdot \nabla w = \int_{\Omega} fw$$

The ultra-weak formulation requires much less regularity for $u$ and searches $u \in L^2(\Omega)$ such that for all $w \in H^2(\Omega) \cap H^1_0(\Omega)$

$$\int_{\Omega} u(\Delta w) = \int_{\Omega} fw$$

However the finite element method would not work as such because of $\Delta w$; one must either use a mixed formulation as in [4] or use a basis to represent the space generated by the $\Delta w$ when $w$ varies. As the second order operator in (1.3) is degenerate, we had to use the second alternative in this paper. For the Dirichlet problem this means that a finite set of test functions $w^j$ is chosen, then $g^j = \Delta w^j$ is computed (in weak form), and then the following linear system is solved:

$$\int_{\Omega} (\sum_i u_i g^i) g^j = \int_{\Omega} f w^j, \quad \forall j$$

(1.4)

The remaining part of the paper is organized as follows:

First, in Section 2 we study two Monte-Carlo algorithms; one based on (1.1) reformulated in a mathematically more correct form with stopping times between the regimes. Trajectories are simulated using the Box-Muller formula and an explicit Euler time finite difference scheme. The probability density for $((Z(T), Y(T)) \in (z, z + dz) \times (y, y + dy)$ is computed
An Ultra Weak Finite Element Method as an Alternative to a Monte Carlo Method for an Elasto-Plastic Problem with Noise

and its limit \( m(z, y) \) when \( T \) is large is the final product of these simulations. Then in Section 3 we proceed with the Kolmogorov partial differential equation (1.2) of the process, and its limit for the invariant measure, and show numerically in Section 4 that a standard finite element method does not reproduce the results of the Monte-Carlo simulations. Finally, in Section 5 we turn to the ultra-weak formulation. To analyze the method we return to the Dirichlet problem and show that we can successively compute the solution of (1.4) when \( f = \Delta 1_A \), for a given subset \( A \) of \( \Omega \); in this case the solution is \( u = 1_A \) and it cannot be computed by standard FEM because it is discontinuous. Then the method is applied to the computation of the invariant measure \( m \) and compared with the results obtained by Monte-Carlo simulations.

In conclusion we remark that the ultra-weak method is certainly harder to program compared with Monte-Carlo algorithms but it is somewhat faster and, most importantly, more precise even though it is hampered by the necessity to choose several important parameters such as the place at which the computational domain is truncated.

2. Monte-Carlo Simulations.

2.1. Reformulation with Stopping Times. We recall from [2] the stochastic differential equations governing the process \((Z(t), Y(t))\) of (1.1) corresponding to the elastic and plastic regimes, respectively.

For \( n = 0, 1, .. \) and starting with \( \theta_0 = \tau_0 = 0 \), we define two sets of stopping times in the following way:

\[
\begin{align*}
\theta_{n+1} &= \inf\{t > \tau_n \mid |Z(t)| = b\} \\
\tau_{n+1} &= \inf\{t > \theta_{n+1} \mid \text{sign}(Y(t)) = -\text{sign}(Z(t))\}
\end{align*}
\]  

(2.1)

When \( t \in]\tau_n, \theta_{n+1}[ \), we have \( |Z(t)| < b \) and \((Z(t), Y(t))\) satisfies:

\[
\begin{align*}
\frac{dY(t)}{dt} &= -(c_0 Y(t) + kZ(t))dt + dW(t) \\
\frac{dZ(t)}{dt} &= Y(t)dt
\end{align*}
\]  

(2.2)

When \( t \in]\theta_{n+1}, \tau_{n+1}[ \), we have \( |Z(t)| = b \) and \((Z(t), Y(t))\) satisfies:

\[
\begin{align*}
\frac{dY(t)}{dt} &= -(c_0 Y(t) + kZ(t))dt + dW(t) \\
\frac{dZ(t)}{dt} &= 0
\end{align*}
\]  

(2.3)

2.2. Analytic formulae. We will use later the fact that in each elastic and plastic time segments there is an analytic formula for the solution.

