VOLATILITY CALIBRATION WITH AMERICAN OPTIONS

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Abstract. In this article we present a new algorithm for the computation of American options and a least-square approach for the calibration of the volatility with market data for these. Optimal control methods are applied and differentiability of American option with respect to volatility is studied.

Strike-Sensitivity versus time and price of an American put option computed with the algorithm presented in the paper. The sensitivity is computed by automatic differentiation.
1 Introduction

Calibration is an important problem in finance. It is a way to extend the validity of the Black-Scholes model. A volatility surface is computed to fit the observed data and then used for new predictions.

Inverse problems on the coefficients of the PDEs are notoriously unstable, so a Tychonov regularization must be used. Then differentiability must be proved so as to use fast gradient algorithms.

For American option the differentiability is difficult to prove because of the free boundary. Below, Optimality conditions are given for the inverse problem.

Alternatively, he problem can be discretized by choosing a volatility surface which depends on a small number of parameters. In this case automatic differentiation can be used numerically (see Figure ??).

All the proofs of the results below can be found in [?], [?] and in the book [?].

2 Pricing American options

2.1 The variational inequality and the free boundary

Calling $t$ the time to maturity, the boundary value problem for pricing an option with payoff function $P_0$ reads

$$\begin{align*}
\frac{\partial P}{\partial t} - \frac{\eta(S,t)S^2}{2} \frac{\partial^2 P}{\partial S^2} - rS \frac{\partial P}{\partial S} + rP & \geq 0, \quad P \geq P_0 \\
\frac{\partial P}{\partial t} - \frac{\eta(S,t)S^2}{2} \frac{\partial^2 P}{\partial S^2} - rS \frac{\partial P}{\partial S} + rP)(P - P_0) & = 0,
\end{align*}$$

(1)

with Cauchy data

$$P|_{t=0} = P_0.$$  \hspace{1cm} (2)

We focus on the case of a vanilla put, i.e. the payoff function is $P_0(S) = (K - S)_+$, but, to a large extent, what follows holds for more general functions.

The squared local volatility $\eta(S,t)$ is the function that must be calibrated. The interest rate $r \geq 0$ is assumed to be known and constant.

To write the variational formulation of (1) (2), we need to use the Sobolev space

$$V = \{ v \in L^2(\mathbb{R}_+) : S \frac{dv}{dS} \in L^2(\mathbb{R}_+) \},$$

(3)

and we call $\mathcal{K}$ the subset of $V$:

$$\mathcal{K} = \{ v \in V, v \geq P_0 \text{ in } \mathbb{R}_+ \}.$$  \hspace{1cm} (4)
Since the function of \( V \) are continuous, the inequality in (4) has a pointwise meaning. The set \( \mathcal{K} \) is a closed and convex subset of \( V \), because convergence in \( V \) implies pointwise convergence. We introduce the bilinear form \( a_t \):

\[
a_t(v, w) = \int_{\mathbb{R}_+} S^2 \eta(S, t) \frac{\partial v}{\partial S} \frac{\partial w}{\partial S} dS + \int_{\mathbb{R}_+} \left( -r(t) + \eta(S, t) + \frac{S}{2} \frac{\partial \eta}{\partial S}(S, t) \right) S \frac{\partial v}{\partial S} w dS + r \int_{\mathbb{R}_+} vw dS
\]

We make the assumptions: there exist two positive constants, \( \eta \) and \( \bar{\eta} \) such that for all \( t \in [0, T] \) and all \( S \in \mathbb{R}_+ \),

\[
0 < \eta \leq \eta(S, t) \leq \bar{\eta}.
\]

There exists a positive constant \( C_\eta \) such that for all \( t \in [0, T] \) and all \( S \in \mathbb{R}_+ \),

\[
|S \frac{\partial \eta}{\partial S}(S, t)| \leq C_\eta.
\]

These imply that the bilinear form \( a_t \) is continuous on \( V \) uniformly in \( t \), and Gårding’s inequality: for a non negative constant \( \lambda \) depending only on \( \bar{\eta}, \eta \) and \( C_\eta \),

\[
a_t(v, v) \geq \frac{\bar{\eta}}{4} |v|^2 - \lambda \|v\|^2_{L^2(\mathbb{R}_+)} , \quad \forall v \in V.
\]

