Introduction to structured equations in biology

Benoît Perthame
email: benoit.perthame@sorbonne-universite.fr
Sorbonne Université
Laboratoire Jacques-Louis Lions
Paris, France

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Chapter 1

What are structured equations? Epidemic spread

Historically, partial differential equations have been introduced to represent the dynamics in space and time of quantities as densities, velocities, electro-magnetic fields... L. Boltzmann goes one step further in 1872 and represents the density of molecules by their position in the *phase space*: position and velocity.

In 1927, W. Kermack and A. McKendrick proposed a partial differential equation to take into account the age-of-infection in the transmission rate of an epidemic. This is considered as the first example of *structured equation*.

The context of epidemic spread is well adapted to present examples of structured equations in biology and medicine.

1.1 The standard SIR system

To describe the initial spread of an epidemic, the simplest (and certainly most effective) class of models is called *compartmental models*. These are Ordinary Differential Equations representing the dynamics of classes of individuals. Among them the SIR system is the simplest and describes, the proportions of

- $S(t)$ = Susceptible individuals (not immune),
- $I(t)$ = Infectiuous individuals (who propagate the epidemic),
- $R(t)$ = Removed individuals (who do not propagate: immune, deceased,...).

The SIR system reads

$$\begin{cases} \dot{S}(t) = -\beta S I, \\ \dot{I}(t) = \beta S I - \gamma I, \\ \dot{R}(t) = \gamma I. \end{cases} \tag{1.1}$$

The initial states are denoted by $S(0) = S^{init} > 0$, $I(0) = I^{init} > 0$ and $R(0) = R^{init} \geq 0$. One readily checks that $S(t)+I(t)+R(t) = N$ is a constant, $N = 1$ when one considers proportions.

This sytem has been popularized even in newspapers with the COVID-19 epidemics and the (no dimension) *basic reproduction number*

$$\mathcal{R}_0(t) = S(t)\frac{\beta}{\gamma}$$

is a good expression to evaluate *secondary infections*, i.e., the number of individuals which will be infected by a typical infected individual. And $1/\gamma$ is a good evaluation of the time it takes to infect these $\mathcal{R}$ individuals (a few days for COVID-19).

Exercise 1.1 *One can check the following facts:*

- *Assuming* $S(0) > 0$, $I(0) > 0$, *then* $S(t) > 0$, $I(t) > 0$
- $S(t)$ *decreases to a limit* S_∞,
- *When* $S(t) < S_{herd} := \frac{S(0)}{\mathcal{R}_0(0)}$, *then* $\dot{I}(t) < 0$ *and the epidemic stops, (the proportion of infected is thus* $1 - \frac{1}{\mathcal{R}_0(0)}$*)*
- $\ln S(t) - \frac{\beta}{\gamma}[S(t) + I(t)] =: A$ *is a constant,*
- S_∞ *is the smallest of the two roots of the equation* $\ln S - \frac{\beta}{\gamma}S = A$*. Draw this curve and explain the trajectory of* $S(t)$ *depending on* $\mathcal{R}_0(0)$*.*

The quantity S_{herd} is called *herd immunity* and has been also popularized with the recent pandemic.

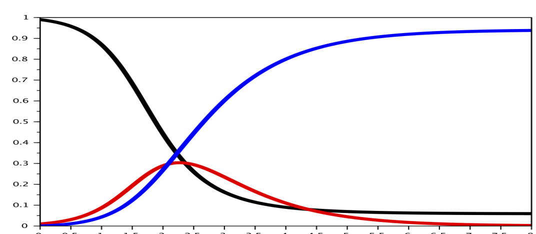

Figure 1.1: SOLUTIONS OF SIR SYSTEM ($\tau_I = 6$ DAYS, $R_0 = 3$. BLACK= SUSCEPTIBLES, RED= INFECTANTS, BLUE= REMOVED. TIME SCALE = 1 WEEK.

Exercise 1.2 *Consider the SIR system with birth*

$$\begin{cases} \frac{d}{dt}S(t) = B - \mu_S S(t) - \beta I(t)S(t), \\ \frac{d}{dt}I(t) = \beta I(t)S(t) - \gamma I(t), \end{cases}$$

with $B > \frac{\mu_S \gamma}{\beta}$ *Show that*

$$\frac{d}{dt}[\bar{S}\ln S(t) - S(t) + \bar{I}\ln I(t) - I(t) = -\frac{\mu_S + \beta\bar{I}}{S(t)}(\bar{S} - S(t))^2.$$

See also Proposition 1.1.

1.2 Heterogeneous population (contact matrices)

The SIR compartmental model is not sufficient because it does not take into account the heterogeneity of the population in terms, for instance, of social contacts or of susceptibility to infection represented by the parameter β in the SIR model (1.1). Many authors have proposed to take into account this feature using a contact matrix $\beta(x,y)$ describing the encounter rate between such classes.

One can consider, for instance, that children have many contacts between them at school and with their parents, adults have many contacts through public transport and work... Then, the number of contacts may be well represented by age groups x for instance. In general, one can introduce a contact matrix $\beta(x,y)$, with x, $y \in \Omega$ some bounded measurable set of parameters, and write

$$\begin{cases} \dot{S}(t,x) = -S(t,x)\int_\Omega \beta(x,y) I(t,y)\, dy, \\ \dot{I}(t,x) = S(t,x)\int_\Omega \beta(x,y) I(t,y)\, dy - \gamma(x) I(t,x), \\ \dot{R}(t,x) = \gamma(x) I(t,x). \end{cases} \tag{1.2}$$

It is convenient to assume that, a.e.,

$$S^{init}(x) > 0, \qquad I^{init}(x) > 0, \qquad \beta(x,y) > 0, \qquad \inf_{x\in\Omega} \gamma(x) > 0,$$

$$\int_\Omega \big[S^{init}(x) + I^{init}(x)\big] dx = 1, \qquad \gamma \in L^\infty(\Omega), \qquad \beta \in L^\infty(\Omega\times\Omega).$$

Existence of solutions still follows from the Cauchy-Lipschitz theorem, with its version in Banach spaces, here $L^1(\Omega)\times L^1(\Omega)$ for (S,I).

For S such that $0 < \inf_{x\in\Omega} S(x)$ and $\sup_{x\in\Omega} S(x) < \infty$, the operator

$$K_S[u](x) := S(x)\int_\Omega \frac{\beta(x,y)}{\gamma(y)} u(y) dy,$$

is such that

$$K_S : L^2(\Omega) \to L^2(\Omega), \qquad \text{for } u \geq 0,\ u \not\equiv 0, \quad \text{then } K_S[u] > 0.$$

The Krein-Rutman theorem, see section 8.4, asserts that there is a simple and positive minimal eigenvalue that we denote by $\mathcal{R}_S > 0$, which represents the basic reproduction number in this framework, associated with a positive eigenfunctions $\varphi_S > 0$,

$$K_S[\varphi_S] = \mathcal{R}_S\, \varphi_S.$$

For a *rank-one matrix*, that is $\beta(x,y) = \beta(x)\sigma(y)$, one can immediately check that

$$\varphi_S(x) = \beta(x) S(x), \qquad \mathcal{R}_S = \int_\Omega \frac{\beta(y)\sigma(y)}{\gamma(y)} S(y) dy.$$

Exercise 1.3 *1. Show that the adjoint operator is defined as*

$$K^*_S[v](y) = \int_\Omega S(x)\frac{\beta(x,y)}{\gamma(y)}v(x)dx.$$

2. For a rank one matrix, compute the dual eigenfunction, defined as $K^*_S[\psi_S] = \mathcal{R}_S\, \psi_S$.

Exercise 1.4 *1. Prove that the following quantity is constant in* t

$$A(x) := \ln S(t,x) - \int_\Omega \frac{\beta(x,y)}{\gamma(y)}\big[S(t,y)+I(t,y)\big]dy.$$

2. Prove that $S(t,x)$ *decreases to a limit* $S_\infty(x)>0$, $S(t,x)+I(t,x)$ *decreases, and* $I(t,x)\to 0$ *as* $t\to\infty$.
3. Prove that $S_1 \le S_2$ *implies* $\mathcal{R}_{S_1} \le \mathcal{R}_{S_2}$ *and deduce that, for solutions of (1.2),* $\mathcal{R}_{S(t)}$ *decreases. We assume it is continuous in* t.
4. Show that [see Exercise 1.3]

$$\frac{d}{dt}\int_\Omega \psi_{S_\infty}(y)I(t,y)dx \ge [\mathcal{R}_{S_\infty}-1]\int_\Omega \psi_{S_\infty}(y)\gamma(y)I(t,y)dx.$$

5. Conclude that $\mathcal{R}_{S(t)} < 1$ *as* $t\to\infty$.

1.3 The Kermack-McKendrick renewal equation

We conclude with an important class of Partial Differential Equations which prototype is the renewal equation. This class of equations is studied in a separate chapter, see Ch. 2. We mention here how it appears in the context of epidemic spread based on the idea from the seminal papers [12, 15].

The modeling assumption here is that the infectivity power of ill individuals depends on the time after infection. Therefore the model shoud keep this information. To do so the proportion $n_I(t,s)$ of infected population is structured by the time s after infection. We arrive at

$$\begin{cases} \frac{d}{dt}S(t) = B - \mu_S S(t) - I_{\text{tot}}(t)S(t), \\ I_{\text{tot}}(t) := \int_0^\infty \beta(s)n_I(t,s)ds, \\ \frac{\partial}{\partial t}n_I(t,s) + \frac{\partial}{\partial s}n_I(t,s) + (\mu_I+\gamma(s))n_I(t,s) = 0, \\ n_I(t,s=0) = I_{\text{tot}}(t)S(t). \end{cases}$$

This a nonlinear equation because of the quadratic terms $F(t)\,S(t)$. However, the linear equation is already interesting, in particular because of the nonlocal boundary condition for $n_I(t,s=0)$.

The endemic steady state. It can be computed by solving

$$\begin{cases} 0 = B - \mu_S \bar{S} - \bar{I}_{\rm tot}\bar{S}, & \bar{I}_{\rm tot} = \int_0^\infty \beta(s)\bar{n}_I(s)ds, \\ \frac{\partial}{\partial s}\bar{n}_I(s) + (\mu_I + \gamma(s))\bar{n}_I(s) = 0, & \bar{n}_I(s=0) = \bar{I}_{\rm tot}\bar{S}. \end{cases}$$

We first observe that

$$\bar{n}_I(s) = \bar{I}_{\rm tot}\,\bar{S}e^{-\mu_I s + \Gamma(s)}, \qquad \Gamma(s) = \int_0^s \gamma(\sigma)d\sigma,$$

and then, the solution $(\bar{S}, \bar{I}_{\rm tot})$ is determined by the two equations

$$B = (\mu_S + \bar{I}_{\rm tot})\bar{S}, \qquad 1 = \bar{S}\int_0^\infty \beta(s)e^{-\mu_I s + \Gamma(s)}ds. \tag{1.3}$$

Since the existence of such an endemic state requires that $\bar{I}_{\rm tot} > 0$, we impose that

$$B > \mu_S \left(\int_0^\infty \beta(s)e^{-\mu_I s + \Gamma(s)}ds\right)^{-1}. \tag{1.4}$$

The dual equation. It is useful to introduce the dual equation (with the normalisation $\psi(0) = 1$)

$$-\frac{\partial}{\partial s}\psi(s) + (\mu_I + \gamma(s))\psi(s) = \bar{S}\beta(s),$$

which we prefer to write as

$$\frac{\partial}{\partial s}[\psi(s)\bar{n}_I(s)] = -\bar{S}\beta(s)\bar{n}_I(s), \qquad \psi(s)\bar{n}_I(s) = \bar{S}\int_s^\infty \beta(\sigma)\bar{n}_I(\sigma)d\sigma.$$

At this stage, the reader should ask: why this complicated dual construction?

The generalized relative entropy. In fact, this is the structure of generalized relative entropy, that arises in all linear positivity preserving equations, see sections §2.5, §3.5 and chapter 4. In the present case, it gives the Lyapunov functional

Proposition 1.1 *With the above constructions and assumption (1.4), we define*

$$\mathcal{E}(t) = \int_0^\infty \psi(s)\bar{n}_I(s)\big[\ln\frac{n_I(t,s)}{\bar{n}_I(s)} - \frac{n_I(t,s)}{\bar{n}_I(s)}\big]ds + \bar{S}\ln S(t) - S(t).$$

Then, we have

$$\frac{d}{dt}\mathcal{E}(t) \geq: D(t) \geq 0, \qquad D(t) = \frac{\mu_S}{S(t)}(\bar{S} - S(t))^2.$$

Notice that $u \mapsto \ln u - u + 1$ is a concave and non-positive function which vanishes at 1 as its derivative.

Proof. [Read it after you have understood Chapter 2]. We compute

$$\frac{\partial}{\partial t}\frac{n_I(t,s)}{\bar n_I(s)} + \frac{\partial}{\partial a}\frac{n_I(t,s)}{\bar n_I(s)} = 0,$$

$$\frac{\partial}{\partial t}\ln\left(\frac{n_I(t,s)}{\bar n_I(s)}\right) + \frac{\partial}{\partial a}\ln\left(\frac{n_I(t,s)}{\bar n_I(s)}\right) = 0,$$

$$\frac{\partial}{\partial t}\psi(s)\bar n_I(s)\ln\left(\frac{n_I(t,s)}{\bar n_I(s)}\right) + \frac{\partial}{\partial a}\psi(s)\bar n_I(s)\ln\left(\frac{n_I(t,s)}{\bar n_I(s)}\right) = -\bar S\beta(s)\bar n_I(s)\ln\left(\frac{n_I(t,s)}{\bar n_I(s)}\right),$$

$$\frac{d}{dt}\int_0^\infty \psi(s)\bar n_I(s)\ln\left(\frac{n_I(t,s)}{\bar n_I(s)}\right)ds = -\bar S\int_0^\infty \beta(s)\bar n_I(s)\ln\left(\frac{n_I(t,s)}{\bar n_I(s)}\right) + \bar n_I(0)\ln\left(\frac{n_I(t,0)}{\bar n_I(0)}\right).$$

Next, we use the Jensen inequality and find

$$\begin{aligned}
\bar n_I(0)\ln\left(\tfrac{n_I(t,0)}{\bar n_I(0)}\right) &= \bar I_{\text{tot}}\bar S\ln\left(\frac{S(t)\int_0^\infty\beta(s)\bar n_I(s)\frac{n_I(t,s)}{\bar n_I(s)}ds}{\bar S\int_0^\infty\beta(s)\bar n_I(s)ds}\right)\\
&\geq \bar I_{\text{tot}}\bar S\left[\ln\frac{S(t)}{\bar S} + \frac{\int_0^\infty\beta(s)\bar n_I(s)\ln\frac{n_I(t,s)}{\bar n_I(s)}ds}{\int_0^\infty\beta(s)\bar n_I(s)ds}\right]\\
&= \bar I_{\text{tot}}\bar S\left[\ln\frac{S(t)}{\bar S} + \frac{\int_0^\infty\beta(s)\bar n_I(s)\ln\frac{n_I(t,s)}{\bar n_I(s)}ds}{\bar I_{\text{tot}}}\right].
\end{aligned}$$

Finally, we obtain, since $\ln x \leq x-1$, with $x = \frac{\bar S}{S(t)}$

$$\frac{d}{dt}\int_0^\infty \psi(s)\bar n_I(s)\ln\left(\frac{n_I(t,s)}{\bar n_I(s)}\right)ds \geq \bar I_{\text{tot}}\bar S\ln\frac{S(t)}{\bar S} \geq \bar I_{\text{tot}}\bar S(1-\frac{\bar S}{S(t)})$$

We compute next

$$\frac{\partial}{\partial t}[\psi(s)n_I(t,s)] + \frac{\partial}{\partial a}[\psi(s)n_I(t,s)] = -\bar S\beta(s)n_I(t,s),$$

$$\frac{d}{dt}\int_0^\infty \psi(s)n_I(t,s)ds- = -I_{\text{tot}}(t)\bar S + I_{\text{tot}}(t)S(t).$$

And finally, we compute, using $B = (\mu_S + \bar I_{\text{tot}})\bar S$,

$$\frac{d}{dt}\mathcal{E}(t) \geq \bar I_{\text{tot}}\bar S(1-\frac{\bar S}{S(t)}) + I_{\text{tot}}(t)\bar S - I_{\text{tot}}(t)S(t) + \big(\frac{\bar S}{S(t)}-1\big)[(\bar S - S(t))\mu_S + \bar S\bar I_{\text{tot}} - S(t)I_{\text{tot}}(t)],$$

which gives the expression of $D(t)$ in Proposition 1.1. □

1.4 Selection, adaptation, evolution

So far, we have considered that the structuring variable was either a fixed parameter as in heterogeneous populations, or a physiological variable changing with the individual's evolution along his life (age in the disease). Another type of dynamics, on a longer time scale, is evolution of species by selection of a fitter phenotypic or genetic trait. This type of question occurs under the name Eco-Evo in ecology, in particular with the adaptation of species to global change.

In Darwinian evolution, the trait x can represent a variety of situations, size of the adults, proportion of certain type of food, level of expression of a given gene in cancer cells, level of resistance to a treatment...

A generic model is to consider a population $n(t,x)$ which reproduces with birth rate $b(x) \geq 0$, with a mutation probability $M(y,x)$ from trait y to x ($M \geq 0$, $\int M(y,x)dx = 1$), and where individuals compete with a competition kernel $C(x,y) \geq 0$. Then one writes

$$\frac{\partial}{\partial t} n(t,x) = \underbrace{\int b(y)M(y,x)n(t,y)dy}_{\text{Birth including mutations}} - \underbrace{\int C(x,y)n(t,x)n(t,y)dy}_{\text{Death by competition}}. \tag{1.5}$$

Nonlinearity and selection comes from competition but evolution comes from mutations. Here, as before, existence of non-negative solutions also follows from the Cauchy-Lipschitz theorem assuming the coefficients are bounded because of the a priori bound proved below..

The main effect is balance between birth and death, which leads to control the total population. We choose $x \in \mathbb{R}$ and define

$$\varrho(t) = \int_{\mathbb{R}} n(t,x)dx.$$

Theorem 1.1 *Assume that $n^{init} \in L^1_+(\mathbb{R})$, and that for all x, $y \in \mathbb{R}$,*

$$0 < C_{min} \leq Cx,y) \leq C_{max} < \infty, \qquad 0 < b_{min} \leq b(y) \leq b_{max} < \infty,$$

then there are two constants such that

$$0 < \varrho_{min} \leq \varrho(t) \leq \varrho_{max} < \infty,$$

Proof. We integrate the equation (1.5) in x and find

$$\frac{d\varrho(t)}{dt} = \int_{\mathbb{R}}\int_{\mathbb{R}} b(y)M(y,x)n(t,y)dydx - \int_{\mathbb{R}}\int_{\mathbb{R}} C(x,y)n(t,x)n(t,y)dydx.$$

For the upper bound, we write

$$\frac{d\varrho(t)}{dt} \leq b_{max}\int_{\mathbb{R}} n(t,y)dy - C_{min}\int_{\mathbb{R}}\int_{\mathbb{R}} n(t,x)n(t,y)dydx$$

$$= b_{max}\varrho(t) - C_{min}\varrho(t)^2 = C_{min}\varrho(t)\big[\frac{b_{max}}{C_{min}} - \varrho(t)\big].$$

As a consequence, we obtain

$$\varrho(t) \leq \max\big(\varrho^{init}, \frac{b_{max}}{C_{min}}\big).$$

The lower bound is left as an exercise. □

Chapter 2

The renewal equation

The linear renewal equation, is to find the solution $n(t,x)$ of the equation

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x)n(t,x) = 0 \quad t \geq 0,\ x \geq 0, \\ N(t) := n(t, x=0) = \displaystyle\int_0^\infty b(y)n(t,y)dy, \end{cases} \tag{2.1}$$

usually completed with an initial data $n(t=0,x) = n^{init}(x)$.

An example of use is the predicition of demography, where $n(t,x)$ describes the population number density of individuals of age x, when they die with rate $d(x) \geq 0$ and give birth with rate $b(x) \geq 0$. Then, $N(t)$ is the number of newborns at time t. We assume that

$$b,\ d \in L^\infty_+(0,\infty).$$

For later purpose we also define

$$D(x) = \int_0^x d(y)dy. \tag{2.2}$$

2.1 Setting the model

Because the age x of the indivuduals progresses as the time t, the population density $n(t,x)$ satisfies, for $s > 0$ small

$$\underbrace{n(t+s, x+s) = n(t,x)}_{\text{Advancement of age as time}} \overbrace{-sd(x)n(t,x)}^{\text{death term}}.$$

As a consequence, dividing by s, and letting $s \to 0$, we find

$$\frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + d(x)n(t,x) = 0.$$

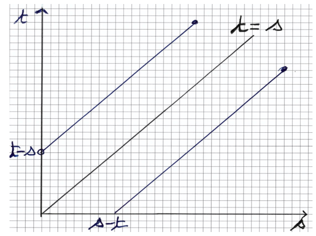

Figure 2.1: Illustration of the method of characteristics for the renewal equation.

This equation has to be complemented by the number of newborns (they have age $x = 0$) at time t; this is given by the quantity

$$n(t, x = 0) = \int_0^\infty b(y) n(t, y) dy,$$

where $b(\cdot)$ denotes the birth rate of the population. This is the renewal equation (2.1).

It is useful to define $N(0) = \int_0^\infty b(x) n^{init}(x)$ which does not necessarily match with $n^{init}(x = 0)$. There might be a discontinuity along the line $\{s = y\}$. This is not surprising for an hyperbolic equation.

2.2 Age structured equations and Volterra equations

The method of characteristics. The renewal equation can be written as an integral equation using the *method of characteristics* illustrated in Fig. 2.1. We write, at least formally, using the chain rule and the notation D in (2.2),

$$\frac{dn(t+s, x+s)}{ds} + d(x+s) n(t+s, x+s) = 0, \qquad \frac{d}{ds}[e^{D(x+s)} n(t+s, x+s)] = 0.$$

And thus, we find for $s \geq \max(-t, -x)$,

$$e^{D(x+s)} n(t+s, x+s) = e^{D(x)} n(t, x), \qquad \forall x \geq 0,\ t \geq 0. \tag{2.3}$$

In other words (take $s = -x$ and then $s = -t$)

$$n(t, x) = \begin{cases} e^{-D(x)} N(t-x) & \text{for } \ t > x, \\ e^{D(x-t) - D(x)} n^{init}(x-t) & \text{for } \ x > t, \end{cases} \tag{2.4}$$

The Volterra integral equation. We may use this formula in the boundary condition that we split as

$$N(t) = \int_0^\infty b(y) n(t, y) dy = \int_0^t b(y) n(t, y) dy + \int_t^\infty b(y) n(t, y) dy,$$

and using $x = y$ in (2.3), first with $s = -y$, and then with $s = -t$, we find

$$N(t) = \underbrace{\int_0^t b(y)e^{-D(y)}N(t-y)dy}_{=\int_0^t b(t-x)e^{-D(t-x)}N(x)dx} \quad + \quad \underbrace{\int_t^\infty b(y)e^{D(y-t)-D(y)}n^{init}(y-t)dy.}_{=\int_0^\infty b(x+t)e^{D(x)-D(x+t)}n^{init}(x)dx} \tag{2.5}$$

This is an integral equation of Volterra type which is well studied and can be solved using the standard Cauchy-Lipschitz theory (it is a contraction in C^0 for t small, and then iterate). A consequence of the method is that $n^{init} \geq 0$ implies that $N \geq 0$.

We may also establish a priori estimates (for later purpose, we do not assume a sign for n^{init} and $N(t)$)

Lemma 2.1 *With $\overline{\mu}(x) = b(x)e^{-D(x)}$, the solution $N(t)$ of (2.5) satisfies, for all $t \geq 0$,*

$$|N(t)| \leq \|\bar{\mu}\|_\infty \int_0^\infty e^{D(x)}|n^{init}(x)|dx \, e^{\|\bar{\mu}\|_\infty t},$$

$$|N(t+s) - N(t)| \leq \sup_{0\leq x\leq t+s}|N(x)| \left[\int_0^\infty |\overline{\mu}(x+s) - \overline{\mu}(x)|dx + \int_0^s \overline{\mu}(x)dx\right]$$
$$+\|e^D n^{init}\|_\infty \int_0^\infty |\overline{\mu}(x+s) - \overline{\mu}(x)|dx, \qquad \forall s \geq 0.$$

The first bound is not sharp and the actual rate of exponential growth is lower than $\|b\|_\infty$ as we see it later. The second bound quantifies compactness of N from that of $\overline{\mu}$ thanks to the Ascoli and Kolmogorov criteria.

Proof. Since $d(x) \geq 0$ and D is non-decreasing, we have

$$|N(t)| \leq \|\bar{\mu}\|_\infty \int_0^t |N(x)|dx + \|\bar{\mu}\|_\infty \int_0^\infty e^{D(x)}|n^{init}(x)|dx$$

and the Gronwall lemma gives us the bound of first inequality.