Let \((Z(0), Y(0)) = (z, y)\) and let \( \omega = \sqrt{c_0^2 - 4k} \). We have

\[
\begin{align*}
Z(t) &= e^{-\omega t} \left\{ z \cos \left( \frac{\omega t}{2} \right) + \frac{2}{\omega} (y + \frac{c_0}{2} z) \sin \left( \frac{\omega t}{2} \right) \right\} \\
&\quad + \frac{2}{\omega} \int_0^t e^{-\omega(t-s)} \sin \left( \frac{\omega s}{2} \right) dW(s)
\end{align*}
\]

\[
\begin{align*}
Y(t) &= -\frac{2}{\omega} Z(t) + e^{-\omega t} \left\{ -\frac{\omega}{2} z \sin \left( \frac{\omega t}{2} \right) + (y + \frac{c_0}{2} z) \cos \left( \frac{\omega t}{2} \right) \right\} \\
&\quad + \int_0^t e^{-\omega(t-s)} \cos \left( \frac{\omega s}{2} \right) dW(s)
\end{align*}
\]  

(2.4)

\( Z(t) \) is a gaussian variable with mean equal to \( m_Z(t, z, y) \) and variance equal to \( \sigma_Z^2(t) \), where

\[
m_Z(t, z, y) = e^{-\omega t} \left\{ z \cos \left( \frac{\omega t}{2} \right) + \frac{2}{\omega} (y + \frac{c_0}{2} z) \sin \left( \frac{\omega t}{2} \right) \right\}
\]

\[
\sigma_Z^2(t) = \frac{4}{\omega^2} \int_0^t e^{-\omega s} \sin^2 \left( \frac{\omega s}{2} \right) ds
\]
$Y(t)$ is a gaussian variable with mean equal to $m_Y(t, z, y)$ and variance equal to $\sigma_Y^2(t)$, where

$$m_Y(t, z, y) = -\frac{c_0}{2}m_Z(t, z, y) + e^{-\omega t} \left\{ -\frac{\omega}{2} z \sin \left( \frac{\omega t}{2} \right) + (y + \frac{c_0}{2}) \cos \left( \frac{\omega t}{2} \right) \right\}$$

$$\sigma_Y^2(t) = \int_0^t e^{-c_0 s} \cos^2 \left( \frac{\omega s}{2} \right) ds - \frac{c_0}{\omega^2} \int_0^t e^{-c_0 s} \sin^2 \left( \frac{\omega s}{2} \right) ds - \frac{2c_0}{\omega^2} e^{-c_0 t} \sin^2 \left( \frac{\omega t}{2} \right)$$

The correlation between $Y(t)$ and $Z(t)$ is given by

$$\sigma_{YZ}(t) = \frac{1}{\omega} \int_0^t e^{-c_0 s} \sin \left( \omega s \right) ds - \frac{2c_0}{\omega^2} \int_0^t e^{-c_0 s} \sin^2 \left( \frac{\omega s}{2} \right) ds$$

When the system is in a plastic state there is an analytic solution also. Let $(Z(0), Y(0)) = (\pm b, y)$ then

$$\begin{cases}
Z(t) = \pm b \\
ye^{-c_0 t} \mp \frac{kb}{c_0} \left( 1 - e^{-c_0 t} \right) + \frac{e^{-c_0 t}}{\sqrt{2c_0}} W(e^{2c_0 t} - 1)
\end{cases} \quad (2.5)$$

### 2.3. Monte Carlo method for computing the invariant measure.

Based on the analytic solution of the first formulation, a C code was written to simulate $(Z(t), Y(t))$. Let $T > 0, N \in \mathbb{N}$, and $(t_n)_{n=0..N}$ a family of time which discretize $[0, T]$, such that $t_n = n\delta t$ where $\delta t := \frac{T}{N}$.

We set $\Sigma \in \mathcal{M}_{2,2}(\mathbb{R}^2)$ such that

$$\Sigma \Sigma^T = \begin{pmatrix}
\sigma_Z^2(\delta t) & \sigma_{Z,Y}(\delta t) \\
\sigma_{Z,Y}(\delta t) & \sigma_Y^2(\delta t)
\end{pmatrix}$$

Let $(G_{n,m})_{n=0..N, m=1,2}$ be random independent Gaussian $\mathcal{N}(0, \sqrt{\delta t})$ variables. Here, all gaussian random variables are generated by the Box-Muller formula [7] and the C function $\text{random()}$. The finite difference scheme for (1.1) is as follows:

if $|Z(t_n)| < b$,

$$\begin{pmatrix}
Z(t_{n+1}) \\
Y(t_{n+1})
\end{pmatrix} = \begin{pmatrix}
m_Z(\delta t, Z(t_n), Y(t_n)) \\
m_Y(\delta t, Z(t_n), Y(t_n))
\end{pmatrix} + \Sigma \begin{pmatrix}
G_{n,1} \\
G_{n,2}
\end{pmatrix} \quad (2.7)$$

if $|Z(t_n)| = b$,

$$\begin{pmatrix}
Z(t_{n+1}) \\
Y(t_{n+1})
\end{pmatrix} = \begin{pmatrix}
\pm b \\
ye^{-c_0 \delta t} \mp \frac{kb}{c_0} \left( 1 - e^{-c_0 \delta t} \right) + e^{-c_0 \delta t} \sqrt{\frac{e^{2c_0 \delta t} - 1}{2c_0}} G_{n,2}
\end{pmatrix} \quad (2.8)$$