The weak form of (1) is to

find \( P \in C^0([0, T]; L^2(\mathbb{R}_+)) \cap L^2(0, T; \mathcal{K}) \) such that \( \frac{\partial P}{\partial t} \in L^2(0, T; V') \), satisfying

\[
P_{t=0} = P_0, \text{ and } \forall v \in \mathcal{K}, \quad \left( \frac{\partial P}{\partial t}(t), v - P(t) \right) + a_t(P(t), v - P(t)) \geq 0.
\]

**Theorem 1** With \( \eta \) satisfying assumptions (6) and (7), the problem (9) has a unique solution \( P \) which belongs to \( C^0([0, T] \times [0, +\infty)) \) with \( P(0, t) = K, \forall t \in [0, T] \), and is such that

\[
S \frac{\partial P}{\partial S}, \frac{\partial P}{\partial t} \in L^2(0, T; V), S \frac{\partial P}{\partial S} \in C^0([0, T]; L^2(\mathbb{R}_+)) \text{ and } \frac{\partial P}{\partial t} \in L^2(0, T; L^2(\mathbb{R}_+)).
\]

The function \( P \) is also greater than or equal to \( P_0 \), the price of the vanilla European put.

The quantities \( \|P\|_{L^2(0, T; V)}, \|P\|_{L^\infty(0, T; L^2(\mathbb{R}_+))}, \|S \frac{\partial P}{\partial t}\|_{L^2(0, T; V')}, \|\frac{\partial P}{\partial t}\|_{L^2(0, T; V')}, \|S \frac{\partial P}{\partial S}\|_{L^\infty(0, T; L^2(\mathbb{R}_+))}, \|\frac{\partial P}{\partial S}\|_{L^2(0, T; L^2(\mathbb{R}_+))} \) are bounded by constants depending only on \( K, \bar{\eta}, \eta \) and \( C_\eta \).

We have that

\[
-1 \leq \frac{\partial P}{\partial S} \leq 0, \quad \forall t \in (0, T], \text{ a.a. } S > 0.
\]
There exists a function $\gamma : (0, T] \rightarrow [0, K)$, such that $\forall t \in (0, T)$, $\{S \text{ s.t. } P(S, t) = P_0(S)\} = [0, \gamma(t)]$. The function $\gamma$ is upper semi-continuous, right continuous in $[0, T)$, and, for each $t \in (0, T)$, $\gamma$ has a left-limit at $t$.

Calling $\mu$ the function $\mu = \frac{\partial P}{\partial t} + A_t P$, where $A_t$ is the linear operator: $V \rightarrow V'$; for all $v, w \in V$, $A_t v = -\frac{\eta(S_t)S^2}{2} \frac{\partial^2 v}{\partial S^2} - rS \frac{\partial v}{\partial S} + rv$, we have

$$\mu = rK1_{\{P = P_0\}}. \quad (11)$$

In other words, a.e., one of the two conditions $P = P_0$ and $\mu = 0$ is not satisfied: there is strict complementarity in (1).

Finally, there exists $\gamma_0 > 0$ depending only on $\bar{\eta}$ and $K$ such that

$$\gamma(t) \geq \gamma_0, \quad \forall t \in [0, T]. \quad (12)$$

### 2.2 A finite element method

We localize the problem on $(0, \bar{S})$ as usual, so $V$ becomes

$$V = \{v \in L^2((0, \bar{S}); S\frac{\partial v}{\partial S} \in L^2((0, \bar{S}); v(\bar{S}) = 0)$$

(where $\bar{S}$ is large enough so that $P_0(\bar{S}) = 0$), and $\mathcal{K} = \{v \in V, v \geq P_0\}$. The variational inequality is (9) with new meanings for $V$, $\mathcal{K}$, and $a_t$.