For the second inequality, we write

$$|N(t+s) - N(t)| \leq \int_0^t |\overline{\mu}(t+s-x) - \overline{\mu}(t-x)|N(x)dx + \left|\int_t^{t+s} \overline{\mu}(t+s-x)N(x)dx\right|$$
$$+ \int_0^\infty |\overline{\mu}(x+t+s) - \overline{\mu}(x+t)|e^{D(x)}n^{init}(x)dx.$$

and the result follows directly thus concluding the proof of Lemma 2.1. □

Once we know $N(t)$ from this Volterra equation, we can plug it as a boundary contition in the renewal equation (2.1) and find a solution $n(t,x)$ by the method of characteristics (2.3).

Exercise 2.1 *Being given b and d, and $N(t)$ a solution of the Volterra equation (2.5). Assuming we know n^{init}, show that the solution (2.4) gives rise to the same $N(t)$.*

2.3 Eigenelements

The eigenelements of linear equations dictate the long term behaviour and a priori bounds, as it is well-known for matrices. Here the property of positivity preservation makes that the renewal equation is relevant to the Perron-Frobenius theory, see Chapter 8 and we explain this here.

Recalling the notation (2.2) for D, for simplicity, we assume that the population is growing, that means the birth rate large enough compared to the death rate in the sense that

$$1 < \int_0^\infty b(x)e^{-D(x)} < \infty. \tag{2.6}$$

Following the Perron-Froebenius theory, we write the problem of finding the first eigenvalue for the renewal equation as a solution $(\lambda_0, \varphi(x))$ of

$$\begin{cases} \frac{\partial}{\partial x}\varphi(x) + \big(\lambda_0 + d(x)\big)\,\varphi(x) = 0, & x \geq 0, \\ \varphi(x=0) = \int b(y)\varphi(y)dy, & \varphi(x) > 0. \end{cases} \tag{2.7}$$

We also normalize it with the condition

$$\varphi(0) = \int_0^\infty b(x)\varphi(x)dx = 1. \tag{2.8}$$

Lemma 2.2 *We assume (2.6). There is a unique solution of (2.7)–(2.8), and it is given by*

$$\varphi(x) = e^{-\lambda_0 x - D(x)} \in L^1 \cap L^\infty(0,\infty), \tag{2.9}$$

with $\lambda_0 > 0$ defined by the relation

$$\int_0^\infty b(x)e^{-\lambda_0 x - D(x)}dx = 1. \tag{2.10}$$

Notice that the property $\lambda_0 > 0$ makes that $\varphi \in L^1 \cap L^\infty(0,\infty)$.

Proof. The formula (2.9) is just the explicit solution of (2.7) normalized by $\varphi(0) = 1$. The formula (2.10) is just the boundary condition and we now check that it is satisfied for a unique value λ_0. To do so, and thanks to assumption (2.6), for $\lambda \in [0,\infty)$, we define the function

$$\lambda \mapsto \mathcal{I}(\lambda) := \int_0^\infty b(x)e^{-\lambda x - D(x)}\,dx.$$

By assumption (2.6), $\mathcal{I}(0) > 1$. Also, using dominated convergence, $\mathcal{I}(\cdot)$ is decreasing and belongs to $C^0\big([0,\infty)\big)$ and $C^\infty(0,\infty)$. Moreover, $\mathcal{I}(\lambda) \to 0$ as $\lambda \to \infty$. Therefore there exists a unique λ_0 satisfying $\mathcal{I}(\lambda_0) = 1$, that is (2.10). □

Here are explicit examples, counter-examples, with or without assumptions (2.6), which show the difficulties related to the non-compactness of the problem since it is set in the half-line.

Example 1. Negative eigenvalue. A typical example is when

$$b(x) = 0 \qquad \text{for } x \geq x_\sharp > 0. \tag{2.11}$$

Then assumption (2.6) is not necessary and one can always solve (2.10). However, it might be that λ_0 is negative and that φ is not necessarily integrable, which is counter-intuitive in terms of interpretation as a population density.

Example 2. Individual aging type, We take b bounded and

$$d(x) \xrightarrow[x\to\infty]{} \infty, \tag{2.12}$$

Assumption (2.6) is not necessary. We always have existence of λ_0, which can be positive or not depending on b and d. In this case, $\int_0^\infty \varphi$ is always finite. We recall that population aging refers to the increase of the mean age $\int x\, n(t,x)dx$ while individual aging refers to increase of death rate $d(x)$ with age x.

Example 3. Non-existence of λ_0. We take $d \equiv 0$, and $b(x) = \frac{a}{(1+x)^2}$. For $a < 1$, there is no solution to the eigenproblem because (2.10) cannot be satisfied.

2.4 Eigenelements (dual)

The *dual eigenproblem* can easily be computed using the definition $(A(\varphi), \psi) = (\varphi, A^*(\psi))$ with $(u,v) = \int_0^\infty u(x)v(x)dx$. We multiply (2.7) by ψ and integrate by parts assuming that $\varphi(x)\psi(x) \to 0$, as $x \to \infty$, we find

$$\int_0^\infty \varphi(x)[-\frac{\partial \psi(x)}{\partial x} + \big(\lambda_0 + d(x)\big)\psi(x)]dx = \varphi(0)\psi(0) = \psi(0)\int_0^\infty b(x)\varphi(x)dx.$$

This has to hold for all φ and we arrive at the equation (we add a normalization)

$$\begin{cases} -\frac{\partial}{\partial x}\psi(x) + \big(\lambda_0 + d(x)\big)\,\psi(x) = \psi(0)b(x), \quad x \geq 0, \\ \psi(x) \geq 0, \qquad \displaystyle\int_0^\infty \psi(x)\varphi(x)dx = 1. \end{cases} \tag{2.13}$$

This is a 'backward problem' and the boundary condition is at $+\infty$, not at $x = 0$.

Exercise 2.2 (Conservative case) *Take $d = b$ and assume $D(x) \xrightarrow[x\to\infty]{} \infty$. Show that $\lambda_0 = 0$ is the only solution of (2.10). Compute φ and show that $\psi \equiv 1$.*

Theorem 2.1 *Assume (2.6). Then there is a unique solution to (2.13) given by*

$$\psi(x) = \frac{\psi(0)}{\varphi(x)} \int_x^\infty b(y)\,\varphi(y)\,dy,$$

with $\psi(0) = \big(\int_0^\infty y\, b(y)\varphi(y)\,dy\big)^{-1}$.

Notice that

- the properties $b \in L^\infty(0,\infty)$ and $\lambda_0 > 0$ ensure that $y\, b(y)\varphi(y) \in L^1(0,\infty)$.
- $\psi(x)$ can vanish for x large but its support is as $[0, x_\sharp]$ with $x_\sharp$ defined by $b(x) = 0$ for $x > x_\sharp$.

Proof. Combined with (2.7), the equation on ψ can also be written

$$\begin{cases} \frac{\partial}{\partial x}[\varphi(x)\psi(x)] = -\psi(0)b(x)\varphi(x), \quad x \geq 0, \\ \psi(x) \geq 0, \quad \displaystyle\int_0^\infty \psi(x)\varphi(x)dx = 1. \end{cases} \tag{2.14}$$

Therefore $\varphi(x)\psi(x) = C + \psi(0)\int_x^\infty b(y)\varphi(y)dy$ and the constant $C = 0$ is fixed by the value $\varphi(0)\psi(0) = \psi(0)$ in view of $\varphi(0) = \int_0^\infty b\varphi = 1$, see (2.6).

Then, we also find

$$\int_0^\infty \varphi\psi = 1 = \psi(0)\int_0^\infty \int_x^\infty b(y)\,\varphi(y)\,dydx,$$

and thus the value $\psi(0)$. □

Exercise 2.3 (Volterra integral eq. revisited) *With the notations* $\widetilde{n}(t,x) = n(t,x)e^{-\lambda_0 t}$, $\widetilde{N}(t) = \int_0^\infty b(x)\widetilde{n}(t,x)dx$, *and the normalization* $\varphi(0) = 1$, *prove that*

$$\widetilde{N}(t) = \int_0^t b\varphi(t-x)\,\widetilde{N}(x)\,dx + \int_0^\infty b\varphi(x+t)\,\frac{\widetilde{n}^{init}(x)}{\varphi(x)}\,dx,$$

and

$$\int_0^\infty \psi(x)\widetilde{n}^{init}(x)dx = \int_0^t \psi\varphi(t-x)\,\widetilde{N}(x)\,dx + \int_0^\infty \psi\varphi(x+t)\,\frac{\widetilde{n}^{init}(x)}{\varphi(x)}\,dx.$$

Exercise 2.4 (Cell cycle type) *Consider the case*

$$b(x) = 2K\mathbb{1}_{\{x \geq x^*\}}, \quad d(x) = K\mathbb{1}_{\{x \geq x^*\}}.$$

Compute φ, *prove that* $\int_0^\infty \varphi$ *is finite and that* $\lambda_0 > 0$ *is given implicitely by*

$$e^{\lambda_0\, x^*} = 2\frac{K}{K+\lambda_0}.$$

Compute the solution ψ *of the dual problem.(Notice that* (2.6) *holds).*

Exercise 2.5 (Distributed birth) *Consider* $q(x) \geq 0$ *with* $\int_0^\infty q(x)dx = 1$ *and the equation*

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + d(x)n(t,x) = q(x)N(t), \\ n(t, x=0) = 0, \\ N(t) = \int_0^\infty b(x)n(t,x)dx. \end{cases}$$

Compute φ, *formula for* $\lambda_0 > 0$, *the condition analogous to* (2.6), *the dual equation and the formula for* ψ.

Show that for a sequence $q_p \rightharpoonup \delta(x)$, *as* $p \to \infty$, *one recovers the renewal equation.*

Exercise 2.6 *Let $x_0 > 0$ and consider for $0 \le x \le x_0$ the equation*

$$\begin{cases} \frac{\partial}{\partial x}[(x_0 - x)\varphi] + \lambda_0 \varphi = 0, \\ \varphi(0) = 1 = \int_0^{x_0} b(x)\varphi(x)dx. \end{cases}$$

Assuming $\int_0^{x_0} b(x)dx > 1$, show that there is a unique solution with $\lambda_0 > 1$.
Give the dual equation and compute its solution.
For $b \equiv 1$ and $x_0 > 1$, compute λ_0.

2.5 Generalized relative entropy

The eigenelements $(\lambda_0, \varphi, \psi)$ give a fundamental tool to analyze the first properties of an equation such as the long term behaviour. The statement uses the renormalized quantity

$$\widetilde{n}(t,x) = n(t,x)e^{-\lambda_0 t},$$

and it is useful to consider in full generality that n^{init} does not have a sign.

Theorem 2.2 (Generalized Relative Entropy) *Under the assumption (2.6), for all convex function $H : \mathbb{R} \to \mathbb{R}$ with $H(0) = 0$, one has*

$$\frac{d}{dt}\int_0^\infty \psi(x)\varphi(x)H\big(\frac{\widetilde{n}(t,x)}{\varphi(x)}\big)dx = -D_H(t) \le 0, \qquad \forall t \ge 0,$$

$$\frac{D_H(t)}{\psi(0)} = \int_0^\infty H\big(\frac{\widetilde{n}(t,x)}{\varphi(x)}\big)d\mu(x) - H\big(\int_0^\infty \frac{\widetilde{n}(t,x)}{\varphi(x)}d\mu(x)\big) \ge 0,$$

for the probability measure $d\mu(x) = b(x)\varphi(x)dx$ (Jensen's inequality).

The name Generalized Relative Entropy (GRE in short), is because in the conservative case, when $\psi = 1$, $b = d$, one recovers the so-called relative entropy which has been widely used, in particular for kinetic and Fokker-Planck equations.

This GRE structure has many consequences. Before proving the theorem, let us mention the elementary L^p estimates.

Corollary 2.1 (Conservation law, estimates) *We have*

(i) $\displaystyle\int_0^\infty \psi(x)n(t,x)dx = e^{\lambda_0 t}\int_0^\infty \psi(x)n^{init}(x)dx,$

(ii) $\displaystyle\int_0^\infty \psi(x)\varphi(x)^{1-p}|n(t,x)|^p dx \le e^{\lambda_0 p t}\int_0^\infty \psi(x)\psi(x)\varphi(x)^{1-p}|n^{init}(x)|^p dx,$ *for* $1 \le p < \infty$,

(iii) $\inf \frac{n^{init}(\cdot)}{\varphi(\cdot)}e^{\lambda_0 t} \le \frac{n(t,x)}{\varphi(x)} \le \sup \frac{n^{init}(\cdot)}{\varphi(\cdot)}e^{\lambda_0 t}.$

This corollary is obtained using
(i) $H(u)=u$ because, clearly, $D_H=0$ for this choice,
(ii) $H(u)=|u|^p$,
(iii) For the upper bound (the lower bound is obtained changing n in $-n$). We take $H(u)=(u-K^+)_+^2$ with $K^+=\sup\frac{n^{init}(\cdot)}{\varphi(\cdot)}$, then we get for $t\geq 0$,

$$\int_0^\infty \psi(x)\varphi(x)\left(\frac{\widetilde{n}(t,x)}{\varphi(x)}-K^+\right)_+^2 dx \leq 0.$$

and thus $\frac{\widetilde{n}(t,x)}{\varphi(x)}\leq K^+$ on the support of ψ, and thus on the support of b. To complete the proof, we notice that this implies $\widetilde{n}(t,0)=\int_0^\infty \frac{\widetilde{n}(t,x)}{\varphi(x)}b(x)\varphi(x)dx\leq K^+$. And we conclude using the equation on $\frac{\widetilde{n}(t,x)}{\varphi(x)}$ below.

Proof. **[of Theorem 2.2].** We successively write

$$\frac{\partial}{\partial t}\widetilde{n}(t,x)+\frac{\partial}{\partial x}\widetilde{n}(t,x)+\big(\lambda_0+d(x)\big)\widetilde{n}(t,x)=0, \tag{2.15}$$

$$\frac{\partial}{\partial t}\frac{\widetilde{n}(t,x)}{\varphi(x)}+\frac{\partial}{\partial x}\frac{\widetilde{n}(t,x)}{\varphi(x)}=0,$$
$$\frac{\partial}{\partial t}H\big(\frac{\widetilde{n}(t,x)}{\varphi(x)}\big)+\frac{\partial}{\partial x}H\big(\frac{\widetilde{n}(t,x)}{\varphi(x)}\big)=0,$$

and finally, thanks to (2.14),

$$\frac{\partial}{\partial t}[\psi(x)\varphi(x)H\big(\frac{\widetilde{n}(t,x)}{\varphi(x)}\big)]+\frac{\partial}{\partial x}[\psi(x)\varphi(x)H\big(\frac{\widetilde{n}(t,x)}{\varphi(x)}\big)]=-\psi(0)b(x)\varphi(x)H\big(\frac{\widetilde{n}(t,x)}{\varphi(x)}\big).$$

After integration in $x\in\mathbb{R}^+$ we find,

$$\begin{aligned}\tfrac{d}{dt}\int\psi(x)\varphi(x)H\big(\tfrac{\widetilde{n}(t,x)}{\varphi(x)}\big)dx &= -\psi(0)\int H\big(\tfrac{\widetilde{n}(t,x)}{\varphi(x)}\big)d\mu(x)+\psi(0)H\big(\tfrac{\widetilde{n}(t,0)}{\varphi(0)}\big)\\ &=\psi(0)\big[-\int H\big(\tfrac{\widetilde{n}(t,x)}{\varphi(x)}\big)d\mu(x)+H\big(\int\tfrac{\widetilde{n}(t,x)}{\varphi(x)}d\mu(x)\big)\big]\\ &=-D_H(t)\leq 0,\end{aligned}$$

for all convex function H. The final inequality follows from Jensen's inequality. The statements follow from this calculation and inequality. □

Exercise 2.7 *Show that for all solutions $n_1(t,x)$, $n_2(t,x)>0$ of the renewal equation (2.1) and $q(t,x)\geq 0$ of the dual equation*

$$-\frac{\partial}{\partial t}q(t,x)-\frac{\partial}{\partial x}q(t,x)+d(x)q(t,x)=q(t,x=0)b(x),\quad x\geq 0,\ t\geq 0,$$

the GRE inequality holds true; for all convex function $H : \mathbb{R} \to \mathbb{R}$ with $H(0) = 0$, one has

$$\frac{d}{dt}\int_0^\infty q(t,x)n_2(t,x)H\big(\frac{n_1(t,x)}{n_2(t,x)}\big)dx = -D_H(t) \leq 0, \qquad \forall t \geq 0,$$

$$\frac{D_H(t)}{q(t,0)n_2(t,0)} = \int_0^\infty H\big(\frac{n_1(t,x)}{n_2(t,x)}\big)b(x)n_2(t,x)dx - H\big(\int_0^\infty \frac{n_1(t,x)}{n_2(t,x)}b(x)n_2(t,x)dx \geq 0,$$

2.6 Long time asymptotic: a general result

In practice one observes the *Stable Age Distribution*, i.e., the long time limit of $\widetilde{n}$ which is expected to be proportional to the steady state φ given by equation (2.7). In this section, we prove a general statement without a rate. Exponential rate of convergence is treated afterwards in section §2.7 where we give a simple method using a restrictive hypothesis. The Doeblin method is explained afterwards.

Again we do not need sign for the solution $n(t,x)$ here. The general result is

Theorem 2.3 *We assume (2.6) and denote $(\lambda_0, \varphi, \psi)$ the first eigenelements. Consider an initial data that $|n^{init}(x)| \leq C\varphi(x)$ and $e^{D(x)+\lambda_0 x}n^{init} \in L^1(0,\infty)$. Then, the solution $n(t,x)$ of (2.1) satisfies*

$$\int_0^\infty \left|ne^{-\lambda_0 t} - m^{init}\varphi(x)\right|\psi(x)dx \searrow 0 \qquad \text{as } t \to \infty, \tag{2.16}$$

with $m^{init} = \int n^{init}(x)\psi(x)dx$ (a conserved quantity).

Proof. We recall that φ is normalized by $\varphi(0) = 1$ and $\int_0^\infty b(x)\varphi(x)dx = 1$. Then, we use the notations

$$\widetilde{n} = ne^{-\lambda_0 t}, \qquad h(t,x) = n(x,t)e^{-\lambda_0 t} - m^{init}\varphi(x), \qquad N^{[h]}(t) = \int_0^\infty b(x)h(x,t)dx.$$

Both $\widetilde{n}$ and h are solutions of the renewal equation (2.15).

First step. Consequences of GRE. Therefore Theorem 2.2 and its Corollary 2.1 apply. So, $h(t,x)$ satisfies

$$\int h(t,x)\psi(x)dx = 0, \qquad |h(t,x)| \leq C_0\varphi(x), \qquad |N^{[h]}(t)| \leq C_0, \tag{2.17}$$

using firstly the conservation property (i), secondly the bound (ii) of Corollary 2.1, with $C_0 = C + |m^{init}|$ and thirdly an immediate control of the boundary term.

We have to prove that such a solution vanishes in $L^1(\psi(x)dx)$ for long times. Notice that, by the GRE property in Theorem 2.2, there is a $L \geq 0$ such that

$$\int_0^\infty |h(t,x)|\,\psi(x)dx \searrow L, \qquad \text{as } t \to \infty.$$

And it remains to show that $L=0$.

Second step. Compactness of $N^{[h]}(t)$. This follows from the Volterra integral equation which here reads

$$N^{[h]}(t)=\int_0^t b\varphi(t-x)\,N^{[h]}(x)\,dx+\int_0^\infty b\varphi(x+t)\,\frac{h^{init}(x)}{\varphi(x)}\,dx.$$

Since we know that $N^{[h]}$ is uniformly bounded, the second estimate of Lemma 2.1, when used in our context. with $\overline{\mu}(x)=\mu(x)=b(x)\varphi(x)\in L^1$, gives us the uniform continuity estimate

$$\omega(s):=\sup_{t\geq 0}|N^{[h]}(t+s)-N^{[h]}(t)|\to 0,\qquad \text{as}\quad s\to 0.$$

Third step. Renormalization. For $k\in\mathbb{N}$, we define the solution $h_k(t,x)$ of equation (2.15), by

$$h_k(t,x)=h(t+k,x),\qquad \int_0^\infty h_k(t,x)\psi(x)dx=0,\qquad |h_k|\leq C_0\varphi,$$

thanks to (2.17). Using Theorem 2.2, for $H(\cdot)$ convex with $H(0)=0$, we have for all $0<T<k$,

$$\int_{-T}^{T}\Big[\int H\big(\frac{h_k(t,x)}{\varphi(x)}\big)d\mu(x)-H\big(N^{[h_k]}(t)\big)\Big]dt:=I_k\to 0,\qquad \text{as}\ \ k\to\infty. \tag{2.18}$$

Indeed, from the very definitions of I_k and h_k, we have

$$I_k=\int_{k-T}^{k+T}\Big[\int H\big(\frac{h(t,x)}{\varphi(x)}\big)d\mu(x)-H\big(\int_0^\infty\frac{h(t,x)}{\varphi(x)}d\mu(x)\big)\Big]dt$$

and the integrand is nonnegative and integrable (and $\frac{h(t,x)}{\varphi(x)}d\mu(x)=b(x)h_k dx$).

We also notice that h_k satisfies equation

$$\begin{cases}\frac{\partial}{\partial t}h_k(t,x)+\frac{\partial}{\partial x}h_k(t,x)+[\lambda_0+d(x)]\,h_k(t,x)=0, & t\geq 0,\ x\geq 0,\\[2mm] N^{[h_k]}(t)=h_k(t,x=0)=\int b(y)h_k(t,y)dy. &\end{cases} \tag{2.19}$$

Fourth step. A weak limit. Next, using the uniform bound, we may extract a subsequence (still denoted h_k) such that, for all $T>0$,

$$\begin{cases}\frac{h_k}{\varphi}\to\frac{g}{\varphi}\quad \text{in}\ \ L^\infty\big((-T,T)\times\mathbb{R}^+\big)-w*,\qquad 0\leq|g|\leq C_0\varphi,\\ \int_0^\infty\psi(x)g(t,x)dx=0\\ N^{[h_k]}(t)=\int b(y)h_k(t,y)dy\to\int b(y)g(t,y)dy:=N^{[g]}(t)\quad \text{in}\ \ C\big([-T,T]\big),\end{cases}$$

the last statement being a consequence of the second step.

Also, from equation (2.19), we conclude that

$$\begin{cases} \frac{\partial}{\partial t} g(t,x) + \frac{\partial}{\partial x} g(t,x) + [\lambda_0 + d(x)]\, g(t,x) = 0, \quad t \geq 0,\, x \geq 0, \\ \\ g(t, x=0) = \int b(y) g(t,y) dy, \end{cases}$$

and thus

$$\frac{\partial}{\partial t} \frac{g(t,x)}{\varphi(x)} + \frac{\partial}{\partial x} \frac{g(t,x)}{\varphi(x)} = 0. \tag{2.20}$$

Fifth step. Entropy and the weak limit. We pass to the limit in the entropy relation (2.18) and thanks to the strong convergence of $N^{[h_k]}(t)$, we obtain, by convexity in weak limits,

$$\begin{aligned} \int_{-T}^{T} \int_0^\infty H\big(\frac{g(t,x)}{\varphi(x)}\big) d\mu(x) dt \; &\leq \lim \int_{-T}^{T} \int_0^\infty H\big(\frac{h_k(t,x)}{\varphi(x)}\big) d\mu(x) dt \\ &= \int_{-T}^{T} H\big(\int_0^\infty \frac{g(t,x)}{\varphi(x)} d\mu(x)\big) dt. \end{aligned} \tag{2.21}$$

But from the Jensen inequality, the reverse inequality is also true thus showing that

$$\int_{-T}^{T} \int_0^\infty H\big(\frac{g(t,x)}{\varphi(x)}\big) d\mu(x) dt = \int_{-T}^{T} H\big(\int_0^\infty \frac{g(t,x)}{\varphi(x)} d\mu(x)\big) dt.$$

This equality for H strictly convex shows that for almost all $t > 0$, on the support of μ, i.e., that of b,

$$\frac{g(t,x)}{\varphi(x)} = C(t) \quad \text{(independent for all } x \in \text{supp b).}$$

Since $\int_0^\infty g(t,x)\psi(x) = 0$, we find that $g = 0$.
Sixth step. Strong convergence. To conclude that L, defined in the first step, vanishes it remains to prove that g_k converges strongly.