Figure 2.1 shows a sample of trajectory of the process at $T=10$. Then, to compute the probability density function of $(Z(T), Y(T))$, we define $L$ sufficiently large, $D_L := (-b, b) \times (-L, L)$ and a uniform grid $\mathcal{G}(N_y, N_z)$ on $D_L$ for given integers $N_y, N_z$. The $i, j$-cell is

$$D_{i,j} = \left[ -Y + \frac{2iY}{N_z}, -Y + \frac{2(i+1)Y}{N_z} \right] \times \left[ -L + \frac{2jL}{N_y}, -L + \frac{2(j+1)L}{N_y} \right]$$

Let us generate numerically $N$ trajectories up to time $T$ and count the number $n_{i,j}$ of trajectories ending in $D_{i,j}$. By the law of large numbers we can approximate the probability
An Ultra Weak Finite Element Method as an Alternative to a Monte Carlo Method for an Elasto-Plastic Problem with Noise

that \(((Z(T), Y(T)) \in \mathcal{D}_{i,j})\) by \(\bar{X}_{i,j}^N := \frac{n_{i,j}}{N}\). By the central limit theorem we also know the error: for instance at 5 % error

\[
P((Z(T), Y(T)) \in \mathcal{D}_{i,j}) \in \left[ \bar{X}_{i,j}^N - \frac{1.96 \bar{X}_{i,j}^N(1 - \bar{X}_{i,j}^N)}{\sqrt{N}}, \bar{X}_{i,j}^N + \frac{1.96 \bar{X}_{i,j}^N(1 - \bar{X}_{i,j}^N)}{\sqrt{N}} \right]
\]

The invariant measure \(m\) of the process is computed as the asymptotic limit for large \(T\) of \(\bar{X}_{i,j}^N\).

In figure 2.2-2.3 \(m\) is shown for different values of \(b\), \(b = 1, 2, 3, 4\), with \(L = 7, T = 10, N = 10^7\). Moreover, for each \(b\), we can estimate the plastic part denoted by \(p_{b}^{MC}\), and the confidence interval for a 5 % error.

**Remark 1. Formulation as Stochastic Differential Inequalities**

In [2] it is shown that (2.1) is equivalent to a set of stochastic differential inequalities (2.9):

\[
\begin{aligned}
dY(t) &= -(c_0 Y(t) + k Z(t))dt + dW(t) \\
(dZ(t) - Y(t)dt)(\zeta - Z(t)) &\geq 0 \\
|\zeta| &\leq b \\
|Z(t)| &\leq b
\end{aligned}
\]

This system is well posed for a given probability distribution \(\psi\) of the initial condition \((Z(0), Y(0))\).

In [2] existence, uniqueness with the law of \(Z(0), Y(0)\) given, and ergodicity are shown. Hence, there exists one and only one invariant law associated to the process \((Z(t), Y(t))\). We also implemented a numerical method based on (2.9) but the results and the computing speed were similar to the standard MC described above.
Fig. 2.2. Plots of $m$ computed by Monte Carlo Method; from the top to the bottom we consider $b = 1, 2$. When $b$ increases the plastic part decreases, indeed $p_1 = 84.9_{[84.7, 85.0]}\%$, $p_2 = 9.62_{[9.61, 9.63]}\%$

3. Kolmogorov equation of the process. The partial differential equation (PDE) ruling the probability density function of the process $(Z(t), Y(t))$, is found by Ito calculus on each interval of time $]\tau_n, \theta_{n+1}[$ and $]\theta_{n+1}, \tau_{n+1}[$.