Moreover, if $\gamma_0 \in (0, K)$ as in (12) is known, one can focus on the smaller interval $[\underline{S}, \bar{S}]$, with $0 \leq \underline{S} < \gamma_0$ and obtain the equivalent weak formulation:

find $P \in L^2((0, T, \mathcal{K}) \cap C^0([0, T]; L^2(\Omega))$, with $\frac{\partial P}{\partial t} \in L^2(0, T; V')$

such that $P(t = 0) = P_0$ and (9) for all $v \in \mathcal{K}$, with the new definition of the closed set $\mathcal{K}$:

$$\mathcal{K} = \{v \in V, v \geq P_0 \text{ in } (0, \bar{S}], P = P_0 \text{ in } (0, \underline{S}]\}. \quad (13)$$

We introduce a partition of the interval $[0, T]$ into subintervals $[t_{n-1}, t_n]$, $1 \leq n \leq N$, with $\Delta t_i = t_i - t_{i-1}$, $\Delta t = \max_i \Delta t_i$ and a partition of the interval $[0, \bar{S}]$ into subintervals $\omega_i = [S_{i-1}, S_i]$, $1 \leq i \leq N_h + 1$, such that $0 = S_0 < S_1 < \cdots < S_{N_h} < S_{N_h+1} = \bar{S}$. The size of the interval $\omega_i$ is called $h_i$ and we set $h = \max_{i=1,\ldots,N_h+1} h_i$. The mesh $\mathcal{T}_h$ of $[0, \bar{S}]$ is the set $\{\omega_1, \ldots, \omega_{N_h+1}\}$. In what follows, we will assume that both the strike $K$ and the real number $\underline{S}$ coincide with nodes of $\mathcal{T}_h$: there exist $\alpha < \kappa$, $0 \leq \alpha < \kappa < N_h + 1$ such that $S_\alpha = K$ and $S_\alpha - 1 = \underline{S}$. We define the discrete space $V_h$ by

$$V_h = \{v_h \in V, \forall \omega \in \mathcal{T}_h, v_h|_\omega \in \mathcal{P}_1(\omega)\}, \quad (14)$$
where $P_1(\omega)$ is the space of linear functions on $\omega$.

Since $K$ is a node of $T_h$, $P_0 \in V_h$, and since $S$ is also a node of $T_h$, we can define the closed subset $K_h$ of $V_h$ by

$$K_h = \{ v \in V_h, \text{ } v \geq P_0 \text{ in } [0, \bar{S}], \text{ } v = P_0 \text{ in } [0, S] \} = \{ v \in V_h, v(S_i) \geq P_0(S_i), i = 0, \ldots, N_h + 1, v(S_i) = P_0(S_i), i < \alpha \}. \quad (15)$$

The discrete problem arising from an implicit Euler scheme is:

$$\text{find } (P^n)_{0 \leq n \leq N} \in K_h \text{ satisfying}$$

$$P^0 = P_0, \quad (16)$$

and for all $n, 1 \leq n \leq N$,

$$\forall v \in K_h, \quad (P^n - P^{n-1}, v - P^n) + \Delta t a_n (P^n, v - P^n) \geq 0. \quad (17)$$

Consider $\lambda$ such that Gårding’s inequality (8) holds, and take $\Delta t < \frac{1}{\lambda}$, there exists a unique $P^n$ satisfying (17).

Let $(w^i)_{i=0, \ldots, N_h}$ be the nodal basis of $V_h$, and let $M$ and $A^m$ in $\mathbb{R}^{(N_h+1) \times (N_h+1)}$ be the mass and stiffness matrices defined by

$$M_{i,j} = (w^i, w^j), \quad A^m_{i,j} = a_m(w^j, w^i), \quad 0 \leq i, j \leq N_h.$$\]

Calling

$$U^n = (P^n(S_0), \ldots, P^n(S_{N_h}))^T \text{ and } U^0 = (P_0(S_0), \ldots, P_0(S_{N_h}))^T,$$

(17) is equivalent to

$$(M(U^n - U^{n-1}) + \Delta t a_n U^n)_i \geq 0, \quad \text{ for } i \geq \alpha,$$

$$U^n_i = U^0_i \quad \text{ for } i < \alpha,$$

$$U^n \geq U^0,$$

$$(U^n - U^0)^T (M(U^n - U^{n-1}) + \Delta t a_n U^n) = 0. \quad (18)$$

We call $M_{\alpha}$, respectively $A^m_{\alpha}$, the block of $M$, respectively $A^m$, corresponding to $\alpha \leq i, j \leq N_h$.