□

2.7 Exponential decay and Poincaré inequality

As can be observed in Figure 2.3, the GRE property does not imply a simple monotone convergence in the long term. Using a Poincaré inequality, that is a control of entropy by entropy dissipation, provides us with a stronger dissipation of entropy which generates exponential decay to the steady state. We recall the notation $\widetilde{n}(t,x) = n(t,x)e^{-\lambda_0 t}$,

Theorem 2.4 *We assume (2.6) and that there is a constant such that $|n^{init}(x)| \leq C\varphi(x)$ and*

$$\exists \lambda_1 > 0, \quad s.t. \qquad b(x) \geq \lambda_1 \frac{\psi(x)}{\psi(0)}, \tag{2.22}$$

we have the Poincaré inequality in L^1, for all integrable function $h(x)$ with $\int_0^\infty h(x)\psi(x)dx = 0$,

$$\frac{\lambda_1}{\psi(0)} \int |h(t,x)|\psi(x)dx \le \int b(x)|h(t,x)|dx - \big| \int [b(x)h(t,x)dx\big|$$

and thus, with m^{init} defined by $m^{init} \int_0^\infty \varphi(x)\psi(x)dx = \int_0^\infty |n^{init}(x)\psi(x)dx$, we have

$$\int |\widetilde{n}(t,x) - m^{init}\varphi(x)|\psi(x)dx \le e^{-\lambda_1 t} \int |n^{init}(x) - m^{init}\varphi(x)|\psi(x)dx. \tag{2.23}$$

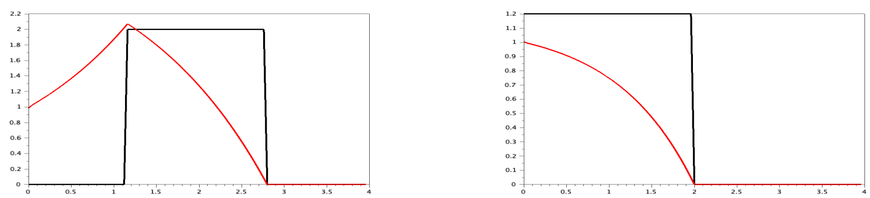

Figure 2.2: BLACK=b. RED= ψ. Illustration of the condition (2.22) for Poincaré inequality. Left: the condition is not satisfied. Right: it is satisfied.

The assumption (2.22) is restrictive if we have in mind that b can vanish for $x \approx 0$ and be positive afterwards because $\psi(x) > 0$ on the convex hull of the support of b. But for x large, or close to the end point of the support of b, in general the quantity $\psi(x)$ vanishes faster than b. See Figure 2.2. There are several proofs of exponential convergence in time that do not use hypothesis (2.22), e.g., using the Laplace transform, see [8]). However the approach using a Poincaré inquality has the advantage of universality.

Proof. *1. Poincaré inequality.* We write

$$\begin{aligned}\int b(x)\ \ |h(t,x)|dx - \big|\int b(x)h(t,x)dx\big| & \\ = \int b(x)|h(t,x)|dx - \big| \int \Big[b(x) - \frac{\lambda_1}{\psi(0)}\psi(x)\Big]\, h(t,x)dx\big| & \\ \ge \int b(x)|h(t,x)|dx - \int \underbrace{\Big[b(x) - \frac{\lambda_1}{\psi(0)}\psi(x)\Big]}_{\ge 0}\ |h(t,x)|dx & \\ = \tfrac{\lambda_1}{\psi(0)} \int |h(t,x)|\psi(x)dx. &\end{aligned}$$

2. Exponential convergence. We define

$$h(t,x) = n(t,x)e^{-\lambda_0 t} - m^{init}\varphi(x),$$

which is a solutions of equation (2.15).

Using the dual equation (2.13), we find

$$\frac{\partial}{\partial t}\big[h(t,x)\psi(x)\big] + \frac{\partial}{\partial x}\big[h(t,x)\psi(x)\big] = -\psi(0)b(x)h(t,x), \quad t \geq 0,\, x \geq 0,$$

Therefore, we also have

$$\frac{\partial}{\partial t}\big[|h(t,x)|\psi(x)\big] + \frac{\partial}{\partial x}\big[|h(t,x)|\psi(x)\big] = -\psi(0)b(x)|h(t,x)|, \quad t \geq 0,\, x \geq 0.$$

After integration in x, we can use the Poncaré inequality since $\int \psi(x)h(t,x)dx = 0$ and obtain

$$\begin{aligned} \frac{d}{dt}\int |h(t,x)|\psi(x)dx &= -\psi(0)\int b(x)|h(t,x)|dx + \psi(0)\Big|\int b(x)h(t,x)dx\Big| \\ &= -\lambda_1 \int |h(t,x)|\psi(x)dx. \end{aligned}$$

We conclude using the Gronwall lemma. □

Exercise 2.8 *Consider the age structured equation*

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n + d(x)n = 0, & x \geq 0, \\ \frac{\partial}{\partial t}v(t) + \mu v = \int b(x)n(t,x)dx, \\ n(t,0) = v(t). \end{cases}$$

Compute the principal eigenelements λ_0, (φ, V) and (ψ, Z). Express the Generalized Relative Entropy principle.
Study the large time behavior of the quantities $(n(t,x), v(t))$ assuming the Poincaré inequality (2.22) is satisfied.

Exercise 2.9 *We take $d \equiv 0$ and*

$$b(x) = \nu\, e^{-\mu x}, \qquad \nu > \mu,$$

1. Prove that (2.6) holds by omputing

$$\lambda_0 = \nu - \mu, \quad \varphi(x) = \lambda_0 e^{-\lambda x}, \quad \psi(x) = \psi(0)e^{-\mu x}.$$

2. Prove that the condition (2.22), for the Poincaré inequality, is satisfied with $\lambda_1 = \nu$.

Exercise 2.10 (Poincaré inequality in L^2 (Compactness)) *We assume there are $a_1 > 0$, $a_2 > 0$ such that $a_1\psi(x) \leq b(x) \leq a_2\psi(x)$. The goal is to prove the inequality*

$$\exists \lambda_1 \text{ such that } \quad \lambda_1 \int_0^\infty \psi(x)\varphi(x)\,(u(x))^2\,dx \leq \int_0^\infty (u(x) - \langle u\rangle)^2\, b(x)\varphi(x)dx, \tag{2.24}$$

for all function u such that $\int_0^\infty u(x)\psi(x)\varphi(x)dx = 0$ with $\langle u\rangle = \int_0^\infty u(x)b(x)\varphi(x)dx$.

1. By contradiction, if this is not true, show that there is a sequence such that $u_n(x) \rightharpoonup u(x)$ in $L^2(\psi(x)\varphi(x)dx) - w$, $\int_0^\infty (u_n(x))^2\psi(x)\varphi(x)dx = 1$ and

$$\int_0^\infty \left(u_n(x) - \langle u_n\rangle\right)^2 b(x)\varphi(x)dx \leq \frac{1}{n}.$$

2. Prove that $\langle u_n\rangle \to \langle u\rangle$.
3. Prove that $u_n(x) \to 0$ in $L^2(\psi(x)\varphi(x)dx)$. Conclude.

Exercise 2.11 (Poincaré inequality in L^2 (Direct)) *We assume there is $\lambda_1 > 0$ such that $b \geq \lambda_1\psi$, we can show*

$$\lambda_1^2 \int_0^\infty \psi(x)\varphi(x) \left(u(x)\right)^2 \leq \int_0^\infty \left(u(x) - \langle u\rangle\right)^2 b(x)\varphi(x)dx, \tag{2.25}$$

for all function u such that $\int_0^\infty u(x)\psi(x)\phi(x)dx = 0$ with $\langle u\rangle = \int_0^\infty u(x)b(x)\varphi(x)dx$.
Hint: *Normalizing ψ by $\int_0^\infty \psi\varphi = 1$, we write*

$$\left|\int_0^\infty u(x)b(x)\varphi(x)dx\right|^2 = \left|\int_0^\infty u(x)[b(x) - a\psi(x)]\varphi(x)dx\right|^2 \leq A\int_0^\infty b(x)\varphi(x)\left(u(x)\right)^2 dx$$

with $A = \int_0^\infty \frac{(b(x)-a\psi(x))^2}{b(x)}\varphi(x)dx \leq 1 - 2a + \frac{a^2}{\lambda_1}$. It remains to choose $a = \lambda_1$.

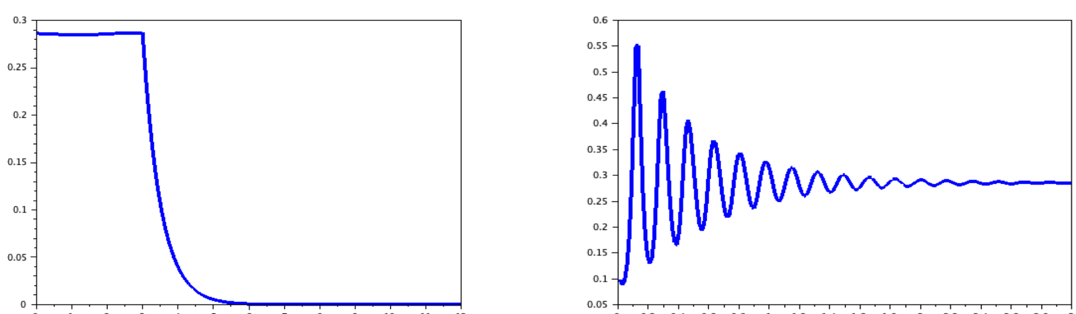

Figure 2.3: THE SOLUTION OF THE RENEWAL EQUATION WITH $b = d = 2 * \mathbf{1}_{\{s>3\}}$. LEFT: THE SOLUTION $s \mapsto n(t,s)$ FOR t LARGE. RIGHT: THE MAP $t \mapsto N(t)$.

2.8 Exponential decay, Bacry-Emery's method

For the Fokker-Planck equation in the full space, Bacry and Emery invented a proof of the Poincaré inequality based on time differentiation of the entropy dissipation. We copy this proof in the framework of the renewal equation.

Theorem 2.5 *Assume $\partial_x(b(x)\varphi(x)) \leq -\lambda_1(b(x)\varphi(x))$ then the Poincaré inequality holds for functions such that $\int_0^\infty \psi\varphi u(x)dx = 0$*

$$\lambda_1 \int_0^\infty \psi\varphi u^2(x)dx \leq 2\int_0^\infty b(x)\varphi(x)(u(x)-\langle u\rangle)^2dx, \quad \langle u(t)\rangle := \int_0^\infty b(x)\varphi(x)u(t,x)dx$$

Consequently, the exponential decay estimate (2.23) holds true.

Because we know that $u(t,x) = \frac{n(x,t)e^{-\lambda_0 t}}{\varphi} - \int_0^\infty \psi n^{init}dx$ converges to 0 as $t \to \infty$ (TO BE DETAILED), we have by GRE

$$\int_0^\infty \psi\varphi(u^{init})^2dx = 2\int_0^\infty\int_0^\infty b(x)\varphi(x)(u-\langle u\rangle)^2dxdt, \tag{2.26}$$

Proof. Since $\partial_t u + \partial_x u = 0$ one finds

$$\begin{aligned}\frac{d}{dt}\int_0^\infty b(x)\varphi(x)(u-\langle u\rangle)^2dx &= 2\int_0^\infty b(x)\varphi(x)(u-\langle u\rangle)(\partial_t u - \langle\partial_t u\rangle)dx \\ &= 2\int_0^\infty b(x)\varphi(x)(u-\langle u\rangle)\,\partial_t u dx \\ &= -2\int_0^\infty b(x)\varphi(x)(u-\langle u\rangle)\,\partial_x u dx\end{aligned}$$

because of the definition of $\langle u\rangle$ and because $\langle\partial_t u\rangle$ does not depend on x. Then,

$$\begin{aligned}\frac{d}{dt}\int_0^\infty b(x)\varphi(x)(u-\langle u\rangle)^2dx &= -2\int_0^\infty b(x)\varphi(x)(u-\langle u\rangle)\,(\partial_x u - \partial_x\langle u\rangle)dx \\ &= -\int_0^\infty b(x)\varphi(x)\partial_x(u-\langle u\rangle)^2dx \\ &= \int_0^\infty \partial_x(b(x)\varphi(x))\,(u-\langle u\rangle)^2dx \\ &\leq -\lambda_1\int_0^\infty (b(x)\varphi(x))\,(u-\langle u\rangle)^2dx\end{aligned}$$

because $u(0) = \langle u\rangle$ for solutions of the renewal equation.

As a consequence, integrating in $t \in (0,\infty)$ we find

$$\lambda_1\int_0^\infty\int_0^\infty (b(x)\varphi(x))\,(u-\langle u\rangle)^2dxdt \leq \int_0^\infty b(x)\varphi(x)(u^{init}-\langle u^{init}\rangle)^2dx \tag{2.27}$$

Combining the two equations (2.26), (2.27) gives the result for u^{init} which is any function satisfying $\int_0^\infty \psi\varphi u^{init}(x)dx = 0$. □

2.9 Exponential decay: Doeblin's method

Doeblin's method is a standard method which can be used in conservative cases and this was recently revisiited in [5, 2]. We present the framework and then, the example of the renewal equation when $b = d$. Notice that $\psi(x)n(t,x)$ solves a conservative equation in general.

The general framework. Consider a linear semigroup on L^1 (resp. M^1) that satisfies (Stochastic semigroup and Doeblin's condition)
1. $S(t).n \geq 0$ for $n \geq 0$,
2. $\int S(t).n = \int n$,
3. $\exists t_0 > 0,\ \alpha > 0$ and $\nu \geq 0$ with $\int \nu = 1$, such that $S(t_0).n \geq \alpha\nu$ for all $n \geq 0$ with $\int n = 1$.

Then one has, for any L^1 function (resp. measure) such that $\int n = 0$

$$\|S(t_0).n\|_1 \leq (1-\alpha)\|n\|_1, \tag{2.28}$$

$$\exists n^* \geq 0, \quad \int n^*(x)dx = 1 \quad \text{such that} \quad S(t).n^* = n^*, \quad \forall t \geq 0, \tag{2.29}$$

and for any L^1 function (resp. probability measure) n

$$\|S(t).(n-n^*)\|_1 \leq (1-\alpha)^k\|n-n^*\|_1, \qquad \text{for } t = kt_0. \tag{2.30}$$

Proof. For a measure n with $\int n = 0$, one has with $M = \int n_+ = \int n_-$,

$$\|S(t_0).n\|_1 \leq \|S(t_0).(n_+ - n_-)\|_1 \leq \|S(t_0).n_+ - \alpha M\nu)\|_1 + \|S(t_0).n_- - \alpha M\nu)\|_1$$
$$= \int [S(t_0).n_+ - \alpha M\nu] + \int [S(t_0).n_- - \alpha M\nu] = 2(1-\alpha)M = (1-\alpha)\|n\|_1;$$

that is (2.28).

Next, from (2.28) we see that $S(t_0)$ is a contraction and has a unique fixed point n^*, $S(t_0).n^* = n^*$. And $S(s).n^*$ being also a fixed point, we have $S(s).n^* = n^*$ for all $s \geq 0$ and (2.29) is proved.

The last result is obvious. □

The renewal equation. We now indicate how to use Doeblin's method for the renewal equation, that is $S(t).n^{init} = n(t,x)$ the solution of (2.1). which preserves non-negativity as in assumption 1. Next, we assume there are $b_\sharp > 0$ and $x_\sharp > 0$ such that

$$b_\sharp \mathbb{I}_{\{x > x_\sharp\}} \leq b(x) = d(x) \leq b_M. \tag{2.31}$$

The first assumption is for the property of mass preservation 2..

It remains to show condition 3, and we argue in two steps. Frist step is to write, by the characteristics (2.4),

$$n(t,x) \geq n^{init}(x-t)e^{-b_M t} \quad \text{for} \quad x > t.$$

Therefore, for $t \geq x_\sharp$, we find

$$N(t) = \int_0^\infty b(x)n(t,x)dx \geq b_\sharp \int_{x_\sharp}^\infty n^{init}(x-t)e^{-b_M t}\mathbb{I}_{\{x>t\}}dx = b_\sharp e^{-b_M t}.$$

In the second step, we use again the characteristics (2.4), and write for $x < t$ and $t > x_\sharp$

$$n(t,x) \geq N(t-x)e^{-b_M x} \quad \text{therefore} \qquad n(t,x) \geq b_\sharp e^{-b_M t}\mathbb{I}_{\{x<t\}}.$$

This proves the third condition in Doeblin's method, choosing any $t_0 > x_\sharp$.

Chapter 3

Growth-fragmentation equation

3.1 Equal division

Another class of structured equation arises in different areas of physics, computer science and biology. The unknown is still denoted by $n(t,x)$ with $x>0$, $t \geq 0$, and it satisfies

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x}\big[g(x)\, n(t,x)\big] + b(x)\, n(t,x) = 4\, b(2x)\, n(t,2x), \\ n(t, x=0) = 0, \\ n(t=0,x) = n^{init}(x). \end{cases} \tag{3.1}$$

Before we study the properties of solutions, let us mention some examples of applications (usually a feed-back controls these processes, which makes the equations nonlinear)

- Biopolymers of length x can split in two pieces of size $\frac{x}{2}$ and can attach monomers with rate $g(x)$.
- TCP connections. Here $n(t,x)$ is the number density of windows of size x routed to the internet. The size of messages increases with a rate $g(x)$. The router can divide by 2 the size of the message in case a bottleneck is reached. See [1].
- Cell division. Cells of size x grow with rate $g(x)$ and divide and generate two cells of size $\frac{x}{2}$. Size is usually a better physiological trait to structure cell populations as bacterium or yeast. Then x is the mass (or length, or volume) of the cell. Assuming equal mitosis, i.e., that cells divide exactly in two equal new cells, The term $\big[g(x)\, n(t,x)\big]$ describes the growth of cells using the (unlimited) nutrients and a usual rule is $g(x) = x^{\frac{2}{3}}$ to take into account surface/volume ratio for food uptake. The term $4\, b(2x)\, n(t,2x)$ describes the division of cells of size $2x$ in two cells of size x, the term $b(x)\, n(t,x)$ takes into account the loose of cells of size x that divide.

The first natural question is to find the growth exponent. The second question is to find the typical repartition of cells (Stable Size Distribution) which results from the two opposite effects, of growth and fragmentaition. Two idendities quantify these effects. The first one is to consider the total number of cells and compute (this is formal at this level, assuming that one can integrate on the half line

$$\frac{d}{dt}\int_0^\infty n(t,x)dx + \int_0^\infty b(x)n(t,x)dx = \int_0^\infty 4\, b(2x)\, n(t,2x)dx = 2\int_0^\infty b(x)n(t,x)dx,$$

therefore

$$\frac{d}{dt}\int_0^\infty n(t,x)dx = \int_0^\infty b(x)n(t,x)dx.$$

In words, the total number of cells only increases thanks to cell division.

One can also compute the biomass. Multiplying by x and integrating by parts we calculate

$$\begin{aligned} \tfrac{d}{dt}\int_0^\infty xn(t,x)dx - \int_0^\infty g(x)n(t,x)dx + \int_0^\infty xb(x)n(t,x)dx &= \int_0^\infty 4\, x\, b(2x)\, n(t,2x)dx \\ &= \int_0^\infty xb(x)n(t,x)dx, \end{aligned}$$

therefore

$$\frac{d}{dt}\int_0^\infty xn(t,x)dx = \int_0^\infty g(x)n(t,x)dx.$$

In words, biomass only increases by use of nutrients.

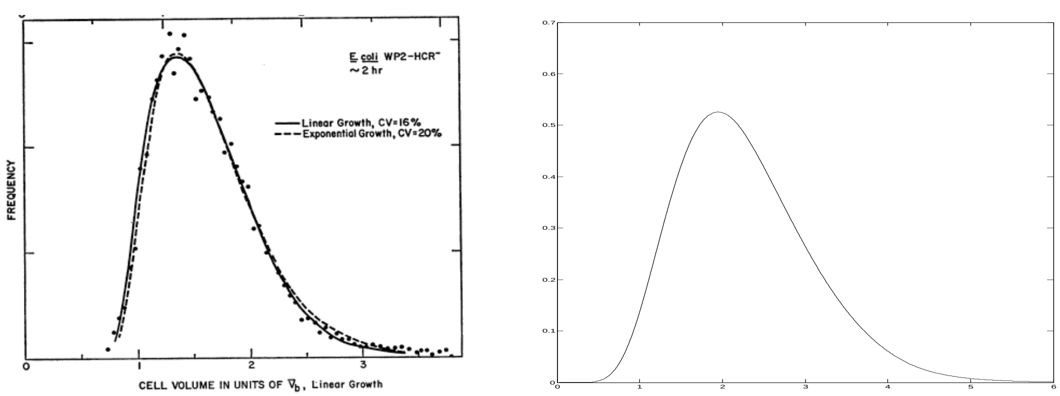

Figure 3.1: Letf: Experimental data for size distribution in *E. coli*. Right: Numerical simulation of equation (3.1).

3.2 Size structured models (asymmetric division)

The division is not always symmetric and a daughter cell can be much smaller that the mother cell. The above model can be generalized to take this into account. We arrive at an equation also called 'growth-fragmentation' because it arises in physics to describe such phenomena e.g.

for polymers.

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x}\big[g(x)\, n(t,x)\big] + b(x)\, n(t,x) = 2\int_x^\infty b(y)\kappa(x,y)\, n(t,y) dy, \\ n(t,x=0) = 0, \\ n(t=0,x) = n^{init}(x). \end{cases} \tag{3.2}$$

Here $b(y)$ is the division rate of cells of sizes y and $\kappa(x,y)$ is the probality that such a cell gives a daughter cell of size $x \leq y$. It is natural to assume that (i) daughter cells are smaller than the mother cell, (ii) the division event gives two cells exactly, (iii) the total mass is conserved during division. These are expresses by the idendities

$$\kappa(x,y) = 0 \qquad \text{for } x > y, \tag{3.3}$$

$$\int_0^y \kappa(x,y)dx = 1, \qquad \int_0^y x\, \kappa(x,y)dx = y/2. \tag{3.4}$$

Notice that this last equality is a consequence of the first two and of the natural symmetry assumption

$$\kappa(x,y) = \kappa(y-x,y).$$

As an exercise, one can notice that the same relations as before hold for growth of the number of cells and total biomass

$$\frac{d}{dt}\int_0^\infty n(t,x)dx = \int_0^\infty b(x)n(t,x)dx, \qquad \frac{d}{dt}\int_0^\infty xn(t,x)dx = \int_0^\infty g(x)n(t,x)dx.$$

For various choices of κ, one can recover models we have already encountered. Let us give examples.

(i) The renewal (age structured) equation (2.1) can be recovered using,

$$\kappa(x,y) = \frac{1}{2}\big(\delta(x=0) + \delta(x=y)\big). \tag{3.5}$$

(ii) Equal mitosis, as in (3.1), is the special case

$$\kappa(x,y) = \delta(x = y/2). \tag{3.6}$$

(iii) Uniform division is the case

$$\kappa(x,y) = \frac{1}{y}\, \mathbb{1}_{\{0\leq x\leq y\}}. \tag{3.7}$$

(iv) More generally, the homogeneous form is used

$$\kappa(x,y) = \frac{1}{y}\, K(\frac{x}{y}), \qquad K \geq 0, \quad \int_0^1 K(\sigma)d\sigma = 1, \quad K(\sigma) = 0 \text{ for } \sigma > 1. \tag{3.8}$$

Exercise 3.1 *It is useful to perform a truncation so as to set the model on an interval* $(0, R)$ *with* $R > 0$. *We introduce a smooth function* $k(x)$ *and a real number* $k_0 > 1$ *with the properties*

$$k'(x) \geq 0, \qquad k(x) = k_0 x \text{ for } x \in \big(0, \frac{R}{2k_0}\big), \qquad k(x) \leq k_0 x, \qquad k(R) = R.$$

Then we consider the equation

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + b(x)\, n(t,x) = \frac{k(x)}{x} k'(x)\, b\big(k(x)\big)\, n\big(t, k(x)\big), \qquad t \geq 0,\ x \in (0,R), \\ n(t, x=0) = 0, \\ n(t=0, x) = n^{init}(x). \end{cases}$$

Show the properties

1. $\displaystyle \frac{d}{dt} \int_0^R n(t,x) dx \leq (k_0 - 1) \int_0^R b(x) n(t,x) dx.$
2. $\displaystyle \frac{d}{dt} \int_0^R x n(t,x) dx \leq \int_0^R n(t,x) dx.$
3. $\displaystyle R \int_0^T n(t,R) dt \leq \int_0^T \int_0^R n(t,x) dx + \int_0^R x n^{init}(x) dx, \quad \forall T > 0.$

3.3 Eigenelements for the growth-fragmentation equation

To simplify the setting and proofs, we consider the simple growth-fragmentation equation

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + b(x)\, n(t,x) = 4\, b(2x)\, n(t, 2x), \\ n(t, x=0) = 0, \\ n(t=0, x) = n^{init}(x). \end{cases} \tag{3.9}$$

The eigenlements are the eigenvalue, eigenfunction and dual eigenfunction $(\lambda_0, \varphi, \psi)$ solution of

$$\begin{cases} \frac{\partial \varphi(x)}{\partial x} + [\lambda_0 + b(x)]\, \varphi(x) = 4\, b(2x)\, \varphi(2x), \qquad x > 0, \\ \varphi(x=0) = 0, \qquad \varphi(x) > 0 \text{ for } x > 0, \qquad \int_0^\infty \varphi(x) dx = 1. \end{cases} \tag{3.10}$$

$$\begin{cases} -\frac{\partial \psi(x)}{\partial x} + [\lambda_0 + b(x)]\, \psi(x) = 2 b(x) \psi(\frac{x}{2}), \qquad x > 0, \\ \psi(x) > 0, \qquad \int_0^\infty \varphi(x) \psi(x) dx = 1. \end{cases} \tag{3.11}$$

In opposition to the renewal equation, there are rarely explicit forms for the solutions and one has to construct them using the Krein-Rutman theorem or variants. The simplest is maybe to write a discrete version based on Exercise 3.1, and pass to the limit after proving uniform bounds. An alternative way is to use the integral representation, see section §8.5.