Consider $(Z(t), Y(t))$ verifying (2.2)-(2.3), let $\tau_n < t < t' < \theta_{n+1}$, then any regular function
Fig. 2.3. Plots of $m$ computed by Monte Carlo Method; from the top to the bottom we consider $b = 3, 4$. When $b$ increases the plastic part decreases, indeed $p_3 = 0.04764\, [0.04760, 0.04768] \%$, $p_4 = 0.0 \%$.

verifies

$$u(t', Z(t'), Y(t')) = u(t, Z(t), Y(t)) + \int_t^{t'} \partial_y u dW(s)$$

$$+ \int_t^{t'} \left[ \partial_t u + \frac{1}{2} \partial_{yy} u - (c_0 Y(s) + k Z(s)) \partial_y u + Y_s \partial_z u \right] ds$$
Similarly when $\theta_{n+1} < t < t' < \tau_{n+1}$, by definition, $z = \pm b$ and so

$$u(t', \pm b, Y(t')) = u(t, \pm b, Y(t)) + \int_t^{t'} \partial_y u dW(s) + \int_t^{t'} [\partial_t u + \frac{1}{2} \partial_{yy} u - (c_0 Y(s) \pm kb) \partial_y u] ds$$

Let us introduce new notations:

**Notations.**

$$D = (-b, +b) \times \mathbb{R}, \quad D_{\pm b} = \partial D \cap \{(\pm b) \times \mathbb{R}\}$$

$$D_{\pm b}^+ = D_{\pm b} \cap \{(z, y) : \pm y > 0\}.$$  

$$A u = -\frac{1}{2} \partial_{yy} u + (c_0 y + k z) \partial_y u - y \partial_z u$$

$$B_{\pm b} u = -\frac{1}{2} \partial_{yy} u + (c_0 y \pm kb) \partial_y u$$

**Proposition 3.1.** Let $g \in L^2(D)$, consider $v(t, z, y) := \mathbb{E}[g(Z_t, Y_t)|(Z_0, Y_0) = (z, y)]$ then, $v$ satisfies the following equation (see [1])

$$\begin{cases}
\partial_t v + A v = 0 & \text{in } [0, T] \times D \\
\partial_t v + B_{\pm b} v = 0 & \text{in } [0, T] \times D_{\pm b}^+
\end{cases}$$

Consequently, the infinitesimal generator of $(Z(t), Y(t))$, denoted $\Lambda$ satisfies:

$$\Lambda : \phi \mapsto \begin{cases}
A\phi & \text{if } z \in (-b, b] \\
B_{\pm b}\phi & \text{if } z = \pm b, \pm y > 0
\end{cases}$$

**3.1. Derivation of the dual equation for the invariant measure.** By definition the invariant measure denoted $\mu$ of the process $(Z(t), Y(t))$ is such that $\int \Lambda(\phi) d\mu = 0$, $\forall \phi$ regular. We assume that $\mu$ has a probability density, denoted by $m$. Then, $m$ has to satisfy the following equation:

$$\int_D m.A\phi + \int_0^{+\infty} m.B_0\phi + \int_{-\infty}^{0} m.B_{-b}\phi = 0 \quad \forall \phi$$

which is,

$$\int_{-b}^{b} dz \int_{-\infty}^{+\infty} m(z, y) \{y \partial_z \phi - (c_0 y + k z) \partial_y \phi + \frac{1}{2} \partial_{yy} \phi\} dy + \int_{0}^{+\infty} m(b, y) \{- (c_0 y + kb) \partial_y \phi(b, y) + \frac{1}{2} \partial_{yy} \phi(b, y)\} dy + \int_{-\infty}^{0} m(-b, y) \{- (c_0 y - kb) \partial_y \phi(-b, y) + \frac{1}{2} \partial_{yy} \phi(-b, y)\} dy = 0$$
Integrated by part formally, it is:

\[
\int_D \left( -\phi y \partial_z m + \phi \partial_y [m(c_0 y + k z)] + \frac{1}{2} \partial_{yy} m \right) dy + \int_{-\infty}^{+\infty} m \phi y \big|_{z=-b}^{z=b} + \\
\int_0^{+\infty} \phi(b, y) \{ \partial_y [m(b, y)(c_0 y + kb)] + \frac{1}{2} \partial_{yy} m(b, y) \} dy + \\
\int_{-\infty}^{0} \phi(-b, y) \{ \partial_y [m(-b, y)(c_0 y - kb)] + \frac{1}{2} \partial_{yy} m(-b, y) \} dy = 0
\]

which implies, in the sense of distribution, that \( m \) has to satisfy the following (see [2]) :

**PROPOSITION 3.2.** The invariant probability \( m \) of \((Z(t), Y(t))\) is solution of

\[
\begin{cases}
-y \partial_z m + \partial_y [m(c_0 y + k z)] + \frac{1}{2} \partial_{yy} m = 0 \text{ in } D \\
|y| m + \partial_y [m(c_0 y + k z)] + \frac{1}{2} \partial_{yy} m = 0 \text{ on } D^{-}_b \cup D^{+}_b \\
m = 0 \text{ on } D^{-}_b \cup D^{+}_b \\
f_D m + \int_0^{\infty} m + \int_{-\infty}^0 m = 1
\end{cases}
\]  