### 2.3 A penalized problem

We consider a smooth non increasing convex function $\mathcal{V}$ such that

$$\mathcal{V}(0) = 1$$

$$\mathcal{V}(y) = 0 \quad y \geq 1$$

$$0 \geq \mathcal{V}'(y) \geq -2 \quad 0 \leq y \leq 1. \quad (19)$$

and we call $\tilde{V}_h = \{ v_h \in V_h, \text{ } v_h(S_i) = 0 \ \forall i < \alpha \}$ and we define the discrete penalized problem:
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find \((P^n_\epsilon)_{0 \leq n \leq N}, P^n_\epsilon \in V_h\) satisfying

\[ P^0_\epsilon = P_0, \]

and for all \(n, 1 \leq n \leq N,\)

\[ P^n_\epsilon - P_\epsilon \in \tilde{V}_h, \quad \text{and for any } v \in \tilde{V}_h, \]

\[ 0 = (P^n_\epsilon - P^{n-1}_\epsilon, v) + \Delta t_n \left( a_{T-t_n}(P^n_\epsilon, v) - rK \sum_{\alpha \leq i \leq \kappa} \frac{|\tilde{\Omega}_i|}{2} V'_i(P^n_\epsilon(S_i) - P_\epsilon(S_i))v(S_i) \right), \]

where \(\tilde{\Omega}_i = \Omega_i \cap (0, K)\) and \(\Omega_i \subset (0, \bar{S})\) is the support of \(w^i\), so \(\frac{|\tilde{\Omega}_i|}{2} = \int_0^K w^i.\)

To see that the sequence \((P^n_\epsilon)_{0 \leq n \leq N}\) converges to \((P^n_\epsilon)_{0 \leq n \leq N}\) as \(\epsilon \to 0\), the main point consists in proving that the limit belongs to \((K)^{N+1}.\) For that, we need to use a discrete maximum principle. This amounts to looking for monotonicity properties of the matrix \(M + \Delta t_n A^n.\) The matrix \(M + \Delta t_n A^n\) cannot be a M-matrix, since the diffusion coefficient \(\eta S^2\) vanishes at \(S = 0\). However, it is very reasonable to assume that the block of \(M + \Delta t_n A^n\) associated to the nodes \(S_i\) greater than a given value is a M-matrix. For a matrix \(A\), we call \(A_\ell\) the block of \(A\) corresponding to row and column indices greater than or equal to \(\ell.\)

**Proposition 1** Let \(\eta\) verify (6) and (7), and choose \(\Delta t < \frac{1}{2M}\), with \(\lambda\) given in (8). We assume that \(S > 0\), i.e. \(\alpha > 0\), and that the parameters \(h\) and \(\frac{h^2}{\min_n \Delta t_n}\) are small enough so that the matrices \(A^n_\ell\) and \(M_\ell + \Delta t_n A^n_\ell\) are tridiagonal irreducible M-matrices for all \(n, 1 \leq n \leq N\) and \(l, \alpha \leq l < N_h\). The sequence \((P^n_\epsilon)_{0 \leq n \leq N}\) given by (20) (21) converges to \((P^n_\epsilon)_{0 \leq n \leq N}\) given by (16) (17) in \((V_h)^{N+1}\) as \(\epsilon\) goes to 0.