Let us observe that, by integration of (3.10), one readily obtains

$$\inf_{x\in(0,\infty)} b \le \lambda_0 \le \sup_{x\in(0,\infty)} b(x).$$

The eigenfunction φ is very smooth and decays at infinity as e^{-ax} witn a positivie constant a computed by the dominant terms at infinity $-a+\lambda_0+b_\infty=0$, assuming $b_\infty = \lim\limits_{x\to\infty} b(x)$ exists.

Concerning the dual eigenfunction ψ, it can have algebraic growth x^k with, still equilibriating the dominant factors at infinity, $\lambda_0 + b_\infty = 2^{1-k} b_\infty$.

Exercise 3.2 (Largest mother cell is smaller than twice the smallest daughter cell) *Consider the eigenvalue problem for the size structured model*

$$\begin{cases} \frac{\partial}{\partial x}\varphi(x) + (\lambda_0 + b(x))\varphi(x) = 4b(2x)\varphi(2x), & x \ge 0, \\ \\ \varphi(0) = 0, \qquad \varphi(x) \ge 0. \end{cases}$$

Assume that $b(x)$ vansihes for $x \le x_-$ and $x \ge x_+$ with $\frac{x_+}{2} \le x_-$.
1. Write a system of two functions $\varphi_-(x)$ for $\frac{x_-}{2} \le x \le x_-$, and $\varphi_+(x)$ for $x_- \le x \le x_+$.
2. Show that $\varphi_-(\frac{x_-}{2}) = 0$ and compute $\varphi_-(x_-)$ as a function of $\varphi_+(x)$.
3. Show that $\varphi_+(x)$ satisfies a renewal equation.
4. Prove that there is a unique possible λ_0 in (3.10) and discuss the sign of λ_0 depending on the value $\int_{x_-}^{x_+} b(z)dz$.
5. Give a condition on x_-, x_+ and b such that $\int \varphi < \infty$.
Hint: in 4, it depens on $2e^{\int_{x_-}^{x_+} b(z)dz}$.

3.4 An example: b constant

For b constant a solution is known explicitly, see [1].

Lemma 3.1 *For $b(x) \equiv b$ a constant, the solution $(\lambda_0, \varphi(x), \psi(x))$ of (3.10)–(3.11) is given by*

$$\lambda_0 = b, \qquad \psi(x) \equiv 1, \qquad \varphi(x) = \bar\varphi \sum_{n=0}^{\infty} (-1)^n \, \alpha_n \, e^{-2^{n+1}bx}, \tag{3.12}$$

with $\alpha_0 = 1$, $\alpha_n = \frac{2}{2^n-1}\,\alpha_{n-1}$ and $\bar\varphi > 0$ an appropriate normalization constant.

This function is depicted in Figure 3.1.

Proof. The statement on λ_0 and ψ is obvious and we only consider the construction on φ being given that $\lambda_0 = b$.

We firstly prove that the equation is satisfied. We have

$$\begin{aligned}
\tfrac{\partial}{\partial x}\varphi(x) &= \bar{\varphi}\sum_{n=0}^{\infty}(-1)^n\alpha_n 2^{n+1}be^{-2^{n+1}bx}\\
&= -2b\varphi + \bar{\varphi}\sum_{n=0}^{\infty}(-1)^n\alpha_n 2(2^n-1)be^{-2^{n+1}bx}\\
&= -2b\varphi + 2\bar{\varphi}\sum_{n=1}^{\infty}(-1)^n 2\alpha_{n-1}be^{-2^n b2x}\\
&= -2b\varphi + 4b\varphi(2x).
\end{aligned}$$

Secondly, we prove that $\varphi(0) = 0$. We have, for $n \geq 2$,

$$\alpha_n = \frac{2^n}{(2^n-1)\dots(2^1-1)} = \frac{1}{(2^{n-1}-1)\dots(2^1-1)} + \frac{1}{(2^n-1)\dots(2^1-1)}$$

and $\alpha_0 = 1$, $\alpha_1 = 2$, so that

$$\alpha_0 - \alpha_1 = 1 - \frac{2}{2-1} = \frac{-1}{2^1-1}, \quad \alpha_0 - \alpha_1 + \alpha_2 = \frac{1}{(2^2-1)(2^1-1)}.$$

One can readily check by induction that

$$\sum_{n=0}^{k}(-1)^n\ \alpha_n = \frac{(-1)^k}{(2^k-1)\dots(2^1-1)},$$

which proves the result as $k \to \infty$.

Thirdly, we prove the positivity of $\varphi(x)$. Multiplying (3.10) by $\text{sgn}(\varphi(x))$ we obtain

$$\frac{\partial}{\partial x}|\varphi(x)| + 2b|\varphi(x)| = 4b\varphi(2x)\ \text{sgn}(\varphi(x)).$$

After integration over the half line $x \geq 0$, we find

$$2b\int_0^\infty |\varphi(x)|dx = 4b\int_0^\infty \varphi(2x)\text{sgn}\big(\varphi(x)\big)dx.$$

Thus, dividing by $2b$ and changing variable $y = 2x$ in the second integral, we obtain

$$\int_0^\infty |\varphi(x)|dx = \int_0^\infty \varphi(y)\text{sgn}\big(\varphi(\frac{y}{2})\big)dy.$$

This proves that $\text{sgn}(\varphi(x)) = \text{sgn}\big(\varphi(\frac{x}{2})\big)$ for all $x > 0$.

On the other hand the series (3.12) defining φ is alternate for, say, $2bx \geq 1$, and thus $\varphi(x) > 0$ for large x, combined with the above sign property we conclude the positivity of φ. □

Uniqueness of the positive solution follows from the generalized relative entropy inequality.

Exercise 3.3 *For the following growth-fragmentation equation, with $k \in \mathbb{N}^*$,*

$$\begin{cases} \frac{\partial}{\partial x}\varphi(x) + kb\,\varphi(x) = k^2 b\,\varphi(kx), \quad x \geq 0, \\ \varphi(0) = 0, \end{cases}$$

determine the solution with a formula analogous to (3.12).

Exercise 3.4 *Read the general method to build the solution for a large class of growth-fragmentation equations in [16], §6.3.*

3.5 Generalized relative entropy for the growth-fragmentation equation

The GRE inequalities can be written for equation (3.9). We use again the notation $\widetilde{n} = e^{-\lambda_0 t} n$, a function that solves

$$\begin{cases} \frac{\partial}{\partial t}\widetilde{n}(t,x) + \frac{\partial}{\partial x}\widetilde{n}(t,x) + \big(\lambda_0 + b(x)\big)\,\widetilde{n}(t,x) = 4b(2x)\,\widetilde{n}(t,2x), \quad t > 0,\ x \geq 0, \\ \widetilde{n}(t, x=0) = 0, \quad t > 0, \\ \widetilde{n}(0,x) = n^{init}(x) \geq 0. \end{cases} \tag{3.13}$$

Theorem 3.1 *Assume there exist eigenelements $(\lambda_0, \varphi, \psi)$ solution to (3.10)–(3.11), then for all convex function $H : \mathbb{R} \to \mathbb{R}$ with $H(0) = 0$, we have,*

$$\frac{d}{dt}\int H\left(\frac{\widetilde{n}(t,x)}{\varphi(x)}\right)\varphi(x)\,\psi(x)dx \leq -D_H(t) \leq 0, \qquad \forall t > 0,$$

where the entropy dissipation $D_H(t) \geq 0$ is given by

$$D_H(t) = 4\int \varphi(2x)b(2x)\psi(x)\ \Big[H\Big(\frac{\widetilde{n}(t,2x)}{\varphi(2x)}\Big) - H\Big(\frac{\widetilde{n}(t,x)}{\varphi(x)}\Big) \\ -H'\Big(\frac{\widetilde{n}(t,x)}{\varphi(x)}\Big)\Big(\frac{\widetilde{n}(t,2x)}{\varphi(2x)} - \frac{\widetilde{n}(t,x)}{\varphi(x)}\Big)\Big]dx \geq 0.$$

Proof. On the one hand, using (3.13), we have

$$\frac{\partial}{\partial t}\frac{\widetilde{n}(t,x)}{\varphi(x)} + \frac{\partial}{\partial x}\frac{\widetilde{n}(t,x)}{\varphi(x)} = 4b(2x)\frac{\varphi(2x)}{\varphi(x)}\Big[\frac{\widetilde{n}(t,2x)}{\varphi(2x)} - \frac{\widetilde{n}(t,x)}{\varphi(x)}\Big],$$

$$\frac{\partial}{\partial t}H\left(\frac{\widetilde{n}(t,x)}{\varphi(x)}\right) + \frac{\partial}{\partial x}H\left(\frac{\widetilde{n}(t,x)}{\varphi(x)}\right) = 4b(2x)\frac{\varphi(2x)}{\varphi(x)}H'\left(\frac{\widetilde{n}(t,x)}{\varphi(x)}\right)\Big[\frac{\widetilde{n}(t,2x)}{\varphi(2x)} - \frac{\widetilde{n}(t,x)}{\varphi(x)}\Big].$$

On the other hand

$$\frac{\partial}{\partial x}\left(\varphi(x)\psi(x)\right) = 4\psi(x)b(2x)\varphi(2x) - 2\varphi(x)b(x)\psi(\frac{x}{2}).$$

Therefore

$$\frac{\partial}{\partial t}\left[\varphi(x)\psi(x)H\left(\frac{\widetilde{n}(t,x)}{\varphi(x)}\right)\right] + \frac{\partial}{\partial x}\left[\varphi(x)\psi(x)H\left(\frac{\widetilde{n}(t,x)}{\varphi(x)}\right)\right]$$

$$= 4b(2x)\varphi(2x)\psi(x)H'\left(\frac{\widetilde{n}(t,x)}{\varphi(x)}\right)\left[\frac{\widetilde{n}(t,2x)}{\varphi(2x)} - \frac{\widetilde{n}(t,x)}{\varphi(x)}\right]$$

$$+ \left[4\psi(x)b(2x)\varphi(2x) - 2\varphi(x)b(x)\psi(\tfrac{x}{2})\right] H\left(\frac{\widetilde{n}(t,x)}{\varphi(x)}\right).$$

After integration in x we arrive at

$$\frac{d}{dt}\int_0^\infty \varphi\psi H\left(\frac{\widetilde{n}(t,x)}{\varphi(x)}\right) = \int_0^\infty 4\psi(x)B(2x)\varphi(2x)H'\left(\frac{\widetilde{n}(t,x)}{\varphi(x)}\right)\left[\frac{\widetilde{n}(t,2x)}{\varphi(2x)} - \frac{\widetilde{n}(t,x)}{\varphi(x)}\right]$$

$$+\int_0^\infty 4\psi(x)B(2x)\varphi(2x)\left[H\left(\frac{\widetilde{n}(t,x)}{\varphi(x)}\right) - H\left(\frac{\widetilde{n}(t,2x)}{\varphi(2x)}\right)\right]dx.$$

This is exactly the announced result. □

Exercise 3.5 *Construct many functions such that $h(2x) = h(x)$ and deduce that the Poincaré inequality cannot hold true for the entropy/entropy dissipation of Theorem 3.1 .*

Exercise 3.6 *For the equation (3.2) with $\kappa(x,y) = 0$ when $y < x$, we assume there are eigenelements $(\lambda_0, \varphi, \psi)$ which behave 'nicely'*
Find the equation on $u(t,x) = \frac{\widetilde{n}}{\varphi}$.
Write the entropy dissipation D_H.
Solution.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \int b(x)\kappa(x,y)\frac{\varphi(y)}{\varphi(x)}[u(y) - u(x)]dy,$$

$$D_H = 2\iint \psi(x)\varphi(y)b(y)\kappa(x,y)\left[H\left(\frac{\widetilde{n}(y,t)}{\varphi(y)}\right) - H\left(\frac{\widetilde{n}(x,t)}{\varphi(x)}\right) - H'\left(\frac{\widetilde{n}(x,t)}{\varphi(x)}\right)\left(\frac{\widetilde{n}(y,t)}{\varphi(y)} - \frac{\widetilde{n}(x,t)}{\varphi(x)}\right)\right]dxdy$$

3.6 Exponential decay (Poincaré inequality)

We recall that $\int_0^\infty \psi(x)n(t,x)dx = e^{\lambda_0 t}\int_0^\infty n^{init}(y)\psi(y)dy$ and we assume that the eigenelements exist, normalized by $\int_0^\infty \psi(x)\varphi(x)dx = 1$. We set again

$$\widetilde{n}(t,x) = n(t,x)e^{-\lambda_0 t}, \qquad h(t,x) = \widetilde{n}(t,x) - \varphi(x)\int_0^\infty n^{init}(y)\psi(y)dy.$$

Using a Poincaré inequality, which requires strong assumptions on the fragmentation rate $\kappa(x,y)$, we prove that $h(t,x)$ decays exponentially to 0, therefore justifying that φ is the Stable Steady Size Distribution. The framework is the general form (3.2), (3.8). The hypothesese are that there are positive constants such that

$$b(\cdot) \geq b_m > 0, \qquad \kappa(x,y) = \frac{1}{y}K\left(\frac{x}{y}\right), \qquad K(\cdot) \geq 0 \tag{3.14}$$

and there is a constant $\lambda_1 > 0$ such that

$$\lambda_1 \, \varphi(x) \, \psi(y) \leq b(y)\kappa(x,y) \qquad \forall x, \, y > 0. \tag{3.15}$$

Assumption (3.15) means that for y large $\psi(y)$ decays at least as $\frac{1}{y}$ which is possible when b_∞ is small (see Section §3.3). However, the following theorem is based on [4] where extensions can be found.

Theorem 3.2 *Under the asumptions (3.14), (3.15), the quadratic Poincaré inequality holds, for functions $h(x)$ such that $\int_0^\infty \psi(x)h(x)dx = 0$,*

$$\iint \psi(x)\varphi(y)b(y)\kappa(x,y) \left| \frac{h(y)}{\varphi(y)} - \frac{h(x)}{\varphi(x)} \right|^2 dxdy \geq \lambda_1 \int_0^\infty \psi(x)\varphi(x) \Big| \frac{h(x)}{\varphi(x)} \Big|^2 dx.$$

Consequently, we have the exponential decay

$$\frac{d}{dt} \int \int_0^\infty \psi(x)\varphi(x) \Big| \frac{h(t,x)}{\varphi(x)} \Big|^2 dx \leq e^{-2\lambda_1 t} \int_0^\infty \psi(x)\varphi(x) \Big| \frac{h^{init}(x)}{\varphi(x)} \Big|^2 dx.$$

Proof. We write

$$\begin{aligned}
&\iint \psi(x)\varphi(y)b(y)\kappa(x,y) \left| \frac{h(y)}{\varphi(y)} - \frac{h(x)}{\varphi(x)} \right|^2 dxdy \\
&\qquad \geq \lambda_1 \iint_{x \leq y} \psi(x)\varphi(y)\varphi(x)\psi(y) \left| \frac{h(y)}{\varphi(y)} - \frac{h(x)}{\varphi(x)} \right|^2 dxdy \\
&\qquad \geq \lambda_1 \iint \psi(x)\varphi(y)\varphi(x)\psi(y) \left| \frac{h(y)}{\varphi(y)} - \frac{h(x)}{\varphi(x)} \right|^2 dxdy \\
&\qquad = \frac{\lambda_1}{2} \iint \psi(x)\varphi(x)\psi(y)\varphi(y) \left[\left| \frac{h(y)}{\varphi(y)} \right|^2 + \left| \frac{h(x)}{\varphi(x)} \right|^2 - 2 \frac{h(y)}{\varphi(y)} \frac{h(x)}{\varphi(x)} \right] dxdy \\
&\qquad = \frac{\lambda_1}{2} \iint \psi(x)\varphi(x)\psi(y)\varphi(y) \left[\left| \frac{h(y)}{\varphi(y)} \right|^2 + \left| \frac{h(x)}{\varphi(x)} \right|^2 \right] dxdy
\end{aligned}$$

where we have used the condition $\int_0^\infty \psi(x)h(x)dx = 0$. Since $\int_0^\infty \psi(x)\varphi(x)dx = 1$, we conclude

$$\iint \psi(x)\varphi(y)b(y)\kappa(x,y) \left| \frac{h(y)}{\varphi(y)} - \frac{h(x)}{\varphi(x)} \right|^2 dxdy \geq \lambda_1 \int_0^\infty \psi(x)\varphi(x) \left| \frac{h(x)}{\varphi(x)} \right|^2 dx.$$

This is the Poincaré inequality.

The exponential convergence follows from the GRE principle (see Exercise 3.6) which asserts that

$$\begin{aligned}
\frac{d}{dt} \int_0^\infty \psi(x)\varphi(x) \left| \frac{h(t,x)}{\varphi(x)} \right|^2 dx &= -2 \iint \psi(x)\varphi(y)b(y)\kappa(x,y) \left| \frac{h(y)}{\varphi(y)} - \frac{h(x)}{\varphi(x)} \right|^2 dxdy \\
&\leq -2\lambda_1 \int_0^\infty \psi(x)\varphi(x) \left| \frac{h(t,x)}{\varphi(x)} \right|^2 dx.
\end{aligned}$$

□

3.7 Exponential decay (method of the integral eq.)

We give a direct exponential rate of convergence for the case $b(x) \equiv b$ (constant), The following theorem is based on [20] where one can also find an extension of the method to non constant division rates b.

Theorem 3.3 *Assume $b(x) \equiv b$ a constant, i.e., $\lambda_0 = b$, $\psi = 1$, then solutions of (3.13) satisfy*

$$\|h(t,x)\|_{L^1(\mathbb{R}^+)} \leq e^{-bt}\big[\|h^0(x)\|_{L^1(\mathbb{R}^+)} + 6b\|H^0\|_{L^1(\mathbb{R}^+)}\big], \tag{3.16}$$

with

$$H^0(x) = \int_0^x h^0(y)\, dy \to 0 \quad \textit{as} \quad x \to \infty.$$

Proof. We set

$$H(t,x) = \int_0^x h(t,y)dy,$$

and notice that $H(t,) \to 0$ ad $x \to \infty$. The functions h and H satisfy

$$\begin{cases} \dfrac{\partial}{\partial t}h(t,x) + \dfrac{\partial}{\partial x}h(t,x) + 2bh(t,x) = 4bh(t,2x), \quad t > 0,\ x \geq 0, \\ h(t,x=0) = 0, \quad \displaystyle\int_0^\infty h(t,x)dx = 0, \qquad \forall t > 0, \end{cases} \tag{3.17}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t}H(t,x) + \dfrac{\partial}{\partial x}H(t,x) + 2bH(t,x) = 2bH(t,2x), \quad t > 0,\ x \geq 0, \\ H(t,x=0) = 0, \quad H(t,\infty) = 0, \qquad \forall t > 0. \end{cases} \tag{3.18}$$

First step. We begin with a study of H. We have

$$\frac{\partial}{\partial t}[H(t,x)e^{bt}] + \frac{\partial}{\partial x}[H(t,x)e^{bt}] + b[H(t,x)e^{bt}] = 2b[H(t,2x)e^{bt}],$$

and thus

$$\frac{\partial}{\partial t}|H(t,x)e^{bt}| + \frac{\partial}{\partial x}|H(t,x)e^{bt}| + b|H(t,x)e^{bt}| \leq 2b|H(t,2x)e^{bt}|.$$

We find after integration in x, using that H vanishes at infinity that

$$\frac{d}{dt}\int_0^\infty |H(t,x)e^{bt}|dx \leq 0, \qquad \int_0^\infty |H(t,x)|dx \leq e^{-bt}\int_0^\infty |H^0(x)|dx. \tag{3.19}$$

Second step. We work on $K(t,x) = \frac{\partial}{\partial t}H(t,x)$. We have

$$\begin{cases} \dfrac{\partial}{\partial t}K(t,x) + \dfrac{\partial}{\partial x}K(t,x) + 2bK(t,x) = 2bK(t,2x), \quad t > 0,\ x \geq 0, \\ K(t,x=0) = 0, \quad K(t,\infty) = 0, \qquad \forall t > 0. \end{cases} \tag{3.20}$$

Therefore, as in the first step, since

$$K^0(x) = -h^0(x) - 2bH^0(x) + 2bH^0(2x),$$

we deduce that

$$\begin{aligned} \int_0^\infty |K(t,x)|dx &\leq e^{-bt} \int_0^\infty |K^0(x)|dx \\ &\leq e^{-bt} \int_0^\infty \big[|h^0(x)| + 2b|H^0(x)| + 2b|H^0(2x))|\big]dx \\ &= e^{-bt} \int_0^\infty \big[|h^0(x)| + 3b|H^0(x)|\big]dx. \end{aligned} \tag{3.21}$$

Third step. We deduce the time decay of h from this time decay property of H. Indeed, we compute from (3.18)

$$h(t,x) = \frac{\partial}{\partial x} H = -\frac{\partial}{\partial t} H(t,x) - 2bH(t,x) + 2bH(t,2x),$$

and thus

$$\begin{aligned} \int_0^\infty |h(t,x)|dx &\leq \int_0^\infty |K(t,x)|dx + 3b \int_0^\infty |H(t,x)|dx \\ &\leq e^{-bt} \left\{ \int_0^\infty \big[|h^0(x)| + 6b \int_0^\infty |H^0(x)|dx \right\}. \end{aligned}$$

From this, we directly deduce the estimate of the theorem. □

3.8 Singular case: multiple dominant eigenvalues

For the model with growth rate proportional to the size of cells, a singular phenomena occurs related to multiple eigenvalues with the minimal real part. This is well explained in [3] for a wide class of models including

$$\frac{\partial}{\partial t} n(x,t) + \frac{\partial}{\partial x}\big(xn(x,t)\big) + b(x)n(x,t) = 4b(2x)\, n(2x,t). \tag{3.22}$$

There is a set of principal eigenelements $(\lambda_0, \varphi_0, \psi_0)$ and the dual equation is

$$\lambda\psi(x) - x\psi'(x) + b(x)\psi(x) = 2b(x)\psi(\frac{x}{2}), \qquad \lambda_0 = 1, \quad \psi_0(x) = x.$$

However there are also other eigenelements for $k \in \mathbb{Z}$

$$\lambda_k = 1 + \frac{2i\pi k}{\ln 2}, \qquad \varphi_k(x) = e^{\frac{2i\pi k}{\ln 2}} \varphi_0(x), \qquad \psi_k(x) = x^{\lambda_k}.$$

In this situation, using the GRE principle, it remains that, with $n^{init}(x) = \sum_{k=0}^{\infty} \alpha_k \varphi_k(x)$, one has

$$n(t,x)e^{-t} - \sum_{k\in\mathbb{Z}} \alpha_k \varphi_k(x) e^{\frac{2i\pi k}{\ln 2} t} \to 0 \qquad \text{as } t \to \infty.$$

Chapter 4

Generalized relative entropy for other equations

4.1 The difficulty for L^p norms

We now explain the notion of Generalized Relative Entropy on continuous models. We begin with the most classical equation, namely the parabolic equation for the unknown $n(t,x)$,

$$\frac{\partial n}{\partial t} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(a_{ij}\frac{\partial n}{\partial x_j}\big) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\big(b_i n\big) + dn = 0, \qquad x \in \mathbb{R}^d, \tag{4.1}$$

where the coefficients depend on t and x, $d \equiv d(t,x)$ (no sign assumed), $b_i \equiv b_i(t,x)$, and the symmetric matrix $A(t,x) = \big(a_{ij}(t,x)\big)_{1\leq i,j\leq d}$ satisfies $A(t,x) \geq 0$. We could possibly set the equation on a domain and assume Dirichlet, zero-flux, mixed or periodic boundary conditions and then include them in the above calculation.

Here, it is not obvious to derive a priori bounds on the solution $n(t,x)$, by opposition to the case $A \geq \nu Id > 0$, $b_i \equiv 0$, $d(x) \geq 0$ where we have, multiplying the equation by $n|n|^{p-2}$ with $p > 1$,

$$\frac{d}{dt}\int \frac{|n(t,x)|^p}{p}dx + \frac{4\nu(p-1)}{p^2}\int |\nabla n^{p/2}|^2 dx \leq 0.$$

Indeed the only remarkable property of (4.1) is the mass conservation and L^1 contraction principle

$$\frac{d}{dt}\int n(t,x)dx + \int d(t,x)n(t,x)dx = 0,$$

$$\frac{d}{dt}\int \big|n(t,x)\big|dx + \int d(t,x)\big|n(t,x)\big|dx \leq 0.$$

However, the conservative Fokker-Planck equation is very standard when $d = 0$, $A = Id$, $b = -\nabla V$ for some convex potential with enough growth at infinity

$$\frac{\partial n}{\partial t} - \Delta n - \operatorname{div}(\nabla V\ n) = 0.$$

Then, the prinicpal eigenvalue is $\lambda_0 = 0$ and the steady state is $\varphi = e^{-V}$, the dual eigenfunction is $\psi \equiv 1$ and the relative entropy is a standard object

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi H\left(\frac{n}{\varphi}\right) dx = -\int_{\mathbb{R}^d} \varphi(x) H''\big(\frac{n(t,x)}{\varphi(x)}\big) \big|\nabla \frac{n(t,x)}{\varphi(x)}\big|^2 dx.$$

In particulat the *log* entropy $\int_{\mathbb{R}^d} n \ln\left(\frac{n}{\varphi}\right) dx$ plays an important role for logarithmic Sobolev inequalities.