**(3.3)**

**REMARK 2.** Not much is known on the regularity of \( m \). However, thanks to ergodic property of \((Z_t, Y_t)\), \( \lim_{t \to -\infty} v(t, z, y) = \int m(z, y) g(z, y) dzdy. \)

**4. Finite element method on the problem for \( m \).** A finite element method was used on (3.3) for solving \( m \). Some smoothness is assumed otherwise the finite element method is not likely to work. To implement the boundary conditions we defined an auxiliary problem:

\[
|y| \tilde{m} + \partial_y [\tilde{m}(c_0 y + k z)] + \frac{1}{2} \partial_{yy} \tilde{m} = 0 \text{ on } D^{+}_b \cup D^{-}_b, \quad \tilde{m}(\pm b, 0) = 0,
\]

with \( \tilde{m}(\pm b, \pm \infty) = c \) constant, and then solved

\[
\begin{cases}
-y \partial_z m + \partial_y [m(c_0 y + k z)] + \frac{1}{2} \partial_{yy} m = 0 \text{ in } D \\
m = \tilde{m} \text{ on } D^{+}_b \cup D^{-}_b, \\
m = 0 \text{ on } D^{-}_b \cup D^{+}_b
\end{cases}
\]  

**(4.1)**

and choose \( c \) such that

\[
\int_D m + \int_0^{\infty} m + \int_{-\infty}^0 m = 1
\]

**4.1. Numerical result.** We solved (4.1) by the \( P^1 \)-finite element method using the package **freefem++** [6]. We took \( L=10, b=1.0 \). The solution disagree with those obtained with the Monte Carlo method and does not converge when the mesh size decreases. The solution is not in \( H^1(D) \) and so the finite element method is illicite (see figure 4.1).

**5. Ultra Weak Finite Element Methods.**

**5.1. Introduction to ultra weak method on a simple problem.** In order to illustrate with a simple case the principle of ultra weak methods, we consider a Laplace equation in \( \Omega \subset \mathbb{R}^2 \) with right hand side \( f \in H^{-2}(\Omega) \). Thus we search for the unique \( u \in L^2(\Omega) \) satisfying [4]

\[
-\int_\Omega u \Delta w = \int_\Omega f w, \forall w \in H^2(\Omega) \cap H^1_0(\Omega).
\]  

**(5.1)**
Let \( \{g^i, i \in I\} \) a set of \( N \)-independent functions of \( L^2(\Omega) \). Problem (5.1) is approximated in the subspace generated by the \( g^i \)

\[
  u \approx u_N := \sum_{i \in I} u_i g^i \text{ with } u_i \in \mathbb{R}, \forall i \leq N. \tag{5.2}
\]

For each \( i = 1, \ldots, N \) denote \( v^i \) the solution of

\[
  -\Delta v^i = g^i \text{ in } \Omega \quad v^i = 0 \text{ on } \partial \Omega \tag{5.3}
\]

Consequently

\[
  \int_{\Omega} u g^i = \sum_j \left( \int_{\Omega} g^i g^j \right) u_j = \int_{\Omega} f v^i \tag{5.4}
\]

Finally, denoting

\[
  \tilde{F} = (\int_{\Omega} f v^i), \quad U = (u_i), \quad A = (\left( \int_{\Omega} g^i g^j \right))
\]

we have to solve \( AU = \tilde{F} \).

For the test cases, we choose \( \Omega := [-L_x, L_x] \times [-L_y, L_y] \) triangulated uniformly with \( N_x \times N_y \) vertices and with \( L_x = L_y = 5 \).
5.2. Numerical test case 1. We set, $f := 1$ and, with $\sigma = 0.3$:

$$g^{i,j}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left( -\frac{1}{2} \left( \frac{x - x_i}{\sigma} \right)^2 + \left( \frac{y - y_j}{\sigma} \right)^2 \right), i = 0..N_x, j = 0..N_y$$

We compare $u_N$ with $u_h$ the solution of the standard finite element method of degree 1 on the same mesh. Figure 5.1 shows relative $L^2$-error $\frac{||u_h - u_N||_{L^2}}{||u_h||_{L^2}}$.