### 2.4 The discrete exercise boundary

One may ask if there is a well defined exercise boundary \(t \to \gamma_h(t)\) also in the discrete problem. A positive answer has been given by Jaillet et al [?] in the case of a constant volatility, an implicit Euler scheme and a uniform mesh in the logarithmic variable. The main argument of the proof lies in the fact that the solution to the discrete problem is non decreasing with respect to the variable \(t.\) With a local volatility, this may not hold (see the numerical example below). The result of Jaillet et al has been completed for a local volatility in [?], in the special case when the mesh is uniform in the variable \(S:\) here too, the discrete problem has a free boundary. The proof does not rely any longer on the monotonic character of the discrete solution with respect to \(t\) but on the discrete analogue of the bounds (10), i.e. \(-1 \leq \frac{\partial P}{\partial S} \leq 0.\) This is proved by studying the penalized problem (21) and by using a discrete maximum principle on the partial derivative with respect to \(S\) (for this reason, a uniform mesh is needed). We can summarize this by
Theorem 2 Assume that the grid $T_h$ is uniform and that $S > 0$. Assume also that the parameters $h$ and $\frac{h^2}{\Delta t}$ are small enough so that the matrices $A^n$ and $M + \Delta t A^n$ are tridiagonal irreducible $M$-matrices for all $n$, $1 \leq n \leq N$.

There exist $N$ real numbers $\gamma^n_h$, $1 \leq n \leq N$, such that

\[
\begin{align*}
S &\leq \gamma^n_h < K, \\
\gamma^n_h &\text{ is a node of } T_h, \\
\forall i, 0 \leq i \leq N_h, \quad P^n(S_i) = P_o(x_i) \Leftrightarrow S_i \leq \gamma^n_h.
\end{align*}
\]

(22)

We believe that this may be extended to somewhat more general meshes.

2.5 A front-tracking algorithm

Here, we propose an algorithm for computing the solution of (17) assuming that the free boundary is the graph of a function. In our experience, this algorithm, based on tracking the free boundary, is more robust (and slightly more expensive) than the Brennan and Schwartz algorithm (see [?]). Since the free boundary is the graph of a function, the idea is to look for $\gamma^n_h$ by

- Start from $\gamma^n_h = \gamma^{n-1}_h$,
- solve the discrete problem corresponding to

\[
\frac{P^n - P^{n-1}}{\Delta t} - \frac{\eta(S, t) S^2}{2} \frac{\partial^2 P^n}{\partial S^2} - r S \frac{\partial P^n}{\partial S} + r P^n = 0 \text{ for } \gamma^n_h < S < \bar{S},
\]

\[
P^n = P_o \text{ for } 0 \leq S \leq \gamma^n_h,
\]

and $P^n(\bar{S}) = 0$,

- if $P^n$ satisfies (17), stop else shift the point $\gamma^n_h$ to the next node on the mesh left/right according to which constraint is violated by $P^n$.

With the notations introduced above, the algorithm for computing $P^n_h$ is as follows:

**Algorithm**

choose $k$ such that $\gamma^{n-1}_h = S_k$; set found=false;

while(not found) 
  .. solve
    \[
    (M(U^n - U^{n-1}) + \Delta t A^n U^n)_i = 0, \quad \text{for } i \geq k, \\
    U^n_i = U^0_i, \quad \text{for } i < k.
    \]
    (23)
  .. if $((U^n - U^0)_{k+1} < 0 )$
    .. found=false; $k = k + 1$
  .. else
.. compute $a = (M(U^n - U^{n-1}) + \Delta t_n A^n U^{n-1})_{k-1}$;
.. if $(a < 0)$
.. found = false; $k = k - 1$;
.. else found = true

In our tests, we have computed the average (over the time steps) number of iterations to obtain the position of the free boundary: it was found that (with a rather fine time-mesh), this number is smaller than 2.

3 Calibration with American options: A least square problem

The calibration problem consists in finding $\eta$ from the observations of

- the spot price $S_0$ today,
- the prices $(\bar{P}_i)_{i \in I}$ of a family of American puts with different maturities and different strikes $(T_i, K_i)_{i \in I}$.