Also the related Poincaré inequality has been widely studied and holds for V with quadratic growth. There is a constant $\nu > 0$ such that, when $\int_{\mathbb{R}^d} \varphi(x) u(x) dx = 0$, it holds

$$\nu \int_{\mathbb{R}^d} \varphi(x) |u(x)|^2 dx \leq \int_{\mathbb{R}^d} \varphi(x) |\nabla u(x)|^2 dx.$$

This is the inequality for the quadratic entropy and u stands for $\frac{n}{\varphi}$. See [13].

4.2 Parabolic eq. with coefficients independent of time

In the case of coefficients independent of time, and depending on the values of $a_{ij}(x)$, $b(x)$ and $d(x)$, the solution can exhibit exponential growth or decay as $t \to \infty$. Therefore, we will assume that 0 is the first eigenvalue and, following the Krein-Rutman theorem (see [6]), we also assume that we can find two functions $\varphi(x) > 0$, $\psi(x) > 0$, such that

$$\begin{cases} -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial \varphi}{\partial x_j}\big) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\big(b_i(x)\varphi\big) + d(x)\varphi = 0, \\ \qquad \varphi(x) > 0, \qquad \int \varphi(x) dx = 1, \end{cases} \tag{4.2}$$

$$\begin{cases} -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial \psi}{\partial x_j}\big) - \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}\psi + d(x)\psi = 0, \\ \qquad \psi(x) > 0, \qquad \int \varphi(x)\psi(x) dx = 1. \end{cases} \tag{4.3}$$

These are the first eigenvectors; φ for the direct problem and ψ for the dual operator. Notice that such eigenelements do not always exist but there are standard examples, namely when $d \equiv 0$, $A = Id$ and there is a potential V such that $b = -\nabla V$. Then, one can readily check that solutions to (4.2)–(4.3) are

$$\varphi = e^{-V} \qquad \psi \equiv 1,$$

when $V(x) \to \infty$ as $|x| \to \infty$ fast enough in order to fulfill the integrability conditions.

The generalized relative entropy property of the parabolic equation (4.1) can be expressed as

Lemma 4.1 *For coefficients independent of t, assume that there exist eigenelements φ, ψ satisfying (4.2)–(4.3). Then for all convex function $H : \mathbb{R} \to \mathbb{R}$, and all solutions n to (4.1) with sufficient decay in x to zero at infinity ($|n^{init}| \leq C\varphi$), we have*

$$\frac{d}{dt} \int \psi(x)\, \varphi(x)\, H\big(\tfrac{n(t,x)}{\varphi(x)}\big) dx$$

$$= - \int \psi\, \varphi\, H''\big(\tfrac{n(t,x)}{\varphi(x)}\big) \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i}\big(\frac{n}{\varphi}\big)\, \frac{\partial}{\partial x_j}\big(\frac{n}{\varphi}\big)\, dx \leq 0.$$

For conservative equations, i.e., $d \equiv 0$, it is usual to take $\psi \equiv 1$, and then the corresponding principle is classical (especially related to stochastic differential equations and Markov processes, [25]).

Proof of Lemma 4.1. We just calculate (leaving the intermediary steps to the reader)

$$\frac{\partial}{\partial t}\big(\frac{n}{\varphi}\big) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big[a_{ij} \frac{\partial}{\partial x_j}\big(\frac{n}{\varphi}\big)\big] + 2\varphi \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i}\big(\frac{n}{\varphi}\big)\, \frac{\partial}{\partial x_j}\big(\frac{1}{\varphi}\big) + b \cdot \nabla\big(\frac{n}{\varphi}\big) = 0.$$

Therefore, for any smooth function H, we arrive at

$$\tfrac{\partial}{\partial t} H\big(\tfrac{n}{\varphi}\big) \quad - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big[a_{ij} \frac{\partial}{\partial x_j} H\big(\frac{n}{\varphi}\big)\big] + 2\varphi \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i} H\big(\frac{n}{\varphi}\big)\, \frac{\partial}{\partial x_j}\big(\frac{1}{\varphi}\big)$$

$$+ b \cdot \nabla H\big(\tfrac{n}{\varphi}\big) + H''\big(\tfrac{n}{\varphi}\big) \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i}\big(\frac{n}{\varphi}\big)\, \frac{\partial}{\partial x_j}\big(\frac{n}{\varphi}\big) = 0.$$

At this stage we can 'undo' the calculation that lead from an equation on n to an equation on n/φ and we arrive at

$$\tfrac{\partial}{\partial t} \varphi H\big(\tfrac{n}{\varphi}\big) \quad - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big[a_{ij} \frac{\partial}{\partial x_j} \varphi H\big(\frac{n}{\varphi}\big)\big] + \varphi H''\big(\frac{n}{\varphi}\big) \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i}\big(\frac{n}{\varphi}\big)\, \frac{\partial}{\partial x_j}\big(\frac{n}{\varphi}\big)$$

$$+ \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\, \big[b_i \varphi H\big(\frac{n}{\varphi}\big)\big] + d\varphi H\big(\frac{n}{\varphi}\big) = 0.$$

Finally, combining it with the equation on ψ, we deduce that

$$\tfrac{\partial}{\partial t} \psi\varphi H\big(\tfrac{n}{\varphi}\big) \quad - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big[\psi a_{ij} \frac{\partial}{\partial x_j} \varphi H\big(\frac{n}{\varphi}\big)\big] + \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big[a_{ij} \varphi H\big(\frac{n}{\varphi}\big) \frac{\partial}{\partial x_j} \psi\big]$$

$$+ \psi\varphi H''\big(\tfrac{n}{\varphi}\big) \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i}\big(\frac{n}{\varphi}\big)\, \frac{\partial}{\partial x_j}\big(\frac{n}{\varphi}\big) = 0.$$

After integration in x (because we have assumed sufficient decay in x to zero at infinity), we arrive at the result stated in Lemma 4.1. □

This lemma can be used as indicated before for a priori estimates, long time convergence to a steady state and we refer to [20, 17, 18, 3] for specific examples.

As far as long time convergence is concerned, we notice that, as in Lemma 8.2, a control of entropy by entropy dissipation is useful for exponential convergence in as $t \to \infty$ as in (8.7). For the quadratic entropy, this follows from the Poincaré inequality

$$\nu \int \psi\varphi \left(\frac{m}{\varphi}\right)^2 \leq 2 \int \psi\varphi \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i}\big(\frac{m}{\varphi}\big)\ \frac{\partial}{\partial x_j}\big(\frac{m}{\varphi}\big), \quad \text{when} \quad \int \psi m = 0.$$

Such inequalities, as well as log-Sobolev inequalities, are classical when $\varphi = e^{-V}$ for a potential $V(x)$ with superlinear growth at infinity ([13]). The change of unknown function to $n\psi$ and $\varphi\psi$ gives the condition $\varphi\psi = e^{-V}$ for $V(x)$ with superlinear growth to ensure the Poincaré inequality. We are not aware of any general condition on d, b and A in this direction.

4.3 Parabolic eq. with time dependent coefficients

In fact the above manipulations are also valid for time dependent coefficients. A situation similar to the Floquet theory and which is therefore useful for periodic coefficients for instance. We now consider solutions to

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t,x) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(a_{ij}(t,x)\frac{\partial \varphi}{\partial x_j}\big) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\big(b_i(t,x)\varphi\big) + d(t,x)\varphi = 0, \\ \varphi(t,x) > 0, \end{cases} \tag{4.4}$$

$$\begin{cases} \frac{\partial}{\partial t}\psi(t,x) - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(a_{ij}(t,x)\frac{\partial \psi}{\partial x_j}\big) - \sum_{i=1}^{d} b_i(t,x)\frac{\partial}{\partial x_i}\psi + d(t,x)\psi = 0, \\ \psi(t,x) > 0. \end{cases} \tag{4.5}$$

Then we have

Lemma 4.2 *For all convex function $H : \mathbb{R} \to \mathbb{R}$, and all solutions n to (4.1) with sufficient decay in x to zero at infinity, we have*

$$\frac{d}{dt} \int \psi(t,x)\ \varphi(t,x)\ H\big(\tfrac{n(t,x)}{\varphi(t,x)}\big)dx$$

$$= -\int \psi\ \varphi\ H''\big(\tfrac{n(t,x)}{\varphi(t,x)}\big) \sum_{i,j=1}^{d} a_{ij}\frac{\partial}{\partial x_i}\big(\frac{n}{\varphi}\big)\ \frac{\partial}{\partial x_j}\big(\frac{n}{\varphi}\big)\ dx \leq 0.$$

Again we leave the proof of this variant to the reader.

4.4 Scattering equations

To exhibit another class of equation where the Generalized Relative Entropy inequality holds true, let us mention the scattering (also called linear Boltzman) equation

$$\frac{\partial}{\partial t} n(t,x) + k_T(x)\, n(t,x) = \int_{\mathbb{R}^d} k(x,y)\; n(t,y)\; dy. \tag{4.6}$$

Here we restrict ourselves to coefficients independent of time for simplicity, but the same inequality holds in the time dependent case as before. We assume that

$$0 \leq k_T(\cdot) \in L^\infty(\mathbb{R}^d), \qquad 0 \leq k(x,y) \in L^1 \cap L^\infty\big(\mathbb{R}^{2d}\big).$$

And we do not make special assumption on the symmetry of the cross-section $k(x,y)$.

Again, changing k_T in $k_T + \lambda$ if necessary in order to have a zero first eigenvalue, we assume that there are solutions $\varphi(x)$ and $\psi(x)$ to the stationary equation and its dual, namely

$$k_T(x)\,\varphi(x) = \int_{\mathbb{R}^d} k(x,y)\; \varphi(y)\; dy, \qquad \varphi(x) > 0. \tag{4.7}$$

$$k_T(x)\,\psi(x) \;= \int_{\mathbb{R}^d} k(y,x)\;\; \psi(y)\; dy, \qquad \psi(x) > 0. \tag{4.8}$$

These two steady state solutions allow us to derive the Generalized Relative Entropy inequality

Lemma 4.3 *With the above notations, we have*

$$\begin{aligned}
\tfrac{\partial}{\partial t} \quad & \big[\psi(x)\; \varphi(x)\; H\big(\tfrac{n(x)}{\varphi(x)}\big)\big] \\
&+ \int_{\mathbb{R}^d} k(x,y)\big[\psi(y)\varphi(x) H\big(\frac{n(t,x)}{\varphi(x)}\big) - \psi(x)\varphi(y) H\big(\frac{n(t,y)}{\varphi(y)}\big)\big] dy \\
&= \int k(x,y)\psi(x)\varphi(y) \Big[H\big(\frac{n(t,x)}{\varphi(x)}\big) - H\big(\frac{n(t,y)}{\varphi(y)}\big) \\
&\qquad\qquad + H'\big(\tfrac{n(t,x)}{\varphi(x)}\big)\big[\tfrac{n(t,y)}{\varphi(y)} - \tfrac{n(t,x)}{\varphi(x)}\big]\Big]\, dy,
\end{aligned}$$

and also (after integration in x),

$$\begin{aligned}
\tfrac{d}{dt} \quad & \int_{\mathbb{R}^d} \big[\psi(x)\; \varphi(x)\; H\big(\frac{n(x)}{\varphi(x)}\big)\big] \\
&= \int k(x,y)\psi(x)\varphi(y) \Big[H\big(\frac{n(t,x)}{\varphi(x)}\big) - H\big(\frac{n(t,y)}{\varphi(y)}\big) \\
&\qquad\qquad + H'\big(\tfrac{n(t,x)}{\varphi(x)}\big)\big[\tfrac{n(t,y)}{\varphi(y)} - \tfrac{n(t,x)}{\varphi(x)}\big]\Big]\, dy \\
&\leq 0.
\end{aligned}$$

Again we leave to the reader the easy computation that leads to this result and just indicate a class of classical examples where φ and ψ can be computed explicitly.

Example 1. We consider the case where the cross-section in the scattering equation is given by

$$k(x,y) = k_1(x)k_2(y).$$

Then we arrive at (up to a multiplicative constant)

$$\varphi(x) = \frac{k_1(x)}{k_T(x)}, \qquad \psi(x) = \frac{k_2(x)}{k_T(x)},$$

and we need the compatibility condition

$$\int_{\mathbb{R}^d} \frac{k_2(x)k_1(x)}{k_T(x)^2} dx = 1.$$

As in the case of the Perron-Frobenius thorem in section §8.1, this means that 0 is the first eigenvalue, a condition that can always be met changing if necessary k_T in $\lambda + k_T$.

Example 2. We consider the more general case where there exists a function $\varphi(x) > 0$ such that the scattering cross-section satisfies the symmetry condition (usually called detailed balance or microreversibility)

$$k(y,x)\varphi(x) = k(x,y)\varphi(y).$$

Then the choice $k_T(y) = \int_{\mathbb{R}^d} k(x,y)dx$ gives the solutions $\varphi(x)$ to (4.7), and $\psi(x) = 1$ to equation (4.8).

Again we conclude on long time convergence and the possibility to prove exponential time decay to the steady state. As in Lemma 8.2, this follows from a control of entropy by entropy dissipation and thus for the quadratic entropy, from the Poincaré inequality

$$\nu \int \psi(x)\varphi(x)\big(\frac{h}{\varphi}\big)^2 dx \le \int_{\mathbb{R}^d} k(y,x)\psi(x)\varphi(y) \left[\frac{h(x)}{\varphi(x)} - \frac{h(y)}{\varphi(y)}\right]^2 dy\, dx,$$

whenever

$$\int_{\mathbb{R}^d} \psi(x)h(x)dx = 0.$$

This is not always true but holds whenever there is a function $\psi > 0$ such that

$$\nu_1 = \int \varphi\psi^2/\psi < \infty, \qquad \nu_2\psi(y)\varphi(x) \le k(x,y), \qquad \nu = (\nu_1\nu_2)^{-1},$$

a condition that is fulfilled for instance if we work on a bounded domain in velocity and k positive (the difficulties in practical examples as cell division equation is that ψ needs not be bounded in unbounded domains and φ can vanish at infinity).

We write, for any function $\psi > 0$, and $\nu_1 = \int \varphi/\psi$,

$$\begin{aligned}\int \psi(x)\varphi(x)\big(\tfrac{h}{\varphi}(x)\big)^2 dx &= \int \psi(x)\varphi(x)\left(\int [\tfrac{h}{\varphi}(x) - \tfrac{h}{\varphi}(y)]\psi(y)\varphi(y)dy\right)^2 dx \\ &\leq \nu_1 \int\int \psi(x)\varphi(x)[\tfrac{h}{\varphi}(x) - \tfrac{h}{\varphi}(y)]^2\psi(y)\varphi(y)dydx \\ &\leq \nu_1\nu_2 \int\int \psi(x)k(x,y)\varphi(y)[\tfrac{h}{\varphi}(x) - \tfrac{h}{\varphi}(y)]^2 dydx\end{aligned}$$

Notice that a large class of the examples above enter this condition but not with the choice $\psi = \psi$.

Chapter 5

Weak solutions of the renewal equation via a semi-discrete approximation

5.1 Distributional solutions

We consider again the renewal equation

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}[g(t,x)n(t,x)] + d(t,x)\, n(t,x) = 0, \quad t \geq 0,\, x \geq 0, \\ g(t,0)\ n(t,x=0) = \int_0^\infty b(t,y)n(t,y)dy, \\ n(t=0,x) = n^{init}(x). \end{cases} \tag{5.1}$$

We assume that

$$0 \leq d \leq d_M < \infty, \quad 0 \leq b \leq b_M < \infty, \tag{5.2}$$

$$g(t,x) \in C^1_{\mathrm{b}}(\mathbb{R}^+ \times \mathbb{R}^+), \quad g(t,0) \geq g_m > 0, \tag{5.3}$$

$$n^{init} \in L^1 \cap L^\infty(\mathbb{R}^+). \tag{5.4}$$

We define the weak solutions (or distributional solutions), as follows:

Definition 5.1 *A function $n \in L^1_{\mathrm{loc}}\big(\mathbb{R}^+ \times \mathbb{R}^+\big)$ satisfies the renewal equation (5.1) in the weak (distribution) sense, if for all $T > 0$ and all* test function $\Psi \in C^1_{\mathrm{comp}}\big([0,T] \times [0,\infty[\big)$ *such that $\psi(T,x) \equiv 0$, we have*

$$\begin{aligned} -\int_0^T \int_0^\infty n(t,x) \quad & \left\{ \frac{\partial \Psi(t,x)}{\partial t} + g(t,x)\frac{\partial \Psi(t,x)}{\partial x} - d(x)\Psi(t,x) + \Psi(t,0)b(t,x) \right\} dx\, dt \\ &= \int_0^\infty n^{init}(x)\Psi^0(x)dx. \end{aligned} \tag{5.5}$$

A motivation for such a definition is that

Theorem 5.1 *Whenever $n \in C^1\big([0,\infty[\times[0,\infty[\big)$ is a classical solution to the renewal equation (5.1), it is also a weak solution.*

Proof. Multiply (5.1) by the test function Ψ and integrate by parts on $[0,T]\times\mathbb{R}^+$. □

It turns out that C^1 regularity is usually too strong for practical purposes. For instance $n(t,x)$ is obviously discontinous if

$$g(0,0)\ n^{init}(0) \neq \int_0^\infty b(x) n^{init}(x) dx.$$

Distributional solutions are the natural way to define weak solutions of PDEs. They are the good concept because they exist and are unique in a wide class for the coefficients and the solution itself. For instance they are well adapted to study how discrete versions of the renewal equation converge to a 'continuous' solution. We prove here the following result in this direction,

Theorem 5.2 (Existence and bounds) *We assume (5.2)–(5.4), then there is a unique weak solution $n \in L^\infty\big(0,T;L^1(\mathbb{R}^+)\big)$, for all $T>0$, to the renewal equation (5.1) and it satisfies*

$$\int_0^\infty |n(t,x)| dx \leq \int_0^\infty |n^{init}(x)| dx\ e^{\|(b-d)_+\|_\infty t} := M(t),$$

$$|n(t,x)| \leq \max\big(\sup_y |n^{init}(y)|, \frac{b_M}{g_m} M(t)\big) e^{(\|\frac{\partial g}{\partial x}\|_\infty - d_m)t}.$$

The end of this Chapter is devoted to the proof of this Theorem. We begin with uniqueness and turn to existence through a discrete version.

5.2 The dual problem

We use the Hilbert Uniqueness Method which is based on the dual equation to (5.1), with a source term $S(t,x)$ on a given time interval $[0,T]$. It is defined, for $x\geq 0,\ t\in[0,T]$, as

$$\begin{cases} -\frac{\partial}{\partial t}\psi(t,x) - g(t,x)\frac{\partial}{\partial x}\psi(t,x) + d(t,x)\,\psi(t,x) = \psi(t,0) b(t,x) + S(t,x), \\ \psi(t=T,x) = 0. \end{cases} \tag{5.6}$$

This problem is backward in t and x, therefore it does not use a boundary condition at $x=0$.

Lemma 5.1 *Assume (5.2)–(5.3) and $d,b \in C^1(\mathbb{R}^+)$, $S \in C^1_{\text{comp}}\big([0,T)\times\mathbb{R}^+\big)$, b with compact support, then there is a unique C^1 solution to the dual equation (5.6). Moreover $\psi(t,x)$ vanishes for $x\geq R>0$ for some R depending on the data. and the bound holds*

$$\sup_{0\leq t\leq T, x\in\mathbb{R}^+} |\psi(t,x)| \leq C(T,b_M,\bar{d})\|S\|_\infty.$$

Proof. We use the method of characteristics based on the solution to the differential system parametrized by the Cauchy data (t,x) which is fixed

$$\begin{cases} \frac{d}{ds}X(s) = g(s,X(s)), \quad t \leq s \leq T, \\ X(t) = x \geq 0. \end{cases}$$

The solutions exist thanks to the Cauchy-Lipschitz theorem and $X(s) \geq 0$ thanks to assumption (5.3). It might be useful to keep in mind that $X(s)$ depends on (t,x) and thus the notation $X(s) \equiv X(s;t,x)$.

Then, setting

$$\widetilde{\psi}(s) = \psi(s,X(s))e^{\int_s^t d(\sigma,X(\sigma))d\sigma}, \quad \widetilde{B}(s) = b(s,X(s))e^{\int_s^t d(\sigma,X(\sigma))d\sigma},$$
$$\widetilde{S}(s) = S(s,X(s))e^{\int_s^t d(\sigma,X(\sigma))d\sigma},$$

we rewrite equation (5.6) as

$$\begin{aligned} \frac{d}{ds}\widetilde{\psi}(s) &= [\frac{\partial}{\partial t}\psi + g\frac{\partial}{\partial x}\psi - d\,\psi]e^{\int_s^t d(\sigma,X(\sigma))d\sigma}\Big|_{(s,X(s))} \\ &= -[\psi(s,0)b(s,X(s)) + S(s,X(s))]e^{\int_s^t d(\sigma,X(\sigma))d\sigma} \\ &= -\psi(s,0)\widetilde{B}(s) - \widetilde{S}(s), \end{aligned}$$

Next, we integrate between $s=t$ and $s=T$, use the Cauchy data at $t=T$ and the idendity $\widetilde{\psi}(t) = \psi(t,x)$, and we obtain

$$\psi(t,x) = \int_t^T [\psi(s,0)\widetilde{B}(s;t,x) + \widetilde{S}(s;t,x)]ds. \tag{5.7}$$

In order to make it more clear, we have recorded that the $\widetilde{\ }$ quantities depend also on (t,x), i.e., $\widetilde{B}(s) = \widetilde{B}(s;t,x)$, $\widetilde{S}(s) = \widetilde{S}(s;t,x)$.

This integral equation can be solved first for $x=0$. Then, equation (5.7) is reduced to the Volterra equation

$$\psi(t,0) = \int_t^T [\psi(s,0)\widetilde{B}(s;t,0) + \widetilde{S}(s;t,0)]ds, \qquad 0 \leq t \leq T$$

which, thanks to the (backward) Cauchy-Lipschitz theorem, has a unique solution that vanishes for $t=T$. By the C^1 regularity of the data, we also have $\psi(t,0) \in C^1$.

Since $\psi(t,0)$ is now known, formula (5.7) gives us the explicit form of the solution for all (t,x). Notice that, in the compact support statement, $\widetilde{\psi}(t,x)$ vanishes for $x \geq R$ where R denotes the size of the support of B and S in x, plus $T\|g\|_\infty$.

The uniform bound on ψ also follows from formula (5.7), and the C^1 regularity of the data shows that $\psi(\cdot,\cdot) \in C^1$. □

5.3 Uniqueness

With the help of the dual problem that we have studied in the previous section, we can prove uniqueness of weak solutions. We use the classical Hilbert Uniqueness Method. The idea is simple: when the coefficients d, b satisfy the assumptions of Lemma 5.1, we can use the solution ψ to (5.6) as a test function in the weak formulation (5.5). For the difference $n = n^2 - n^1$ between two possible solutions n^2, n^1 with the same initial data, we arrive at

$$-\int_0^T \int_0^\infty n(t,x) \quad \left\{\tfrac{\partial\Psi(t,x)}{\partial t} + g(t,x)\tfrac{\partial\Psi(t,x)}{\partial x} - d(x)\Psi(t,x) + \Psi(t,0)b(t,x)\right\} dx\, dt$$

$$= \int_0^\infty n^{init}(x)\Psi^0(x)dx = 0.$$

Taking into account (5.6) , for all $T > 0$ and all functions $S \in C^1_{\text{comp}}$, we arrive at

$$\int_0^T \int_0^\infty n(t,x)S(t,x)dx\, dt = 0,$$

and this implies $n \equiv 0$.

When the coefficient b, d are merely bounded and b does not have compact support, we consider a regularized family $d_p \to d$, $b_p \to b$ where the convergence holds a.e. with uniform bounds and d_p, b_p satisfying the assumptions of Lemma 5.1. Then, for a given function $S \in C^1_{\text{comp}}$, we solve (5.6) and call ψ_p its solution. Inserting it in the solution of weak solutions, we obtain

$$\int_0^T \int_0^\infty n(t,x)S(t,x)dx\, dt = R_p,$$

$$R_p = \int_0^T \int_0^\infty n(t,x)\left\{[d_p - d(x)]\Psi_p(t,x) + \Psi_p(t,0)[b(t,x) - b_p(t,x)]\right\} dx\, dt,$$

and using that ψ_p is uniformly bounded, we deduce that

$$|R_p| \leq C \int_0^T \int_0^\infty |n(t,x)| \left\{|d_p - d(x)| + |b(t,x) - b_p(t,x)|\right\} dx\, dt.$$

Because $n \in L^1([0,T] \times \mathbb{R}^+)$ and because of the uniform bounds on d_p, b_p and the convergence a.e., we deduce by dominated convergence that

$$R_p \xrightarrow[p\to\infty]{} 0.$$

Therefore, we have recovered the idendity $\displaystyle\int_0^T \int_0^\infty n(t,x)S(t,x)dx\ dt = 0$, for all functions $S \in C^1_{\text{comp}}$, and this implies again $n \equiv 0$.