![Figure 5.1](image.png)

**Fig. 5.1.** Relative $L^2$ error in a log-log scale, showing a $O(N^{-2.62})$ behavior.

5.3. Numerical test case 2. We set, $A = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 1\}$ and choose $f \in H^{-2}$ such that $\forall \phi \in H_0^2(\Omega), \langle f, \phi \rangle = \int_A \Delta \phi$. The basis function we chose are

$$g^{i,j}(x, y) = \frac{1}{\sqrt{|K_{i,j}|}} K_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}], i = 0..N_x - 1, j = 0..N_y - 1$$

Figure 5.2 shows the relative $L^2$-error between the analytical solution $u = 1_A$ and $u_N$.

5.4. Duality with an elliptic problem: switching time by $\lambda$. In this section, we explain the ultra weak formulation for the invariant measure. Consider the Laplace transform for $\lambda > 0$,

$$u_\lambda(z, y) = \int_0^\infty e^{-\lambda t} v(t, z, y) dt$$

we observe that

$$\lambda u_\lambda(z, y) = -\int_0^\infty (-\lambda e^{-\lambda t}) v(t, z, y) dt$$

integration by parts gives

$$\lambda u_\lambda(z, y) = \int_0^\infty e^{-\lambda t} \partial_t v(t, z, y) dt - [e^{-\lambda t} v(t, z, y)]_{t=0}^{t=\infty}$$
Fig. 5.2. Relative $L^2$-error in log-log, showing a $O(N^{-0.52})$ behavior

then

$$\lambda u_\lambda(z, y) = -\int_0^{\infty} e^{-\lambda t} Av(t, z, y) dt + g(z, y)$$

Finally, using the fact that $A$ is independant of $t$, we obtain

$$\lambda u_\lambda(z, y) + Au_\lambda(z, y) = g(z, y)$$

In the same way, on boundaries we have

$$\lambda u_\lambda(\pm b, y) + Bu_\lambda(\pm b, y) = g(\pm b, y)$$

This last formulation is numerically superior to (3.1) because we can compute $\lim_{t\to\infty} v(t, z, y)$ in one step.

Indeed, thanks to initial value theorem for Laplace transform,

$$\lim_{\lambda\to 0} \lambda u_\lambda = \lim_{t\to\infty} v(t, z, y) \quad (5.5)$$

Hence,

$$\lim_{\lambda\to 0} \lambda u_\lambda = \int_{\Omega} mg \quad (5.6)$$

Next, we consider $(g^i)_{i \in I}$ a set of independent piecewise continuous functions in $L^2(\overline{D})$ and we approximate $m \in L^2(D)$, by $\sum_{i \in I} m_i g^i$. To each basis function $g \in L^2(D)$, we associate $u$ solution of the following problem.

$$\begin{cases}
\lambda u + Au = g \text{ in } D \\
\lambda u + Bu^+ = g^+ \text{ in } D^+_b \\
\lambda u + Bu^- = g^- \text{ in } D^-_b
\end{cases} \quad (5.7)$$

Where $g^\pm$ may coincide with the trace of $g$ when defined.
5.5. Computation of $m$. We shall build $m$ by inverting (5.6); this means solving a linear system with matrix

$$A_{i,j} = \int_D g^i g^j dz dy + \int_{-\infty}^0 g^i(-b,y)g^j(-b,y) dy + \int_0^\infty g^i(b,y)g^j(b,y) dy$$

Thus, denoting $u_{\lambda,i}$ the solution of (5.7), we have to find it for each $g^i$.

Note that (5.6) is also $(Am)_i = (\lim_{\lambda \to 0} \lambda u_{\lambda,i})$.

6. Numerical result with the Ultra Weak formulation.

6.1. Elliptic problem with $\lambda$ parameter. From the computations, we need to approximate $D$ by $D_L$ and add an artificial condition on $y = \pm L$. We denote $D_{L,b}^+ = \{z = b, 0 < y < L\}$ and $D_{L,-b}^- = \{z = -b, -L < y < 0\}$ and add a Neumann condition on the rest of the boundary:

$$\begin{cases}
\lambda u + A u &= g \text{ in } D_L \\
\lambda u + B_b u &= g_+ \text{ in } D_{L,b}^+ \\
\lambda u + B_{-b} u &= g_- \text{ in } D_{L,-b}^- \\
\partial_n u &= 0 \text{ on } -b < z < b, y = \pm L
\end{cases}$$

(6.1)

We found that the choice of $L$ is critical, it need to be small for numerical speed and large for precision. After that, the main difficulty was to deal with boundaries conditions, they are nonlocal Dirichlet conditions at the border $z = \pm b$.