We call $T = \max_{i \in I} T_i$. We consider the least square problem: find $\eta \in \mathcal{H}$ minimizing

$$J(\eta) + J_R(\eta) = \sum_{i \in I} |P_i(S_0, T_i) - \bar{P}_i|^2,$$

where $\mathcal{H}$ is a suitable closed subset of a possibly infinite dimensional function space, $J_R$ is a suitable Tychonoff regularization functional, and

\begin{align*}
\text{find } (P^n_i)_{0 \leq n \leq N_i}, & \ P^n_i \in \mathcal{K}_{h,i} \text{ satisfying} \\
& P^0_i = P_{0,i},
\end{align*}

and for all $n, 1 \leq n \leq N_i$,

$$\forall v \in \mathcal{K}_{h,i}, \ (P^n_i - P^{n-1}_i, v - P^n_i) + \Delta t_n a_{T_i - t_n} (P^n_i, v - P^n_i) \geq 0,$$

We call $\mu^n_{i,j}$ the real number

$$\mu^n_{i,j} = (P^n_i - P^{n-1}_i, w^j - P^n_i) + \Delta t_n a_{T_i - t_n} (P^n_i, w^j - P^n_i).$$

3.1 Optimality conditions

In [?] the inverse problem corresponding to the continuous counterpart of (26) is studied and optimality conditions are given for suitable choices of $\mathcal{H}$ and $J_R$.

In order to find optimality conditions for the present least-square problem, we replace the state equations (25) (26) by the penalized problem (20) (21). Doing so, we obtain a new least square problem, for which necessary optimality conditions are easily found. Then, we pass to the limit as $\epsilon$ goes to zero: we obtain the following result:
Theorem 3 Let η* be a minimizer of (24) which can be found as a limit of a sequence ηε of minimizers for the penalized problem, and let (P_i^n)_{i∈I} be the solutions to (25) (26) with \( \eta = \eta^* \). There exist \( y_i^{*,n} \in \tilde{V}_h \), and \( \alpha^n_{i,j} \in \mathbb{R} \), 1 ≤ n ≤ N_i, \( \rho \leq j \leq N_h \), \( i \in I \), such that
\[ (y_i^{*,N_i}, v) + \Delta t_{N_i} \left( a^*_0(v, y_i^{*,N_i}) + \sum_{j=\rho}^{N_h} \alpha^N_{i,j} v(S_j) \right) = 2(P_i^{*,N_i}(S_0) - \bar{P}_i) v(S_0), \]

\[ (y_i^{*,n} - y_i^{*,n+1}, v) + \Delta t_n \left( a^*_{T_i-t_n}(v, y_i^{*,n}) + \sum_{j=\rho}^{N_h} \alpha^n_{i,j} v(S_j) \right) = 0, \quad 1 \leq n < N, \]

with, for all \( j, n, \rho \leq j \leq N_h, \quad 1 \leq n \leq N_i, \)
\[ \alpha^n_{i,j}(P_i^{*,n}(S_j) - P_0(S_j)) = 0, \quad \mu^n_{i,j} y_i^{*,n}(S_j) = 0, \quad \alpha^n_i y_i^{*,n}(S_j) \geq 0, \]
such that for any \( \eta \in \mathcal{H} \), noting by \( \delta \eta = \eta - \eta^* \),
\[ 0 \leq < DJ_R(\eta^*), \delta \eta > \]
\[ -\frac{1}{2} \sum_{i=1}^{N_i} \sum_{n=1}^{N_h} \Delta t_n \sum_{j=\rho}^{N_h} S^2_j \delta \eta(S_j, T_i - t_n) y_i^{*,n}(S_j) \left( \frac{P_i^{*,n}(S_j) - P_i^{*,n}(S_{j-1})}{h_j \bar{P}_i} + \frac{P_i^{*,n}(S_j) - P_i^{*,n}(S_{j+1})}{h_{j+1}} \right). \]