This concludes the uniqueness result stated in the Theorem 5.2. $\square$

Remark 5.1 *The assumption $g \in C^1$ can be relaxed to Sobolev regularity using the Di Perna-Lions theory but we will not go in this extension which we do not use here.*

5.4 A semi-discrete approximation

For proving the existence part of Theorem 5.2, we use a semi-discrete approximation, a problem which has its own interest in view of numerical methods, and pass to the limit. To do so, we need some notations. We fix a grid size h, $0 < h < 1$ and we set

$$x_i = ih, \qquad x_{i+1/2} = (i + 1/2)h, \qquad i \in \mathbb{N}, \tag{5.8}$$

$$d_{i+1/2}(t) = \frac{1}{h}\int_{x_i}^{x_{i+1}} d(t,x)\, dx, \qquad b_{i+1/2}(t) = \frac{1}{h}\int_{x_i}^{x_{i+1}} b(t,x)\, dx, \tag{5.9}$$

$$g_i(t) = g(t, x_i), \qquad i \in \mathbb{N}. \tag{5.10}$$

We may truncate the indices i with some finite number I, such that $x_I = hI \underset{h\to 0}{\longrightarrow} \infty$. This reduces the system (5.11) below to a system of ODEs but the theory also stands for $i \in \mathbb{N}$.

The semi-discrete model is to find a vector function $n \in C^1(\mathbb{R}^+; \mathbb{R}^I)$ solving the differential system, for $0 \leq i \leq I - 1$

$$\begin{cases} h\frac{d}{dt}n_{i+1/2}(t) + g_{i+1}(t)n_{i+1}(t) - g_i(t)n_i(t) + hd_{i+1/2}(t)n_{i+1/2}(t) = 0, \\ g_0(t)\; n_0(t) = h \displaystyle\sum_{0\leq i\leq I-1} b_{i+1/2}(t)\; n_{i+1/2}(t), \\ n_{i+1/2}(0) = n^{init}_{i+1/2} = \frac{1}{h}\displaystyle\int_{x_i}^{x_{i+1}} n^{init}(x)\, dx. \end{cases} \tag{5.11}$$

We use here a standard upwind scheme for values $n_i(t)$ and $1 \leq i \leq I$,

$$n_i(t) = \begin{cases} n_{i-1/2}(t) & \text{for } g_i(t) > 0, \\ n_{i+1/2}(t) & \text{for } g_i(t) < 0. \end{cases} \tag{5.12}$$

Because, here, $g_i(t) > 0$, we only use $n_i(t) = n_{i-1/2}(t)$. The boundary points are special:
- for $i = 0$, $g_0(t) > 0$ by assumption (5.3), and we need a value $n_0(t)$ which has been defined by the second equation of (5.11),
- for $i = I - 1$, $g_I(t) > 0$ and $n_I(t) = n_{I-1/2}(t)$ is well defined.

Exercise 5.1 *For $g \equiv 1$, and using the Euler scheme, $\frac{d}{dt}n_{i+1/2}(t) \approx \frac{1}{h}[n_{i+1/2}(t^{k+1}) - n_{i+1/2}(t^k)]$, the above explicit upwind scheme becomes extremely simple. Write this scheme.*

The analysis we present now follows the traditional two steps of numerical analysis. First, we prove stability (uniform bounds), second we prove consistancy (see Section §5.5).

Theorem 5.3 (A priori estimates) *Assume (5.2)–(5.4). The following estimates hold:*

$$h \sum_{0\leq i\leq I-1} |n_{i+1/2}(t)| \leq h \sum_{0\leq i\leq I-1} |n^{init}_{i+1/2}| e^{\|(b-d)_+\|_\infty t} \leq M(t), \quad \forall t \geq 0, \tag{5.13}$$

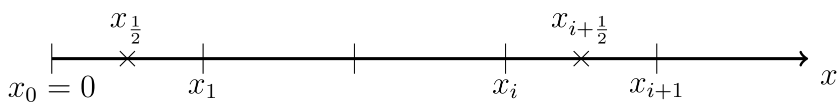

Figure 5.1: Notations for the finite volume grid used in the semi-discrete approximation.

with $M(t) = \|n^{init}\|_1 e^{\|(b-d)_+\|_\infty t}$. *Also, with* $\nu = \|\dfrac{\partial g(x,t)}{\partial x}\|_\infty - d_m$, *we have the* L^∞ *bound*

$$\max_{0\le i\le I-1} |n_{i+1/2}(t)| \le \max\Big(\frac{b_M}{g_m} M(T), \max_{0\le i\le I-1} |n^{init}_{i+1/2}|\Big) e^{\nu t}, \qquad 0 \le t \le T. \tag{5.14}$$

Notice that by linearity, and as a consequence of (5.13), for two different initial data n^{init} and $\tilde{n}^{init}$, the *contraction property* holds

$$h \sum_{0\le i\le I-1} |n_{i+1/2}(t) - \tilde{n}_{i+1/2}(t)| \le h \sum_{0\le i\le I-1} |n^{init}_{i+1/2} - \tilde{n}^{init}_{i+1/2}| e^{\|(b-d)_+\|_\infty t}. \tag{5.15}$$

Exercise 5.2 *For an initial data such that* $\int x\, n^{init}(x)dx < \infty$, *prove that*

$$\frac{d}{dt} h \sum_{0\le i\le I-1} x_i |n_{i+1/2}(t)| \le M(T)\|g\|_\infty.$$

What is the continuous version?

Proof. For the *integrability property* (5.13), we multiply the first equation of (5.11) by $\mathrm{sgn}(n_{i+1/2}(t))$ and obtain, since $n_{i+1}(t) = n_{i+\frac{1}{2}}(t)$,

$$h\frac{d}{dt}|n_{i+1/2}(t)| + g_{i+1}(t)|n_{i+1}(t)| - g_i(t)|n_i(t)| + h d_{i+1/2}(t)|n_{i+1/2}(t)| \le 0.$$

We sum up on i this inequalities and use $n_0(t)$ from (5.11), we find

$$h\frac{d}{dt}\sum_{i=0}^{I-1} |n_{i+1/2}(t)| + g_I(t)|n_I(t)| - g_0(t)|n_0(t)| + h\sum_{i=0}^{I-1} d_{i+1/2}(t)|n_{i+1/2}(t)| \le 0.$$

We deduce

$$\begin{aligned} h\frac{d}{dt}\sum_{i=0}^{I-1} |n_{i+1/2}(t)| + h\sum_{i=0}^{I-1} d_{i+1/2}(t)|n_{i+1/2}(t)| \quad &\le g_0(t)|n_0(t)| \\ &= h\Big|\sum_{i=0}^{I-1} b_{i+1/2}(t) n_{i+1/2}(t)\Big| \\ &\le h\sum_{i=0}^{I-1} b_{i+1/2}(t)|n_{i+1/2}(t)|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} h\tfrac{d}{dt}\sum_{i=0}^{I-1}|n_{i+1/2}(t)| &\leq h\sum_{i=0}^{I-1}[b_{i+1/2}(t)-d_{i+1/2}(t)]|n_{i+1/2}(t)| \\ &\leq h\|(b-d)_+\|_\infty \sum_{i=0}^{I-1}|n_{i+1/2}(t)| \end{aligned}$$

and (5.13) follows.

For the L^∞ *property* stated in (5.14), we consider only the maximum value of $n_{i+1/2}(t)$ and let the reader treat the minimum. We are going to prove that for all $\varepsilon > 0$,

$$\max_{0\leq i\leq I-1} n_{i+1/2}(t) < \max\Big(\frac{b_M}{g_m}M(T)+\varepsilon, \max_{0\leq i\leq I-1}|n^{init}_{i+1/2}|+\varepsilon\Big)e^{(\nu+\varepsilon)t}. \tag{5.16}$$

Then, the result follows taking $\varepsilon \to 0$. To prove this bound, we assume by contradiction that there is a first time $0 < t_0 \leq T$ where it holds as an equality and we let i_0 be an index where the max is achieved, that is

$$n_{i_0+1/2}(t_0) = \max\Big(\frac{b_M}{g_m}M(T)+\varepsilon, \max_{0\leq i\leq I-1}|n^{init}_{i+1/2}|+\varepsilon\Big)e^{(\nu+\varepsilon)t_0}, \tag{5.17}$$

$$n_{i+1/2}(t) < \max\Big(\frac{b_M}{g_m}M(T)+\varepsilon, \max_{0\leq i\leq I-1}|n^{init}_{i+1/2}|+\varepsilon\Big)e^{(\nu+\varepsilon)t}, \quad \forall t<t_0,\ 1\leq i\leq I-1.$$

Consequently, we have

$$\frac{d}{dt}n_{i_0+1/2}(t_0) \geq (\nu+\varepsilon)n_{i_0+1/2}(t_0). \tag{5.18}$$

From the equation (5.11), we know that

$$h\frac{d}{dt}n_{i_0+1/2}(t_0) + g_{i_0+1}(t)n_{i_0+1/2}(t_0) - g_{i_0}(t_0)n_{i_0}(t_0) + hd_{i_0+1/2}(t_0)n_{i_0+1/2}(t_0) = 0.$$

There are two cases.
If $i_0 = 0$, we have two inequalities

$$g_{i_0}(t_0)n_{i_0}(t_0) = h\sum_{0\leq i\leq I-1} b_{i+1/2}(t)\ n_{i+1/2}(t) \leq b_M M(T),$$

$$g_{i_0+1}(t)n_{i_0+1/2}(t_0) \geq g_{i_0}(t)n_{i_0+1/2}(t_0) - h\|\frac{\partial g}{\partial x}\|_\infty n_{i_0+1/2}(t_0).$$

Therefore we find

$$h\frac{d}{dt}n_{i_0+1/2}(t_0) + g_{i_0}(t)\underbrace{\Big[n_{i_0+1/2}(t_0) - \frac{b_M}{g_m}M(T)\Big]}_{>0 \text{ by } (5.17)} \leq h\nu n_{i_0+1/2}(t_0)$$

which contradicts (5.18).

If $i_0 > 0$, since $n_{i_0+1/2}(t_0) \geq n_{i_0}(t_0) = n_{i_0-1/2}(t_0)$, we have

$$h\frac{d}{dt}n_{i_0+1/2}(t_0) + g_{i_0+1}(t)n_{i_0+1/2}(t_0) - g_{i_0}(t_0)n_{i_0+1/2}(t_0) + hd_m n_{i_0+1/2}(t_0) \leq 0,$$

$$h\frac{d}{dt}n_{i_0+1/2}(t_0) - h\|\frac{\partial g}{\partial x}\|_\infty n_{i_0+1/2}(t_0) + hd_m n_{i_0+1/2}(t_0) \leq 0,$$

which, again, contradicts (5.18).

Therefore we have proved the inequality (5.16) for all $\varepsilon > 0$ and the proof of the L^∞ is completed. □

5.5 Limit as $h \to 0$

We need to introduce notations which give a better understanding, in continuous terms, of the semi-discrete inequalities in Theorem 5.3. With the previous notations, we build the piecewise constant functions

$$\begin{cases} n_h^{init}(x) = \sum_{i=0}^{I-1} n_{i+1/2}^{init} \mathbb{1}_{\{x_i<x<x_{i+1}\}}, & n_h(t,x) = \sum_{i=0}^{I-1} n_{i+1/2}(t) \mathbb{1}_{\{x_i<x<x_{i+1}\}}, \\ b_h(t,x) = \sum_{i=0}^{I-1} b_{i+1/2}(t) \mathbb{1}_{\{x_i<x<x_{i+1}\}}, & d_h(t,x) = \sum_{i=0}^{I-1} d_{i+1/2}(t) \mathbb{1}_{\{x_i<x<x_{i+1}\}}. \end{cases} \tag{5.19}$$

We extend these functions by 0 to the half line $x \geq 0$ and we recall that $x_I^h := hI \to \infty$ as $h \to 0$.

We deduce from (5.9) and standard approximation theory, the strong convergence results

$$\begin{cases} d_h(t,x) \xrightarrow[h\to 0]{} d(t,x) \quad \text{a.e. and} \quad 0 \leq d_h \leq d_M, \\ b_h(t,x) \xrightarrow[h\to 0]{} b(t,x) \quad \text{a.e. and} \quad 0 \leq b \leq b_M, \\ n_h^{init}(x) \xrightarrow[h\to 0]{} n^{init}(x) \quad \text{a.e. and in } L^1(\mathbb{R}^+), \quad \|n_h^{init}\|_\infty \leq \|n^{init}\|_\infty. \end{cases} \tag{5.20}$$

Also, from the estimates in Theorem 5.3, we deduce that

Corollary 5.1 *Assume (5.2)–(5.4). Then, for a.e. $t \geq 0$, the function $n_h(t,x) \geq 0$ satisfies*

$$\int_0^\infty |n_h(t,x)|dx \leq M(t),$$

$$\sup_{x>0} |n_h(t,x)| \leq \max\left(\|n^{init}\|_\infty, \frac{b_M}{g_m}M(t)\right)e^{\nu t}, \qquad \nu = \|\frac{\partial g}{\partial x}\|_\infty - d_m.$$

As a consequence, there are functions such that for all $T > 0$,

$$n \in L^\infty\big(0, T; L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)\big),$$

and a sequence $h(k) \to 0$ as $k \to \infty$, so that the function $n_k(t,x) := n_{h(k)}(t,x)$ satisfies for all $T > 0$,

$$n_k \to n \quad L^\infty\big(0, T; L^\infty(\mathbb{R}^+)\big)\text{weak-*}.$$

We are now ready to state our main result

Theorem 5.4 *Assume (5.2)–(5.4). Then, n is the weak solution of (5.1) in the sense of Definition 5.1. Moreover, the full family n_h converges weakly (as above) to n.*

Proof. We have to establish the weak form (5.5). Let $\Psi \in C^2\big([0,T] \times [0,\infty[\big)$ be a test function such that for some $T > 0$ and $R > 0$, $\Psi(T, \cdot) = 0$ and $\Psi(t,x) = 0$ for $x > R$. We define,

$$\Psi_{i+1/2}(t) = \frac{1}{h} \int_{x_i}^{x_{i+1}} \Psi(t,x)dx, \qquad i \in \mathbb{N}. \tag{5.21}$$

We assume that h is small enough so that $x_I^h = Ih > R$. Then, we successively rewrite (5.11) after testing against $\Psi_i(t)$ as:

$$\int_0^T \sum_{i=0}^{I-1} \Big[h \frac{dn_{i+1/2}(t)}{dt} \quad \Psi_{i+1/2}(t) + [g_{i+1}(t)n_{i+1}(t) - g_i(t)n_i(t)]\Psi_{i+1/2}(t)$$
$$+ d_{i+1/2}(t)n_{i+1/2}(t)\Psi_{i+1/2}(t)\Big]dt = 0,$$

$$\int_0^T \sum_{i=0}^{I-1} \Big[- h\frac{d\Psi_{i+1/2}(t)}{dt} \; n_{i+1/2}(t) \quad + g_{i+1}(t)n_{i+1/2}(t)[\Psi_{i+1/2}(t) - \Psi_{i+3/2}(t)]$$
$$- d_{i+1/2}(t)n_{i+1/2}(t)\Psi_{i+1/2}(t)\Big]dt$$
$$= \sum_{i=0}^{I-1} n_{i+1/2}^{init}\Psi_{i+1/2}(t=0) + g_0(t)n_0(t)\Psi_{1/2}(t).$$

We notice that

$$g_0(t)n_0(t) = h\sum_{i=1}^{I-1} b_{i+1/2}(t)n_{i+1/2}(t) = \int_0^\infty b_h(t,x)n_h(t,x)dx.$$

Therefore we find

$$-\int_0^T\int_0^\infty \quad n_h(t,x)\left[\tfrac{d}{dt}\Psi(t,x) - d_h(t,x)\Psi(t,x) + b_h(t,x)\Psi_{1/2}(t)\right]dxdt$$
$$-\int_0^T \sum_{i=0}^{I-1} g_{i+1}(t)n_{i+1/2}(t)[\Psi_{i+3/2}(t) - \Psi_{i+1/2}(t)]dt = \int_0^\infty n_h^{init}(x)\Psi(t=0,x)dx.$$

The flux term, on the right hand side of the quality in the second line, has to be written in a more adapted way

$$h\left[\Psi_{i+3/2}(t)-\Psi_{i+1/2}(t)\right]=\int_{x_i}^{x_{i+1}}[\Psi(x+h)-\Psi(x)]dx=\int_{x_i}^{x_{i+1}}\int_0^h\Psi'(x+y)dxdy,$$

$$\Psi_{i+3/2}(t)-\Psi_{i+1/2}(t)=\int_{x_i}^{x_{i+1}}[\Psi'(x)+O(h)]dx,$$

$$g_{i+1}(t)\left[\Psi_{i+3/2}(t)-\Psi_{i+1/2}(t)\right]=\int_{x_i}^{x_{i+1}}[g(x,t)\Psi'(x)+O(h)]dx,$$

$$\sum_{i=0}^{I-1}g_{i+1}(t)n_{i+1/2}(t)[\Psi_{i+3/2}(t)-\Psi_{i+1/2}(t)]=\int_0^\infty n_h(t,x)[g(x,t)\Psi'(x)+O(h)]dx.$$

Therefore, we arrive at the expression

$$-\int_0^T\int_0^\infty n_h(t,x)\left[\tfrac{d}{dt}\Psi(t,x)+g(x,t)\Psi'(x)-d_h(t,x)\Psi(t,x)+b_h(t,x)\Psi_{1/2}(t)\right]dxdt$$
$$=\int_0^\infty n_h^{init}(x)\Psi(t=0,x)dx+O(h).$$

We can now pass to the weak-strong limit in all terms of this equality for the subsequence n_k, with the convergences we have derived before.

Finally, we prove that the full family converges . Because of the uniqueness result in Section §5.3, any subsequence extracted from n_h converges to the same limit. Therefore the full family converges. □

5.6 The BV bound

We reacall that, because of the possible jump at $(t=0,x=0)$ which will propagate along the characteristics, a BV estimate is the best that can be expected for the renewal equation. We now assume that n^{init} has bounded variation, which may be wrtitten that $\frac{\partial n^{init}}{\partial x}$ is a bounded measure or also

$$\limsup_{h\to 0}\sum_{i=1}^{I-1}|n_{i+1/2}(t)-n_{i-1/2}(t)|:=|n^{init}|_{TV}=\|\frac{\partial n^{init}}{\partial x}\|_{Measure}<\infty.$$

We recall the assumption that $I:=I^h$ satisfies $h\,I\to\infty$. We are going to prove the

Proposition 5.1 *Assume (5.2)–(5.4), $n^{init}\in BV(0,\infty)$ and, to simplify, $b=b(x)$, $d=d(x)$ are independent of t, and $g\equiv 1$, then the total variation of n_h (see (5.19)) is uniformly bounded* *and $n(t,x)$ belongs to $BV\big((0,T)\times(0,\infty)\big)$ with*

$$\begin{cases} \|\frac{\partial n(t,x)}{\partial t}\|_{Measure} \le e^{\|(b-d)_+\|_\infty t}\|\frac{\partial n^{init}(x)}{\partial x} + dn_h^{init}(x)\|_{Measure}, \\ \|\frac{\partial n(t,x)}{\partial x}\|_{Measure} \le e^{\|(b-d)_+\|_\infty t}\|\frac{\partial n^{init}(x)}{\partial x} + dn_h^{init}(x)\|_{Measure} + d_M M(t). \end{cases}$$

Proof.
The semi-discrete version, estimate of the time derivative. Differentiating in t equation (5.11), we find, $n_{-1/2}(t) := n_0(t)$,

$$h\frac{d}{dt}\ \frac{dn_{i+1/2}(t)}{dt} + \frac{dn_{i+1/2}(t)}{dt} - \frac{dn_{i-1/2}(t)}{dt} + hd_{i+1/2}\frac{dn_{i+1/2}(t)}{dt} = 0,$$

$$h\frac{d}{dt}\ |\frac{dn_{i+1/2}(t)}{dt}| + |\frac{dn_{i+1/2}(t)}{dt}| - |\frac{dn_{i-1/2}(t)}{dt}| + hd_{i+1/2}|\frac{dn_{i+1/2}(t)}{dt}| \le 0.$$

$$\begin{aligned} h\frac{d}{dt}\sum_{i=0}^{I-1}|\frac{dn_{i+1/2}(t)}{dt}| + h\sum_{i=0}^{I-1} d_{i+1/2}|\frac{dn_{i+1/2}(t)}{dt}| \quad &\le |\frac{dn_0(t)}{dt}| \\ &= |h\sum_{i=0}^{I-1} b_{i+1/2}\frac{dn_{i+1/2}(t)}{dt}| \end{aligned}$$

We conclude that

$$h\frac{d}{dt}\sum_{i=0}^{I-1}|\frac{dn_{i+1/2}(t)}{dt}| + h\sum_{i=0}^{I-1}[d_{i+1/2} - b_{i+1/2}]\,|\frac{dn_{i+1/2}(t)}{dt}| \le 0,$$

that is, thanks to th eGronwall lemma,

$$h\sum_{i=0}^{I-1}|\frac{dn_{i+1/2}(t)}{dt}| \le h\sum_{i=0}^{I-1}|\frac{dn_{i+1/2}(0)}{dt}|e^{\|(b-d)_+\|_\infty t},$$

Finally, using equation (5.11) at $t=0$, we arrive at

$$h\sum_{i=0}^{I-1}|\frac{dn_{i+1/2}(t)}{dt}| \le e^{\|(b-d)_+\|_\infty t}\sum_{i=0}^{I-1}\big|n_{i+1/2}^{init} - n_{i-1/2}^{init} + d_{i+1/2}n_{i+1/2}^{init}\big| := K_h(t) \tag{5.22}$$

Notice that, on the right hand side, the term $i=0$ contains the possible initial jump at the boundary.

The semi-discrete version, estimate of the x derivative. Equation (5.11) now gives us the relation

$$n_{i+1/2}(t) - n_{i-1/2}(t) = -h\frac{d}{dt}n_{i+1/2}(t) - hd_{i+1/2}(t)n_{i+1/2}(t).$$

That is the discrete estimate is

$$\sum_{i=1}^{I-1} |n_{i+1/2}(t) - n_{i-1/2}(t)| \leq K_h(t) + d_M M(t). \tag{5.23}$$

With our previous notations, these estimates can also be stated, for all $t \geq 0$, as

$$\begin{cases} \|\dfrac{\partial n_h(t,x)}{\partial t}\|_{Measure} \leq e^{\sup(d-b)\,t}\|\dfrac{\partial n_h^{init}(x)}{\partial x} + d_h n_h^{init}(x)\|_{Measure}, \\ \|\dfrac{\partial n_h(t,x)}{\partial x}\|_{Measure} \leq e^{\sup(d-b)\,t}\|\dfrac{\partial n_h^{init}(x)}{\partial x} + d_h n_h^{init}(x)\|_{Measure} + d_M M(t). \end{cases}$$

At the limit we find the same bounds for the solution of the renewal equation with initial data in BV as announced in Theorem 5.1. □

Exercise 5.3 *Consider the renewal equation with $b = b(x)$.*
1. Show that

$$\Big|\frac{d}{dt}N(t)\Big| \leq \left[\|\frac{\partial b}{\partial x}\|_\infty + b_M d_M + b_M^2\right] M(t).$$

One can begin with the semi-discrete version.
2. Assume $b(x)$ is uniformly continuous in x, show that

$$\sup_{0\leq t\leq T} \big|N(t+h) - N(t)\big| \to 0, \quad \text{as } h \to 0.$$

Compare with the compactness estimate in Lemma 2.1.
Hint. $M(T)\big[h(b_M d_M + b_M^2) + C\frac{h}{\varepsilon} + \|b \star \varrho_\varepsilon - b\|_\infty\big]$.

Exercise 5.4 *Consider the renewal or growth fragmentation equation with coeficients independent of time. Assume there are eigenelements $(\lambda_0, \varphi, \psi)$. Show that*

$$\int_0^\infty \psi(x)\left|\frac{\partial n(x,t)}{\partial t}\right| dx \leq e^{\lambda_0 t}\int_0^\infty \psi(x)\left|\frac{\partial n^{init}(x)}{\partial t}\right| dx,$$

$$\sup_{x\in(0,\infty)} \frac{\left|\frac{\partial n(x,t)}{\partial t}\right|}{\varphi(x)} \leq e^{\lambda_0 t} \sup_{x\in(0,\infty)} \frac{\left|\frac{\partial n^{init}(x)}{\partial t}\right|}{\varphi(x)}.$$

Chapter 6

Nonlinear renewal equations

6.1 The Kermack-McKendrick model for epidemic spread

We recall this subject was treated in section §1.3.