6.2. A superposition method for the boundary conditions. We observed the conditions at $z = b, 0 < y < L$ and $z = -b, -L < y < 0$ are autonomous ODE in the $y$ variable. Then, they could be solved separately in order to obtain non homogenous Dirichlet conditions for $u$.

But, we needed to impose appropriate boundaries conditions at the end parts of $[0, L]$ and $[-L, 0]$. For the truncated problem, it seemed compatible to impose homogenous Neumann conditions at $(z, y) = \pm (b, L)$. But, the values of $u$ denoted by $u_+$ in $z = b, y = 0$ and $u_-$ in $z = -b, y = 0$ are unknown.

6.2.1. Dealing with $u_{\pm}$. In [3] Bensoussan and Turi showed $u_\lambda$ is continuous. Then, the difficulty was to guess $u_+$ and $u_-$ such that $u_\lambda$ be continuous.

By linearity the solution of (6.1) is also a linear combination of the three following problems:

$$\begin{cases}
\lambda u_0 + A u_0 &= g \text{ in } D \\
\lambda u_0 + B_b u_0 &= g_+ \text{ in } D_{L,b}^+ \\
\lambda u_0 + B_{-b} u_0 &= g_- \text{ in } D_{L,-b}^- \\
\text{with } u_+ = 0, u_- = 0
\end{cases}$$

$$\begin{cases}
\lambda u_1 + A u_1 &= 0 \text{ in } D \\
\lambda u_1 + B_b u_1 &= 0 \text{ in } D_{L,b}^+ \\
\lambda u_1 + B_{-b} u_1 &= 0 \text{ in } D_{L,-b}^- \\
\text{with } u_+ = 1, u_- = 0
\end{cases}$$
\[
\begin{align*}
\begin{cases}
\lambda u_2 + Au_2 &= 0 \text{ in } D \\
\lambda u_2 + B_b u_2 &= 0 \text{ in } D_{L,b} \\
\lambda u_2 + B_{-b} u_2 &= 0 \text{ in } D_{L,-b}
\end{cases}
\end{align*}
\]

with \(u_+ = 0, u_- = 1\)

We must find \(\alpha\) and \(\beta\) such that \(u = u_0 + \alpha u_1 + \beta u_2\) is continuous in \((-b, 0)\) and \((b, 0)\), i.e.

\[
\begin{align*}
u_0(b, 0^+) + \alpha u_1(b, 0^+) + \beta u_2(b, 0^+) &= u_0(b, 0^-) + \alpha u_1(b, 0^-) + \beta u_2(b, 0^-) \\
u_0(-b, 0^+) + \alpha u_1(-b, 0^+) + \beta u_2(-b, 0^+) &= u_0(-b, 0^-) + \alpha u_1(-b, 0^-) + \beta u_2(-b, 0^-)
\end{align*}
\]

Finally, we solve the following linear system.

\[
\begin{pmatrix}
u_1(b, 0^+) - u_1(b, 0^-) & u_2(b, 0^+) - u_2(b, 0^-) \\
u_1(-b, 0^+) - u_1(-b, 0^-) & u_2(-b, 0^+) - u_2(-b, 0^-)
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
= 
\begin{pmatrix}
u_0(b, 0^+) - u_0(b, 0^-) \\
u_0(-b, 0^+) - u_0(-b, 0^-)
\end{pmatrix}
\]

6.2.2. Test of Convergence of \(\lambda u_\lambda\) to a constant value. We have verified numerically that \(\lambda u_\lambda\) converges, to a constant value. Figure 6.1-6.2 corresponds to the following choice:

\[g(z, y) = \frac{1}{2\pi\sigma_z\sigma_y} e^{-\frac{1}{2} \frac{z^2}{\sigma_z^2}} e^{-\frac{1}{2} \frac{y^2}{\sigma_y^2}} \quad (6.2)\]

In figure 6.3-6.4, \(g\) is similar but non zero on \(D_{\pm b}\):

\[g(z, y) = \frac{1}{2\pi\sigma_z\sigma_y} e^{-\frac{1}{2} \frac{(z-z_i)^2}{\sigma_z^2}} e^{-\frac{1}{2} \frac{(y-y_i)^2}{\sigma_y^2}} \quad (6.3)\]

where \(\sigma_z = \sigma_y = \sqrt{1/200}\). Both plots show that indeed \(\lambda u_\lambda\) tends to a constant when \(\lambda\) tends to zero.