3.2 Differentiability

Proposition 2 Let the assumptions of Proposition 1 be satisfied for all \( \eta \in \mathcal{H} \) verifying (6) (7). Let \( \eta \in \mathcal{H} \) be such that the strict complementarity conditions
\[ P^n_i(S_j) > P_{o,i}(S_j) \iff \mu^n_{i,j} = 0, \quad \text{(29)} \]
for all \( i \in I \) and for all \( j, \rho \leq j \leq N_h \), where \( P^n_i \) is the solution to (3) (4), and \( \mu^n_{i,j} = (P^n_i - P_i^{n-1}, w^j) + \Delta t_n a_{T_i-t_n}(P^n_i, w^j). \) The functional \( J \) is differentiable at \( \eta \), and for any admissible variation \( \chi \) of \( \eta \),
\[ < DJ(\eta), \chi > = \]
\[ -\frac{1}{2} \sum_{i=1}^{N_i} \sum_{n=1}^{N_h} \Delta t_n \sum_{j=\rho}^{N_h} S^2_j \chi(S_j, T_i - t_n) y^n_i(S_j) \left( \frac{P^n_i(S_j) - P^n_i(S_{j-1})}{h_j \bar{P}_i} + \frac{P^n_i(S_j) - P^n_i(S_{j+1})}{h_{j+1}} \right). \]

where \( y^n_i = y^n(\eta_i) \in \tilde{V}_h, \alpha^n_{i,j} \in \mathbb{R}, \rho \leq j \leq N_h, \) are the solution to: \( \forall v \in \tilde{V}_h, \)
\[ (y_i^{N_i}, v) + \Delta t_{N_i} \left( a_0(v, y_i^{N_i}) + \sum_{j=\rho}^{N_h} \alpha^N_{i,j} v(S_j) \right) = 2(P_i^{N_i}(S_0) - \bar{P}_i) v(S_0), \]

\[ (y_i^n - y_i^{n+1}, v) + \Delta t_n \left( a_{T_i-t_n}(v, y_i^n) + \sum_{j=\rho}^{N_h} \alpha^n_{i,j} v(S_j) \right) = 0, \quad 1 \leq n < N, \]
with
\[ \alpha_{i,j}^n (P_i^n(S_j) - P_o(S_j)) = 0, \quad \mu_{i,j}^n y_{i}^n(S_j) = 0, \quad \alpha_{i}^n y_{i}^n(S_j) \geq 0. \]

4 Algorithm

We describe the simplest possible projected descent method in the space \( Y \), where the descent direction is computed thanks to the considerations above. The degrees of freedom of a function \( \chi \in Y \) are the values of \( \chi \) at some nodes of a grid and we call them \( (\Lambda^* \ell(\chi))_{1 \leq \ell \leq L} \). We endow \( Y \) with the basis \( (\Lambda \ell(\chi))_{1 \leq \ell \leq L} \) defined by \( \Lambda^* \ell(\Lambda_k) = \delta_{\ell k}, \) and we define the inner product \( (\sum_{\ell=1}^{L} a_\ell \Lambda_\ell, \sum_{\ell=1}^{L} b_\ell \Lambda_\ell)_Y = \sum_{\ell=1}^{L} a_\ell b_\ell. \)

Algorithm

- Choose \( \eta \in H, \epsilon > 0 \) and \( \rho > 0 \), set \( e = +\infty \).
- While \( e > \epsilon \) do
  1. Compute \( (P_i)_{i \in I} \) by (25) (26), by using for example one of the algorithms proposed in §2.5 and \( J(\eta) + J_R(\eta), J(\eta) = \sum_{i \in I} |P_i^N(S_o) - P_i|^2; \)
  2. For all \( i \in I \), compute \( (y_{i}^n)_{1 \leq n \leq N_i}, y_{i}^n \in \tilde{V}_h \) satisfying (28).
  3. compute \( \zeta \in Y \) such that for all \( \chi \in Y, \)
     \[ (\zeta, \chi)_Y = -\frac{1}{2} \sum_{i \in I} \sum_{n=1}^{N_i} \Delta t_n \sum_{j=\rho}^{N_h} S_j^2 \chi(S_j, T_i - t_n) y_{i}^n(S_j) \left( \frac{u_{i}^n(S_j) - u_{i}^n(S_{j-1})}{h_j} + \frac{u_{i}^n(S_j) - u_{i}^n(S_{j+1})}{h_{j+1}} \right). \] (31)
  4. set \( \tilde{\eta} = \pi_H(\eta - \rho(\nabla J_R(\eta) + \zeta)), e = ||\tilde{\eta} - \eta||, \eta = \tilde{\eta}, \) where \( \pi_H \) is the projection on \( H \).
- end do

The complete justification of the algorithm above is still an open question because it is not proved that \( -\nabla J_R(\eta) - \zeta \) is always a descent direction. However, from Proposition 2, we know that most often, \( \zeta \) is exactly \( \nabla J(\eta) \); in this case, the algorithm coincides with a projected gradient method.