6.2 Poincaré inequality and perturbation theory

Local stability of non zero steady states has been well established in a series of papers based on spectral analysis, see [10, 11, 21]. Roughly speaking it is known that stability of the linearized operator implies local nonlinear stability. It is in fact a general feature of PDEs that exponential decay (by Doeblin method or Poincaré inequality for instance) allows to 'insert' small nonlinearities because of the damping term leading to exponential decay. We illustrate this with a specific conservative renewal equation closely related to the elapsed time model for neural networks.

We consider an environmental quantity $I(t)$ which controls both birth and death terms under the form

$$\begin{cases} \frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}n(t,x) + d(x, I(t))n(t,x) = 0, \\ n(t, x = 0) = \int_0^\infty d(x, I(t))n(t,x)dx, \\ I(t) = \int_0^\infty q(x)n(t,x)dx, \end{cases}$$

always with an initial data $n^{init} \geq 0$. The quantity $q(x)$ measures the uptake of individuals of age x on the environment.

Because it is a conservative system, we set

$$M = \int_0^\infty n^{init}(x)dx = \int_0^\infty n(t,x)dx, \quad \forall t \geq 0.$$

We again assume and define

$$0 < d_m \leq d(x, I) \leq d_M < \infty, \qquad 0 < q(x, I) \leq q_M < \infty, \tag{6.1}$$

$$D(x,I)=\int_0^x d(x,I)dx, \qquad \frac{\partial D(x,I)}{\partial I}>0, \qquad \int_0^\infty \frac{\partial D(x,I)}{\partial I} e^{-D(x,I)}dx < \frac{\int_0^\infty e^{-D(x,\bar I)}dx}{I}, \tag{6.2}$$

The assumption on d_m implies various forms of the Poincaré inequality (see Section §2.7).

The steady state plays the role of the eigenfunction and, the steady state with same integral M is determined by

$$\frac{\partial}{\partial x}\varphi(x)+d(x,\bar I)\varphi(x)=0, \qquad \varphi(x)=\varphi(0)e^{-D(x,\bar I)},$$

and the boundary condition comes automatically

$$\varphi(0)=\int_0^\infty d(x,\bar I)\varphi(x)dx=\varphi(0)\int_0^\infty d(x,\bar I)e^{-D(x,\bar I)}dx=\varphi(0)\int_0^\infty \frac{\partial}{\partial x}e^{-D(x,\bar I)}dx.$$

It remains to impose the two relations for the two unknowns $(M,\bar I)$,

$$M=\varphi(0)\int_0^\infty e^{-D(x,\bar I)}dx, \qquad \bar I=\int_0^\infty q(x)\varphi(x)dx=M\frac{\int_0^\infty q(x)e^{-D(x,\bar I)}dx}{\int_0^\infty e^{-D(x,\bar I)}dx}. \tag{6.3}$$

Notice that assumption (6.2) implies the existence and uniqueness of $\bar I$.

Theorem 6.1 *Assume (6.1)–(6.2), and the smallness assumption*

$$2q_M\int_0^\infty \sup_{I>0}\left|\frac{\partial d(x,I)}{\partial I}\right|\varphi(x)dx<d_m. \tag{6.4}$$

Then, as $t\to\infty$, $\int_0^\infty |n(t,x)-\varphi(x)|dx\to 0$ with an exponential rate.

Proof. We compute

$$\frac{\partial(n(t,x)-\varphi(x))}{\partial t}+\frac{\partial(n(t,x)-\varphi(x))}{\partial x}+d(x,I(t))n(t,x)-d(x,\bar I)\varphi(x)=0,$$

$$\frac{\partial|n(t,x)-\varphi(x)|}{\partial t}+\frac{\partial|n(t,x)-\varphi(x)|}{\partial x}+d(x,I(t))|n(t,x)-\varphi(x)|\le|d(x,I(t))-d(x,\bar I)|\varphi(x),$$

$$\begin{aligned}\frac{d}{dt}\int_0^\infty |n(t,x)-\varphi(x)|dx+ &\int_0^\infty d(x,I(t))|n(t,x)-\varphi(x)|\\ &\le\int_0^\infty |d(x,I(t))-d(x,\bar I)|\varphi(x)+|n(t,0)-\varphi(0)|.\end{aligned}$$

We can estimate

$$\begin{aligned}|n(t,0)-\varphi(0)| &=\left|\int_0^\infty[d(x,I(t))n(t,x)-d(x,\bar I)\varphi(x)]dx\right|\\ &\le\left|\int_0^\infty d(x,I(t))[n(t,x)-\varphi(x)]dx\right|+\int_0^\infty|d(x,I(t))-d(x,\bar I)|\varphi(x)dx\\ &=\left|\int_0^\infty[d(x,I(t))-d_m][n(t,x)-\varphi(x)]dx\right|+\int_0^\infty|d(x,I(t))-d(x,\bar I)|\varphi(x)dx\\ &\le\int_0^\infty[d(x,I(t))-d_m]|n(t,x)-\varphi(x)|dx+\int_0^\infty|d(x,I(t))-d(x,\bar I)|\varphi(x)dx,\end{aligned}$$

where we have used assumption (6.1). And we arrive at

$$\begin{aligned}\frac{d}{dt}\int_0^\infty |n(t,x)-\varphi(x)|dx \quad &+ d_m \int_0^\infty |n(t,x)-\varphi(x)|dx \le 2\int_0^\infty |d(x,I(t))-d(x,\bar I)|\varphi(x)dx \\ &\le \quad 2\int_0^\infty \sup_{I>0}|\tfrac{\partial d(x,I)}{\partial I}|\varphi(x)dx\ |I(t)-\bar I| \\ &\le \quad 2\int_0^\infty \sup_{I>0}|\tfrac{\partial d(x,I)}{\partial I}|\varphi(x)dx\ \int_0^\infty q(x)|n(t,x)-\varphi(x)|dx.\end{aligned}$$

We conclude using the smallness condition (6.4) and the Gronwall lemma. □

Exercise 6.1 *Establish the same type of result based on the Doeblin method.*

6.3 Periodic solutions. Linear instability

There is a generic situation where interesting patterns arise, when all the steady states are unstable and solutions remain bounded. In such cases the solutions can only exhibit some pattern. This phenomena was first observed by A. Turing, for parabolic systems, leading to the theory of diffusion driven instabilities. For the renewal equation, a simple nonlinear version can generate periodic solutions. We just assume the birth rate is controled by the total population

$$\begin{cases} \frac{\partial}{\partial t}n + \frac{\partial}{\partial x}n + d(x)n = 0. \qquad t\ge 0,\ x\ge 0, \\ n(x=0,t) = \displaystyle\int_0^\infty b\big(x,I(t)\big)n(x,t)dx, \qquad I(t) = \int_0^\infty q(x)n(t,x)dx. \end{cases} \tag{6.5}$$

The question is thus to find conditions on the birth rate $b(x,I)$ and on the 'environmental control' $q(\cdot)$ so that the above instability conditions are met.

The steady state $n\equiv 0$ is unstable (that means solutions will stay positive, see Section §6.4) if

$$\int_0^\infty b(x,0)e^{-D(x)}dx > 1, \qquad D(x) = \int_0^x d(y)dy. \tag{6.6}$$

Then we assume that the population controls negatively the birth term

$$\frac{\partial b(x,I)}{\partial I} < 0, \qquad \forall x\ge 0,\ I\ge 0. \tag{6.7}$$

And we finally need a control for large I which will follow from the existence of a positive steady state

$$\exists\, \bar I > 0, \quad \text{such that} \quad \int_0^\infty b(x,\bar I)e^{-D(x)}dx = 1. \tag{6.8}$$

Then, the unique steady state is determined by the value $\varphi(0)$

$$\varphi(x) = \varphi(0)e^{-D(x)}, \qquad \bar I = \varphi(0)\int_0^\infty q(x)e^{-D(x)}dx. \tag{6.9}$$

For a large class of choices of $b(x,I)$ and $q(x)$, beyond the perturbation theory of Section §6.2, the steady state $\bar I$ is stable and solutions converge to φ as $t\to\infty$.

However, a counter-intuitive result is

Theorem 6.2 (Linear instability) *For* $0 < d_m \leq d(x) \leq d_M < \infty$, $b(x, I) = \frac{Q}{1+I}\, \mathbb{I}_{\{1-a_1<x<1+a_2\}}$ *and* $q(x) = \mathbb{I}_{\{\frac{1}{2}-a_3<x<\frac{1}{2}+a_4\}}$*, with* Q *large enough, the assumptions* (6.6)-(6.8) *are satisfied. Moreover, the steady state* φ *is unstable and the linearized equation has a solution growing exponentially for* $a_i > 0$*,* $i = 1, ..., 4$ *small enough.*

The numerical solution of this example is depicted in Fig. 6.1.

Proof.
1. Computing the linearized equation. Around the steady state φ, the linearized equation is

$$\begin{cases} \frac{\partial h(t,x)}{\partial t} + \frac{\partial h(t,x)}{\partial x} + d(x)h(t,x) = 0, \\ h(t,0) = \displaystyle\int_0^\infty \big[b(x,\bar{I})h(t,x) + I_h \frac{\partial b(x,\bar{I})}{\partial I}\varphi(x)\big]dx, \qquad I_h(t) = \int q(x)h(t,x)dx. \end{cases}$$

We look for a time exponential solution $h(t,x) = e^{\lambda t}h(x)$, with $\lambda \in \mathbb{C}$. We arrive, after normalization of the corresponding eigenfunction problem, at

$$\begin{cases} \lambda h(x) + \frac{\partial h(x)}{\partial x} + d(x)h(x) = 0, \\ 1 = h(0) = \int_0^\infty [b(x,\bar{I})h(x) + \frac{\partial b(x,\bar{I})}{\partial I}\varphi(x)I_h]dx, \qquad I_h = \int q(x)h(x)dx. \end{cases}$$

The solution can be computed explicitely, and we arrive at the condition

$$h(x) = e^{-D(x)-\lambda x}, \quad 1 = \int_0^\infty b(x,\bar{I})e^{-D(x)-\lambda x}dx + \varphi(0)I_h \int_0^\infty \frac{\partial b(x,\bar{I})}{\partial I}\, e^{-D(x)}dx. \tag{6.10}$$

In view of condition (6.7) and (6.8), it is hard to believe that $\lambda \in (0,\infty)$ can work to fulfill these conditions. However, we can choose $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$. Below, we just take λ purely imaginary and let the reader convince himself that the argument can be extended to $Re(\lambda) > 0$ small enough.

2. How do $\bar{I}$ *and* $\varphi(0)$ *depend on* Q*?* With the choice of $b(x, I)$ and a smaller than $\frac{1}{2}$, we first notice that (6.6) is satisfied for Q large enough, and (6.8)–(6.9) determine $\bar{I}$ and $\varphi(0)$

$$Q\int_{1-a_1}^{1+a_2} e^{-D(x)}dx = 1 + \bar{I}, \qquad \bar{I} = \varphi(0)\int_{\frac{1}{2}-a_3}^{\frac{1}{2}+a_4} e^{-D(x)}dx > 0.$$

- For Q large we find $\bar{I} = O(Q)$, $\varphi(0) = O(Q)$.
- For Q close to verify (6.6) as an equality, $\bar{I}$ and $\varphi(0)$ are close to 0.

3. Verifying equation (6.10)*.* Then, we take $\lambda = 2i\pi$ so that

$$\int_0^\infty b(x,\bar{I})e^{-D(x)-\lambda x}dx = \frac{Q}{1+\bar{I}}\int_{1-a_1}^{1+a_2} e^{-D(x)}\cos(2\pi x)dx > 0,$$

by relating a_2 to a_1 so that the imaginary part vanishes ($a_1 = a_2$ will do for D locally constant). Also we have, by the same token,

$$I_h = \int_{\frac{1}{2}-a_3}^{\frac{1}{2}+a_4} e^{-D(x)} \cos(2\pi x) dx < 0.$$

Condition (6.10) becomes

$$1 = F(Q, a_1, a_2) := \frac{Q}{1+\bar{I}} \int_{1-a_1}^{1+a_2} e^{-D(x)} \cos(2\pi x) dx + |I_h| \frac{\varphi(0)Q}{(1+\bar{I})^2} \int_{1-a_1}^{1+a_2} e^{-D(x)} dx$$

$$F(Q, a) = \frac{\int_{1-a_1}^{1+a_2} e^{-D(x)} \cos(2\pi x) dx}{\int_{1-a_1}^{1+a_2} e^{-D(x)} dx} + \frac{\bar{I}}{1+\bar{I}} \frac{\int_{\frac{1}{2}-a_3}^{\frac{1}{2}+a_4} e^{-D(x)} |\cos(2\pi x)| dx}{\int_{\frac{1}{2}-a_3}^{\frac{1}{2}+a_4} e^{-D(x)} dx}$$

Varying Q, it is possible to verify that we can ensure this condition. Indeed, on the one hand, for Q close to verify (6.6) as an equality, $\bar{I}$ is close to 0 and $F(Q, a_1, a_2) < 1$. On the other hand, for Q large, a_i small, both terms are close to 1.

Therefore, for Q large enough, one can satisfy (6.10) with $\lambda = \mu + i\frac{\pi}{2}$ and $\mu > 0$. $\square$

Exercise 6.2 *Show that the statement of the theorem can be extended to* $b(x, I) = \frac{Q}{1+I} \mathbb{1}_{\{1-b<x<1+b\}}$, $b < \frac{1}{2}$ *and we can also choose* $Re(\lambda) > 0$ *small enough.*

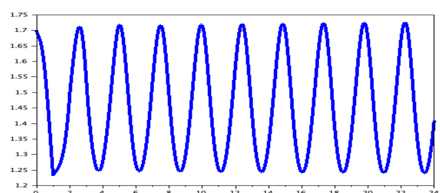

Figure 6.1: PERIODIC SOLUTION OF EQUATION (6.5) WITH DATA FROM THEOREM 6.2. THE QUANTITY $I(t)$ IS SHOWN.

6.4 Non-extinction (persistance)

Consider a nonlinear model

$$\begin{cases} \frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n + d\big(x, I_1(t)\big) n = 0, \qquad x \geq 0, \\ n(t,0) = \int b\big(x, I_2(t)\big) n(t,x) dx, \\ I_j(t) = \int q_j(x) n(t,x) dx, \qquad j = 1, 2, \\ n(t=0, x) = n^{init}(x) \geq 0, \qquad n^{init} \neq 0. \end{cases} \tag{6.11}$$

Since we want to enforce that solutions go away from 0, we assume that the linear problem for $I_j = 0$ has a positive eigenvalue. That is, there are eigenelements $\lambda_0 > 0$, $\varphi > 0$ and $\psi > 0$ such that

$$\begin{cases} \frac{\partial}{\partial x}\varphi + [\lambda_0 + d(x,0)]\varphi = 0, \qquad x \geq 0, \\ \varphi(0) = \displaystyle\int b(x,0)\varphi(x)dx, \qquad \varphi > 0, \\ -\frac{\partial}{\partial x}\psi + [\lambda_0 + d(x,0)]\psi = \psi(0)b(x,0). \end{cases}$$

Technically we also need to assume that there are constants such that

$$0 \leq q_j(x) \leq q_M \psi(x), \qquad j = 1, 2, \tag{6.12}$$

$$d'_I(x, I) \leq K, \qquad b'_I(x, I) \geq -K. \tag{6.13}$$

Theorem 6.3 *With the assumptions* (6.12), (6.13)*, the dynamics* (6.11) *goes uniformly away from* 0

$$\liminf_{t\to\infty} \int n(t,x)\psi(x)dx \geq \frac{\lambda_0}{A}, \qquad A = Kq_M\big(1 + \frac{\psi(0)}{\inf_{(0,\infty)} \Psi}\big).$$

Proof. We write using the conservation law

$$\frac{d}{dt} \quad \int_0^\infty n(t,x)\psi(x)dx = \lambda_0 \int_0^\infty n(t,x)\psi(x)dx$$
$$- \int_0^\infty [d\big(x, I_1(t)\big) - d\big(x, 0\big)]n(t,x)\psi(x)dx - \psi(0)\int_0^\infty [b(x,0) - b\big(x, I_2(t)\big)]n(t,x)dx$$

From assumptions (6.12), (6.13), we control

$$d\big(x, I_1(t)\big) - d\big(x, 0\big) \leq KI_1(t) \leq Kq_M \int_0^\infty n(t,x)\psi(x)dx.$$

Arguing similarly for the term $b(x,0) - b\big(x, I_2(t)\big)$, we find

$$\psi(0)\int_0^\infty [b(x,0) - b\big(x, I_2(t)\big)]n(t,x)dx \leq Kq_M \frac{\psi(0)}{\inf_{(0,\infty)} \Psi} \int_0^\infty n(t,x)\psi(x)dx$$

We conclude that $w(t) = \int_0^\infty n(t,x)\psi(x)dx$ satisfies, with $A = Kq_M\big(1 + \frac{\psi(0)}{\inf_{(0,\infty)} \Psi}\big)$,

$$\frac{d}{dt} w(t,x) \geq \big[\lambda_0 - Aw(t)\big]w(t), \qquad w(0) > 0.$$

This means that $w(t) \geq \min\big(w(0), \frac{\lambda_0}{A}\big)$ and $\liminf_{t\to\infty} w(t) \geq \dfrac{\lambda_0}{A}$ which gives the result. □

Chapter 7

Other linear renewal equations

7.1 The multiple time renewal equation

Several applications use multi-time renewal equations to describe a population density subject to aging or to evolution with time. Recently, the efficiency of tracing softwares has led to evaluate the infected individual as usual in the Kermack-McKendrick system and of secondary infections too (Ferretti et al [**?**]). In the neuroscience, the interpretation is that neurones keep memory of their last discharges in the process of deciding when to fire again. The modeling is based on Hawkes processes as developed in [**?**] .

$$\begin{cases} \frac{\partial}{\partial t}n + \frac{\partial}{\partial s_1}n + \frac{\partial}{\partial s_2}n + p(s_1, s_2)n = 0, \\ n(t, 0, s_1) = \int_{\tau=s_1}^{\infty} p(s_1, \tau)n(t, s_1, \tau)d\tau. \end{cases} \tag{7.1}$$

Our present results on this two-times renewal equation are the existence of a steady state, long time convergence both using the Doeblin method and entropy methods and weakly nonlinear extensions.

The goal of the internship is to extend this model in order to take into account several times, to extend the Doeblin and entropy methods in this framework. An interesting more difficult step is to pas to the limit to an infinite number of past times.

7.2 The renewal equation structured with space

See [**?**]

$$\begin{cases} \varepsilon^2 \frac{\partial}{\partial t}n_\varepsilon(t, a, x) + \frac{\partial}{\partial a}n_\varepsilon + b(a)n_\varepsilon = 0, \qquad t \geq 0,\ a > 0,\ x \in \mathbb{R}^d \\ n_\varepsilon(t, 0, x) = \int_0^\infty \int_{\mathbb{R}^d} k_\varepsilon(x - x')b(a)n_\varepsilon(t, a, x')dadx', \end{cases} \tag{7.2}$$

with $e^{-D} \in L^1(0,\infty)$ and

$$k \geq 0, \quad k(z) = k(|z|), \quad \int_{\mathbb{R}^d} k(z)dz = 1, \quad \int_{\mathbb{R}^d} |z|^2 k(z)dx < \infty, \quad \int_{\mathbb{R}^d} zk(z)dz = 0,$$

$$k_\varepsilon(z) = \frac{1}{\varepsilon^d} k(\frac{z}{\varepsilon}).$$

We set $\rho_\varepsilon(t,x) = \int_0^\infty n_\varepsilon(t,a,x)da$, we find

$$\varepsilon^2 \frac{\partial}{\partial t}\rho_\varepsilon(t,x) + \int_0^\infty b(a)n_\varepsilon(t,a,x)da = \int_0^\infty k_\varepsilon(x-x')b(a)n_\varepsilon(t,a,x')dadx'.$$

It can be rewritten

$$\varepsilon^2 \frac{\partial}{\partial t}\rho_\varepsilon(t,x) = \int_0^\infty\int_{\mathbb{R}^d} k_\varepsilon(x-x')b(a)[n_\varepsilon(t,a,x') - n_\varepsilon(t,a,x)]dadx'.$$

and in the weak form, testing again $\Phi(t,x)$

$$\begin{aligned}
-\int_0^T\int_{\mathbb{R}^d} \rho_\varepsilon(t,x)\frac{\partial}{\partial t}\Phi(t,x)dxdt - \int_{\mathbb{R}^d} \rho_\varepsilon(0,x)\Phi(0,x)dx \\
= \int_0^T\int_0^\infty\int_{\mathbb{R}^{2d}} \frac{k_\varepsilon(x-x')}{\varepsilon^2} b(a)n_\varepsilon(t,a,x')[\Phi(t,x) - \Phi(t,x')]dadx'dxdt \\
= \int_0^T\int_0^\infty\int_{\mathbb{R}^{2d}} \frac{k(z)}{\varepsilon^2} b(a)n_\varepsilon(t,a,x')[\Phi(t,x'+\varepsilon z) - \Phi(t,x')]dadx'dzdt
\end{aligned}$$

where we have used the change of variable $x \mapsto z = \frac{x-x'}{\varepsilon}$.

Expanding the terms in the bracket, we end up with

$$\begin{aligned}
-\int_0^T\int_{\mathbb{R}^d} \rho(t,x)\frac{\partial}{\partial t}\Phi(t,x)dxdt - \int_{\mathbb{R}^d} \rho(0,x)\Phi(0,x)dx \\
= \int_0^T\int_0^\infty\int_{\mathbb{R}^{2d}} b(a)n(t,a,x')k(z)z \otimes zD^2\Phi(t,x')dadx'dzdt.
\end{aligned}$$

And n is identified as

$$\begin{cases} \frac{\partial}{\partial a}n + b(a)n = 0, & t \geq 0,\ a > 0,\ x \in \mathbb{R}^d \\ n(t,0,x) = \int_0^\infty b(a)n(t,a,x)da, \end{cases}$$

7.3 Elapsed time with leaky memory

An elapsed time neurone equation is studied by Fonte and Schmutz [?]. It uses the time a elapsed since the last discharge and a memory variable $x > 0$ which relaxes continuously to 0 but jumps forward at discharges. Here we present a linear version of the system they study.

$$\begin{cases} \frac{\partial}{\partial t} n(t,a,x) + \frac{\partial}{\partial a} n - \frac{\partial}{\partial x}(xn) + b(a,x)n = 0, \qquad t \geq 0,\ a > 0,\ x > 0, \\ n(t,0,x) = \mathbb{1}_{\{x > x_0\}} u'(x) \int_0^\infty b(a,u(x)) n(t,a,u(x)) da, \end{cases} \tag{7.3}$$

Here, for some $x_0 \geq 0$, the mapping $u \in C^1((x_0,\infty);(0,\infty))$ is one-to-one and satisfies $u'(x) > 0$, and thus $u(x_0) = 0$. An example is $u(x) = x - x_0$.

The construction is such that one can readily check the conservation law

$$\frac{d}{dt} \int n(t,a,x) da dx = 0.$$

The mathematical interest lies in conditions for a relevant steady state, i.e., which differs from $N(a)\delta(x)$, with

$$\frac{\partial}{\partial a} N(a) + b(a,0)N(a) = 0, \qquad N(0) = \int_0^\infty b(a,0)N(a)dx.$$

This is a steady state for $x_0 = 0$, $u(0) = 0$ because, $\delta(x) = u'(x)\delta(u(x))$ in that case.

Chapter 8

Perron-Frobenius theory and entropy

8.1 Perron-Frobenius

Pour $x \in \mathbb{R}^d$, on dit que $x \geq 0$ (resp. > 0) si toutes ses coordonnées vérifient $x_i \geq 0$ (resp. > 0), $i = 1, \ldots, d$.

Theorem 8.1 (Théorème de Perron-Frobenius) *Soit $A = (a_{ij}) \in M_{d\times d}(\mathbb{R})$ une matrice à coefficients strictement positifs, $a_{ij} > 0$. On pose*

$$\rho(A) = \sup\{r \geq 0 \quad t.q. \quad \exists x \in \mathbb{R}^d \backslash \{0\},\ x \geq 0,\ A.x \geq r\, x\}.$$

Alors
(i) $\rho(A) > 0$ est valeur propre simple de A associée à un vecteur propre $x^0 > 0$,
(ii) tout autre vecteur propre positif de A est proportionnel à x^0,
(iii) le rayon spectral de A est égal à $\rho(A)$,
(iv) si les coefficients de A sont seulement ≥ 0, alors $\rho(A)$ est encore le rayon spectral de A et une valeur propre ≥ 0 associée à un vecteur propre ≥ 0.