6.3. Computing on \(m\). Given a mesh of \(D\) generated with the software freem+++, we consider a family of gaussian function centered on each node \((z_i, y_i)\) of the mesh.

\[g_i(z, y) = \frac{1}{2\pi\sigma_z\sigma_y} e^{-\frac{1}{2} \frac{(z-z_i)^2}{\sigma_z^2}} e^{-\frac{1}{2} \frac{(y-y_i)^2}{\sigma_y^2}}\]

Then we solve the primal problem (6.1) by a finite element method of degree one, also using freem+++. Finally we solve the linear system for \(m\). The results are shown on figure 6.5,6.6 and the comparison with the Monte-Carlo method is given of Figure 6.7.
An Ultra Weak Finite Element Method as an Alternative to a Monte Carlo Method for an Elasto-Plastic Problem with Noise

7. Conclusion. In this work, we compared a deterministic method to a Monte Carlo computation. The Monte Carlo method is expensive because the stationary state of the process \((Z_t, Y_t)\) is of interest when \(t \to \infty\) for the invariant measure \(m\). This state is also characterized by a PDE. As we saw, classical methods, on this PDE do not work because \(m\) belong to \(L^2\) but not to \(H^1\). So, an Ultra Weak method has been proposed to compute \(m\). The main idea is to solve the dual problem of \(m\) on each function of a basis of \(L^2\). Comparing the result between the two methods, we found less than 3% of \(L^2\) relative difference. This deterministic method is also expensive but it is more precise than the Monte Carlo method at equal computing time. In the future, we shall extend this approach to multidimensional
Fig. 6.2. From the top to the bottom: $\lambda u_\lambda$ for $\lambda = 10^{-4}$ and $\lambda = 10^{-9}$, with $g(z,y) = \frac{1}{2\pi \sigma_x \sigma_y} e^{-\frac{1}{2} \frac{z^2}{\sigma_x^2}} e^{-\frac{1}{2} \frac{y^2}{\sigma_y^2}}$ and $\sigma_x = \sigma_y = \sqrt{1/200}$.

problems more relevant to application to earthquake engineering.
An Ultra Weak Finite Element Method as an Alternative to a Monte Carlo Method for an Elasto-Plastic Problem with Noise

![Graph](image1.png)

**Fig. 6.3.** From the top to the bottom: $\lambda u_\lambda$ for $\lambda = 1.0$ and $\lambda = 10^{-1}$, with $g(z,y) = \frac{1}{2\pi\sigma_z\sigma_y}e^{-\frac{1}{2}\left(\frac{z-b}{\sigma_z}\right)^2} - \frac{1}{2\pi\sigma_y}e^{-\frac{1}{2}\left(\frac{y}{\sigma_y}\right)^2}$ and $\sigma_z = \sigma_y = \sqrt{1/200}$.

REFERENCES

Fig. 6.4. From the top to the bottom: $\lambda u \lambda$ for $\lambda = 10^{-4}$ and $\lambda = 10^{-9}$, with $g(z,y) = \frac{1}{2 \pi \sigma_z \sigma_y} e^{-\frac{1}{2} \left(\frac{(z-b)^2}{\sigma_z^2} + \frac{y^2}{\sigma_y^2}\right)}$ and $\sigma_z = \sigma_y = \sqrt{1/200}$.


An Ultra Weak Finite Element Method as an Alternative to a Monte Carlo Method for an Elasto-Plastic Problem with Noise

<table>
<thead>
<tr>
<th>$b$</th>
<th>plastic part UW (N=80)</th>
<th>plastic part MC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>79.5 %</td>
<td>84.93 %</td>
</tr>
<tr>
<td>2.0</td>
<td>9.38 %</td>
<td>9.63 %</td>
</tr>
<tr>
<td>3.0</td>
<td>0.053 %</td>
<td>0.047 %</td>
</tr>
<tr>
<td>4.0</td>
<td>0.0 %</td>
<td>0.0 %</td>
</tr>
</tbody>
</table>

*Table 6.1*

*Evolution of plastic part versus $b$.  


Fig. 6.5. $m$ computed by Ultra Weak Method, from the top to the bottom we consider $b = 1, 2$. When $b$ increases the plastic part decreases
F\textsc{ig. 6.6.} $m$ computed by Ultra Weak Method, from the top to the bottom we consider $b = 3, 4$. When $b$ increases the plastic part decreases.
Fig. 6.7. Top: Relative error between the result given by the Monte-Carlo method and the results given by the Ultra Weak method, versus $\sqrt{N}$, where $N$ is the number of basis function. Bottom: The same plot in log-log scale.