In the numerical tests below, we have used variants of this algorithm (an interior point algorithm due to J. Herskovits[?]-it is a quasi-Newton algorithm which can handle general constraints), which have proved very robust. In particular, we never experienced breakups caused by the fact that the direction \( \zeta \) is not a descent direction.
Parallelism The algorithm above can be parallelized in a very natural way on a distributed memory machine with \(N_p\) processors, because the computations of the pairs \((P_i, y_i), i \in I\) are independent from each other. We split \(I\) in \(I = \bigcup_{k=1}^{N_p} I_k\) in order to balance the amount of work among the processors, the processor labelled \(k\) being responsible for the sums over \(i \in I_k\) in \(J(\eta)\) and (31). Note that the complexity of the computation of \(P_i, y_i\) depends on \(i\), so load balancing is not straightforward. The data for \(\eta\) and \(\zeta\) are replicated on the \(N_p\) processors. The processor labelled \(k\) computes its own contribution to \(J(\eta)\) and to (31), i.e. the sums over \(i \in I_k\), in an independent manner, then communications are needed for assembling the sums over \(i \in I\) in \(J(\eta)\) and in (31).
For programming, we have used C++ with the message passing library mpi.

5 Results with American Puts on the Footsie Index

In this paragraph, we consider American puts on the footsie index. The data correspond to June 6, 2001. We thank José Da Fonseca for providing us with the data. The price of the underlying asset is \(x_o = 5890\). The American puts correspond to four different maturities: 0.122, 0.199, 0.295, 0.55 years. We set \(T = 0.55\). The interest rate \(r\) varies with time, so \(r\) is replaced by \(r(t)\) in (26), and this function is known. For these maturities, the prices of the observed options vs strike are plotted on Figure 1. The aim is to find the volatility surface from these prices. The volatility is discretized by functions that are the sum of

- a piecewise affine function in the \(S\)-variable which is constant in the regions \(S < 1000\) and \(S > 9000\) and affine in the region \(1000 < S < 9000\)

![Figure 1: The data for the inverse problem: the prices of a family of American puts on the footsie index](image-url)
• a bicubic spline in the region $1000 < S < 9000$, $|t - T/2| < T/2 + 0.1$, whose value and derivatives vanish on the boundary of this rectangle. The control points of the spline are plotted on Figure 2, where the time variable is $T - t$. We see that the control points are not uniformly distributed: the mesh is refined for small times $t$ and at the money.

The grid for $u$ is non uniform with 745 nodes in the $S$-direction and 210 nodes in the $t$ direction. For simplicity, the grid is chosen in such a way that the points $(T_i, K_i)_{i \in I}$ coincide with some grid nodes.

The (squared) volatility obtained at convergence is displayed on Figure 3: the surface has a smile shape. The relative errors between the observed prices and those computed at convergence are plotted on Figure 4, top. They are rather large for small values of $K$. However, we have to realize that the available observed prices are themselves given with a round-off error, which is exactly 0.5. On Figure 4, bottom, we have plotted the relative round-off error on the observed prices. Doing so, we see that the relative errors on the prices at convergence are of the same order as the round-off error on the observed prices. Therefore, it is very natural that the optimization programm cannot improve on this level of error.

Figure 2: The control points of the bicubic splines
Figure 3: The squared volatility surface obtained by running the calibration program
Figure 4: Top: relative errors between the observed prices and those obtained with $\eta$ found after running the calibration program. A curve corresponds to a given maturity. Bottom: relative round-off error on observed prices. The two errors are of the same order.