Cette expression pour $\rho(A)$ est appelée formule du max-min de Collatz-Wielandt, remarquer que $A.x \geq rx$ s'écrit aussi $\min_i \frac{(A.x)_i}{x_i} \geq r$. On a donc

$$\rho(A) = \max_x \min_i \frac{(A.x)_i}{x_i}.$$

Elle est équivalente (pour les matrices irréductibles comme c'est le cas ici) à

$$\rho(A) = \inf\{r \geq 0 \quad t.q. \quad \exists x \in \mathbb{R}^d \backslash \{0\},\ x \geq 0,\ A.x \leq r\, x\}.$$

Posons $T(y) = \log\big(A.\exp(y)\big)$, où *exp* et *log* sont pris coordonnée par coordonnée. Alors, avec $y = \log(x)$, $s = \log(r)$, on a

$$\log \rho(A) = \inf\{s \in \mathbb{R} \quad t.q. \quad \exists y \in \mathbb{R}^d,\ T(y) \leq s + y\} = \inf_y \max_i (T_i(y) - y_i).$$

On est alors ramené à un problème de 'minimisation' classique car

Exercise 8.1 *Montrer que pour tout $i = 1, \dots, d$, l'application $y \mapsto T_i(y)$ est convexe, ainsi que $y \mapsto \max_i(T_i(y) - y_i)$.*

Exercise 8.2 (Seconde formule de Collatz-Wielandt (discrète)) *On considère les itérations*

$$x^{k+1} = A.x^k, \qquad x^0 > 0.$$

Montrer que $x^k > 0$ et $\max_i \big(\frac{x_i^{k+1}}{x_i^k}\big)$ décroit vers λ, $\min_i \big(\frac{x_i^{k+1}}{x_i^k}\big)$ croit vers λ.

Exercise 8.3 (Dépendance en un paramètre) *Soit deux matrices positives telles que (coefficient par coefficient) $A^2 \geq A^1$ alors $\rho(A^2) \geq \rho(A^1)$.*

Montrer que $\rho(A^1) \geq \rho(A^2)\big[1 - \frac{\max_{ij}(a_{ij}^2 - a_{ij}^1)}{\min_{ij} a_{ij}^2}\big]$.

Proof. *Preuve de (i).* Commençons par montrer que $\rho(A)$ est valeur propre. On pose

$$(A.x)_i = \sum_{1 \leq j \leq d} a_{ij} x_j,$$

$$m = \min_{1 \leq i \leq d} \sum_{1 \leq j \leq d} a_{ij}, \qquad M = \max_{1 \leq i \leq d} \sum_{1 \leq j \leq d} a_{ij}.$$

Pour $x \geq 0$, on a

$$(A.x)_i \leq M \max_{1 \leq j \leq d} x_j.$$

Ceci entraîne que $\rho(A) \leq M$. Par ailleurs $\rho(A) \geq m$ car pour le vecteur $x = (1, 1, \dots, 1)$ on a

$$(A.x)_i = \sum_{1 \leq j \leq d} a_{ij} \geq m = m\, x_i,$$

Soit alors une suite de réels $r_n < \rho(A)$ telle que $r_n \underset{n\to\infty}{\longrightarrow} \rho(A)$ et une suite de vecteurs $x^n \geq 0$ tels que

$$A.x^n \geq r_n\, x^n, \qquad \|x^n\| = 1.$$

Par compacité de la boule unité en dimension finie, on en déduit qu'il existe une sous-suite convergente: $x^{n(k)} \underset{k\to\infty}{\longrightarrow} x \geq 0$. On obtient à la limite

$$A.x \geq \rho(A)\, x, \qquad \|x\| = 1. \tag{8.1}$$

Montrons que x est en fait le vecteur propre recherché. Ceci découle du lemme suivant

Lemma 8.1 *On suppose $a_{ij} > 0$. Si*

$$A.x \geq \rho(A)\, x, \qquad x \geq 0, \qquad et\ \ \|x\| = 1,$$

alors $x > 0$ et $A.x = \rho(A)\, x$.

Proof. Posons

$$A.x - \rho(A)\, x = z \geq 0,$$

et supposons $z \neq 0$, nous allons obtenir une contradiction. Posons $y = A.x > 0$, alors, puisque les coefficients de A sont stictement positifs, on a

$$A.y - \rho(A)\, y = A.z > 0.$$

Comme toutes les coordonnées de $A.z$ sont > 0, il existe $\varepsilon > 0$ tel que $A.z \geq \varepsilon y$. On arrive bien à une contradiction avec la définition de $\rho(A)$ car $A.y \geq (\rho(A) + \varepsilon)\, y$. Ceci prouve que $\rho(A)$ est valeur propre pour le vecteur propre x et que $x = A.x/\rho(A) > 0$. □

Montrons maintenant la simplicité de ce vecteur propre $x^0 > 0$. Soit un autre vecteur propre associé à la valeur propre $\rho(A)$. Par addition avec μx^0 on peut supposer que ce vecteur propre, notons le y, est strictement positif aussi. On peut aussi les normer pour la norme l^1:

$$\sum_{i=1}^{d} x_i^0 = \sum_{i=1}^{d} y_i = 1. \tag{8.2}$$

On a

$$\rho(A)(x_i^0 - y_i) = \sum_{j=1}^{d} a_{ij}\, (x_j^0 - y_j),$$

$$\rho(A)|x_i^0 - y_i| = \sum_{j=1}^{d} a_{ij}\, (x_j^0 - y_j)\, \mathrm{sgn}(x_i^0 - y_i) \leq \sum_{j=1}^{d} a_{ij}\, |x_j^0 - y_j|.$$

Mais le lemme 8.1 ci-dessus appliqué à (8.1) montre que ceci implique

$$\rho(A)|x_i^0 - y_i| = \sum_{j=1}^{d} a_{ij}\, |x_j^0 - y_j|,$$

ce qui signifie que $\mathrm{sgn}(x_i^0 - y_i) = \mathrm{sgn}(x_j^0 - y_j)$ pour tout $j = 1, \ldots, d$. Donc $x^0 \geq y$ (ou le contraire), ce qui implique $x^0 = y$ compte tenu de (8.2).

Preuve de (ii). Soit $y \geq 0$ un vecteur propre, alors l'égalité $A.y = \lambda y$ et la définition de $\rho(A)$ montrent que $0 < \lambda \leq \rho(A)$. Ensuite, par multiplication de y par un scalaire, on suppose à nouveau que $x^0 - y > 0$. On a alors

$$A.(x^0 - y) = \rho(A)x^0 - \lambda y \geq \rho(A)(x^0 - y),$$

et, suivant à nouveau le lemme 8.1, ceci montre que $x^0 - y$ est proportionnel à x^0, donc y aussi.

Preuve de (iii). Soit $\lambda \in \mathbb{C}$ tel que $A - \lambda I$ n'est pas inversible. Soit $y \in \mathbb{C}^d$ un vecteur non nul du noyau. Alors

$$A.y = \lambda y \Longrightarrow |\lambda|\, |y_i| = \sum_{1 \leq j \leq d} a_{ij} y_j \frac{\bar{\lambda} \bar{y}_i}{|\lambda y_i|} \leq \sum_{1 \leq j \leq d} a_{ij} |y_j|.$$

Par définition de $\rho(A)$ ceci prouve que soit $|\lambda| \leq \rho(A)$. Supposons $|\lambda| = \rho(A)$, alors (ii) prouve que $|y|$ est proportionnel à x^0 et l'inégalité étant une égalité, ce vecteur y est également vecteur propre.

Preuve de (iv). On considère une matrice A_ε à coefficients > 0 convergents vers ceux de A. Soit x_ε, de norme unité, vérifant $A(x_\varepsilon) = \rho(A_\varepsilon)x_\varepsilon$. Par compacité de la boule unité en dimension finie, on peut extraire de x_ε une sous-suite convergente vers x, $\|x\| = 1$ et de $\rho(A_\varepsilon)$ une sous-suite (extraite à nouveau) convergent vers le rayon spectral de A, $\rho(A) \geq 0$. À la limite on trouve $A(x) = \rho(A)\, x$, $x \geq 0$, $x \neq 0$, ce qui prouve (iv) avec $\rho(A)$ ndéfini par le sup. Le début de la preuve de (iii) montre alors qu'il s'agit encore du rayon spectral. □

L'hypothèse de stricte positivité de l'énoncé (i)-(iii) peut aussi être affaiblie en supposant que A est ≥ 0 et irréductible (voir [22]).

Notice that using the dual eigenvector provides us with an alternative proof of parts the Perron-Frobenius theorem. See section §8.4

8.2 The Generalized Relative Entropy

Let $a_{ij} > 0$, $1 \leq i, j \leq d$, be the coefficients of a matrix $A \in M_{d\times d}(\mathbb{R})$ (there are interesting issues with the case $a_{ij} \geq 0$, for simplicity, we do not treat that cases). Using the Perron-Frobenius theorem, let $\lambda_0 > 0$ denote the eigenvalue associated with a positive right eigenvector $\varphi \in \mathbb{R}^d$, and a positive left eigenvector $\psi \in \mathbb{R}^d$

$$\begin{cases} A.\varphi = \lambda_0 \varphi, & \varphi_i > 0 \quad \text{for} \quad i = 1, \dots, d, \\ \psi.A = \lambda_0 \psi, & \psi_i > 0 \quad \text{for} \quad i = 1, \dots, d. \end{cases}$$

For later purposes, it is convenient to normalize these vectors, so that they are now uniquely defined. We choose

$$\sum_{i=1}^{d} \varphi_i = 1, \qquad \sum_{i=1}^{d} \varphi_i \, \psi_i = 1.$$

We set $\widetilde{A} = A - \lambda_0 Id$ and consider the evolution equation

$$\frac{d}{dt} n(t) = \widetilde{A}.n(t), \qquad n(0) = n^{init}. \tag{8.3}$$

The solutions to this system converge as $t \to \infty$ with an exponential rate. Indeed, the following result is classical

Proposition 8.1 (Long term convergence) *For positive matrices A and solutions to the differential system (8.3), we have,*

$$\rho := \sum_{i=1}^{d} \psi_i n_i(t) = \sum_{i=1}^{d} \psi_i n_i^0, \tag{8.4}$$

$$\sum_{i=1}^{d} \psi_i |n_i(t)| \leq \sum_{i=1}^{d} \psi_i |n_i^0|, \tag{8.5}$$

$$\underline{C}\varphi_i \leq n_i(t) \leq \overline{C}\varphi_i \quad \textit{with constants given by} \quad \underline{C}\varphi_i \leq n_i^0 \leq \overline{C}\varphi_i, \tag{8.6}$$

and there is a constant $\alpha > 0$ such that, with ρ given in (8.4), we have

$$\sum_{i=1}^{d} \psi_i\varphi_i \big(\frac{n_i(t) - \rho\varphi_i}{\varphi_i}\big)^2 \leq \sum_{i=1}^{d} \psi_i\varphi_i \big(\frac{n_i^0 - \rho\varphi_i}{\varphi_i}\big)^2 e^{-\alpha t}. \tag{8.7}$$

Exercise 8.4 (Seconde formule de Collatz-Wielandt (continue))

$$\max_i \frac{d\log(n_i(t))}{dt} \ \textit{croit vers} \ \lambda_0$$

On pourra considérer $\tilde{u} = \frac{d}{dt}[n_i - \rho\varphi_i e^{-\lambda_0 t}]$.

Here, we wish to justify the long term behaviour with an entropy inequality.

Proposition 8.2 (Generalized Relative Entropy) *Let $H(\cdot)$ be a convex function on $\mathbb{R}$, then the solution to (8.3) satisfies*

$$\begin{aligned} \frac{d}{dt} & \sum_{i=1}^{d} \psi_i\varphi_i H\big(\frac{n_i(t)}{\varphi_i}\big) \\ & = \sum_{i,j=1}^{d} \psi_i a_{ij}\varphi_j \Big[H'\big(\frac{n_i(t)}{\varphi_i}\big)\big[\frac{n_j(t)}{\varphi_j} - \frac{n_i(t)}{\varphi_i}\big] - H\big(\frac{n_j(t)}{\varphi_j}\big) + H\big(\frac{n_i(t)}{\varphi_i}\big) \Big] \\ & \leq 0. \end{aligned}$$

Proof. [of Proposition 8.2]. We denote by $\widetilde{a_{ij}}$ the coefficients of the matrix $\widetilde{A}$ and compute

$$\begin{aligned} \tfrac{d}{dt} \textstyle\sum_i \psi_i\varphi_i H\big(\frac{n_i(t)}{\varphi_i}\big) & = \textstyle\sum_{i,j} \psi_i H'\big(\frac{n_i(t)}{\varphi_i}\big)\widetilde{a_{ij}} n_j(t) \\ & = \textstyle\sum_{i,j} \psi_i \widetilde{a_{ij}}\varphi_j H'\big(\frac{n_i(t)}{\varphi_i}\big)\big[\frac{n_j(t)}{\varphi_j} - \frac{n_i(t)}{\varphi_i}\big], \end{aligned}$$

because the additional $\frac{n_i(t)}{\varphi_i}$ term vanishes since $\widetilde{A}.\varphi = 0$. But we also have, again thanks to the equation on φ and ψ, that

$$\sum_{i,j} \psi_i \widetilde{a_{ij}}\varphi_j \big[H\big(\frac{n_j(t)}{\varphi_j}\big) - H\big(\frac{n_i(t)}{\varphi_i}\big)\big] = 0.$$

Combining these two identities, we arrive to the equality in Proposition 8.2. The inequality follows because only the coefficients out of the diagonal, that satisfy $\widetilde{a_{ij}} = a_{ij} \geq 0$, enters here.
□

Proof. [of Proof of Proposition 8.1.] Notice that, as a special case of H in Proposition 8.2, we can choose $H(u) = u$, which being convex together with $-H$ gives the equality

$$\frac{d}{dt}\sum_{i=1}^{d}\psi_i n_i(t) = 0.$$

And (8.4) follows. In particular this identifies the value ρ mentioned in (8.4).

The second statement (8.5) follows immediately by choosing the (convex) entropy function $H(u) = |u|$.

As for the third statement (8.6), let us consider for instance the upper bound. It follows choosing the (convex) entropy function $H(u) = (u - \overline{C})_+^2$ because for this nonnegative function we have

$$\sum_{i=1}^{d}\psi_i\varphi_i H\big(\frac{n_i^0}{\varphi_i}\big) = 0.$$

Therefore, because the Generalized Relative Entropy decays, it remains zero for all times,

$$\sum_{i=1}^{d}\psi_i\varphi_i H\big(\frac{n_i(t)}{\varphi_i}\big) = 0,$$

which proves the result.

It remains to prove the exponential time decay statement (8.7). To do that, we work on

$$h(t,x) = n(t,x) - \rho\varphi,$$

which verifies $\int \psi[n(t,x) - \rho\varphi]dx = 0$ and satisfies the same equation as n. Then, we use the quadratic entropy function $H(u) = u^2$ and the Generalized Entropy Inequality gives

$$\frac{d}{dt}\sum_{i=1}^{d}\psi_i\varphi_i\left(\frac{h_i(t)}{\varphi_i}\right)^2 = -\sum_{i,j=1}^{d}\psi_i a_{ij}\varphi_j\left(\frac{h_j(t)}{\varphi_j} - \frac{h_i(t)}{\varphi_i}\right)^2 \leq 0.$$

Then, we need a discrete Poincaré inequality

Lemma 8.2 (Poincaré inequality) *Being given $\psi_i > 0$, $\varphi_i > 0$, $a_{ij} > 0$ for i and $j = 1, \ldots, d$, $i \neq j$, there is a constant $\alpha > 0$ such that for all vector m satisfying*

$$\sum_{i=1}^{d}\psi_i m_i = 0,$$

we have

$$\sum_{i,j=1}^{d}\psi_i a_{ij}\varphi_j\left(\frac{m_j}{\varphi_j} - \frac{m_i}{\varphi_i}\right)^2 \geq \alpha\sum_{i=1}^{d}\psi_i\varphi_i\left(\frac{m_i}{\varphi_i}\right)^2.$$

With this lemma, we conclude

$$\frac{d}{dt}\sum_{i=1}^{d}\psi_i\varphi_i\left(\frac{h_i(t)}{\varphi_i}\right)^2 \leq -\alpha\sum_{i=1}^{d}\varphi_i\left(\frac{h_i(t)}{\varphi_i}\right)^2,$$

and then, (8.7) follows by a simple use of the Gronwall lemma. □

Proof. [of Lemma 8.2]. After renormalizing the vector m (when it does not vanish, otherwise the result is obvious), we may suppose that

$$\sum_{i=1}^{d}\psi_i m_i = 0, \qquad \sum_{i=1}^{d}\psi_i\varphi_i\left(\frac{m_i}{\varphi_i}\right)^2 = 1.$$

Then we argue by contradiction. If such a α does not exist, this means that we can find a sequence of vectors $(m^k)_{(k\geq 1)}$ such that

$$\sum_{i=1}^{d}\psi_i m_i^k = 0, \qquad \sum_{i=1}^{d}\psi_i\varphi_i\left(\frac{m_i^k}{\varphi_i}\right)^2 = 1, \qquad \sum_{i,j=1}^{d}\psi_i a_{ij}\varphi_j\left(\frac{m_j^k}{\varphi_j}-\frac{m_i^k}{\varphi_i}\right)^2 \leq 1/k.$$

After extraction of a subsequence, we may pass to the limit $m^k \to \bar{m}$ and this vector satisfies

$$\sum_{i=1}^{d}\psi_i \bar{m}_i = 0, \qquad \sum_{i=1}^{d}\psi_i\varphi_i\left(\frac{\bar{m}_i}{\varphi_i}\right)^2 = 1, \qquad \sum_{i,j=1}^{d}\psi_i a_{ij}\varphi_j\left(\frac{\bar{m}_j}{\varphi_j}-\frac{\bar{m}_i}{\varphi_i}\right)^2 = 0.$$

Therefore, from this last relation, for all i and $j = 1,\dots,d$, we have

$$\frac{\bar{m}_i}{\varphi_i} = \frac{\bar{m}_j}{\varphi_j} := \nu.$$

By the zero sum condition, we have $\nu = 0$ because

$$\nu\sum_{i=1}^{d}\psi_i = 0.$$

In other words, $\bar{m} = 0$ which contradicts the normalization and thus such a α should exist. □

Exercise 8.5 *Assume* $\sum_{i=1}^{d}\varphi_i\psi_i = 1$. *Prove that we can choose* α *in the Poincaré inequality such that*

$$a_{ij} \geq \alpha\varphi_i\psi_j \qquad \forall i \neq j.$$

Remark 8.1 *1. The matrix with (positive) coefficients $b_{ij} = \psi_i\ a_{ij}\ \varphi_j$ is doubly stochastic, i.e., the sum of the lines and columns is 1 (see for instance[22]).*
2. Notice that $a_{ii} - \lambda_0 < 0$ because $\sum_j \widetilde{a_{ij}}\varphi_j = 0$. Therefore the matrix C with coefficients $c_{ij} = \frac{1}{\varphi_i}\widetilde{a_{ij}}\ \varphi_j$ is that of a Markov process. In other words, we set $y_i = x_i/\varphi_i$, then it satisfies

$$\frac{d}{dt} y_i(t) = c_{ij} y_j(t),$$

and the vector $(1, 1, \dots, 1)$ is the (positive) eigenvector associated to the eigenvalue 0 of the matrix C, i.e., $c_{ii} = \sum_{j\neq i} c_{ij}$ and $c_{ij} \geq 0$. Then, $(\varphi_i\psi_i)_{(i=1,\dots,d)}$ is the invariant measure of the Markov process. In particular this explains the entropy property which is classical for Markov processes ([25]).

8.3 The Floquet theory

We now consider T-periodic coefficients $a_{ij}(t) > 0$, $1 \leq i, j \leq d$, i.e., $a_{ij}(t+T) = a_{ij}(t)$. And we denote by $A(t) \in M_{d\times d}$ the corresponding matrix. Again our motivation comes from several questions in biology where such structures arise as seasonal rhythm, circadian rhythm, see [9] for instance.

The Floquet theorem tells us that there is a first 'Floquet eigenvalue' $\lambda_{per} > 0$ and two positive T-periodic functions $\varphi(t) \in \mathbb{R}^d$, $\psi(t) \in \mathbb{R}^d$ that are periodic solutions (uniquely defined up to multiplication by a constant) to the differential systems

$$\frac{d}{dt}\varphi(t) = [A(t) - \lambda_{per} Id].\varphi(t), \tag{8.8}$$

$$\frac{d}{dt}\psi(t) = \psi(t).[A(t) - \lambda_{per} Id]. \tag{8.9}$$

Up to a normalization, these elements $(\lambda_{per} > 0, \varphi(t) > 0, \psi(t))$ are unique and we normalize again as

$$\int_0^T \sum_{i=1}^d \varphi_i(t)dt = 1, \qquad \int_0^T \sum_{i=1}^d \psi_i(t)\varphi_i(t)dt = 1.$$

We recall that this case of Floquet theory (which applies to more general situations than positive matrices) is an application of Perron-Frobenius theorem to the resolvent matrix

$$S(t) = e^{\int_0^t A(s)ds},$$

which has positive coefficients also.

Again, we set $\widetilde{A(t)} = A(t) - \lambda_{per} Id$ and consider the differential system

$$\frac{d}{dt} n(t) = \widetilde{A}.n(t), \qquad n(0) = n^{init}.$$

In the present context we obtain the following version of Proposition 8.1,

Proposition 8.3 *For positive matrices A we have,*

$$\rho := \sum_{i=1}^{d} \psi_i(t) n_i(t) = \sum_{i=1}^{d} \psi_i(t=0) n_i^0, \tag{8.10}$$

$$\sum_{i=1}^{d} \psi_i(t)|n_i(t)| \leq \sum_{i=1}^{d} \psi_i(t=0)|n_i^0|, \tag{8.11}$$

if for some constants, we have $\underline{C}\varphi_i(t=0) \leq n_i^0 \leq \overline{C}\varphi_i(t=0)$, then

$$\underline{C}\varphi_i(t) \leq n_i(t) \leq \overline{C}\varphi_i(t), \tag{8.12}$$

and there is a constant $\alpha > 0$ such that

$$\sum_{i=1}^{d} \psi_i(t)\varphi_i(t)\big(\frac{n_i(t) - \rho\varphi_i(t)}{\varphi_i(t)}\big)^2 \leq \sum_{i=1}^{d} \psi_i^0 \varphi_i^0 \big(\frac{n_i^0 - \rho\varphi_i^0}{\varphi_i^0}\big)^2 e^{-\alpha t}. \tag{8.13}$$

Again, this can be justified thanks to entropy inequalities.

Proposition 8.4 *Let $H(\cdot)$ be a convex function on $\mathbb{R}$, then we have*

$$\begin{aligned} \tfrac{d}{dt} \quad & \sum_{i=1}^{d} \psi_i(t)\varphi_i(t) H\big(\frac{n_i(t)}{\varphi_i(t)}\big) \\ & = \sum_{i,j=1}^{d} \psi_i a_{ij} \varphi_j \left[H'\big(\frac{n_i}{\varphi_i}\big)\big[\frac{n_j}{\varphi_j} - \frac{n_i}{\varphi_i}\big] - H\big(\frac{n_j}{\varphi_j}\big) + H\big(\frac{n_i}{\varphi_i}\big)\right] \leq 0. \end{aligned}$$

These two propositions are variants of the corresponding ones in the Perron-Frobenius theorem and we leave the proofs to the reader. Adapting Lemma 8.2 requires an additional compactness argument based on the Ascoli-Arzela Theorem.

8.4 Eigenelements for integral operators

The Perron-Frobenius theorem can be extended to Banach spaces, this is the Krein-Rutman theorem. In its strong form, the natural extension is to work in C^0 which is not always adapted to the situations encountered here. We present an L^2 version for a specific form of operators.

For simplicity, we consider integral operators set in a bounded open set $\Omega \subset \mathbb{R}^d$, with a kernel $K(x,y) > 0$, $K \in L^2(\Omega \times \Omega)$, and we define the operator

$$\mathcal{K}[u](x) = \int_\Omega K(x,y)u(y)dy, \qquad \mathcal{K} : L^2(\Omega) \to L^2(\Omega),$$

thanks to the bound

$$\|\mathcal{K}[u]\|_{L^2(\Omega)} \leq \int_\Omega \|K(\cdot, y)\|_{L^2(\Omega)} u(y) dy \leq \|K(\cdot,\cdot)\|_{L^2(\Omega\times\Omega)} \|u\|_{L^2(\Omega)}.$$

The dual operator is readily computed as

$$\mathcal{K}^*[v](y) = \int_\Omega K(x,y) v(x) dx, \qquad \mathcal{K}^* : L^2(\Omega) \to L^2(\Omega),$$

One can check that $\mathcal{K}$ is a compact operator. Indeed, $K(x,y)$ itself, being a single element, is compact and the Kolmorov criteria with $h \in \mathbb{R}^d$ and $\Omega_h = \{x \in \Omega,\ s.\ t.\ x+h \in \Omega\}$ gives

$$\omega(h) := \|K(x+h,y) - K(x,y)\|_{L^2(\Omega_h\times\Omega)} \to 0 \quad \text{ as } h \to 0.$$

This implies that

$$\|\mathcal{K}[u](x+h) - \mathcal{K}[u](x)\|_{L^2(\Omega_h)} = \big\| \int_\Omega [K(x+h,y) - K(x,y)] u(y) dy \big\|_{L^2(\Omega_h)} \leq \omega(h) \|u\|_{L^2(\Omega)},$$

and thus the compactness. This is enough to get the weak form of the Krein-Rutman theorem:

Theorem 8.2 *There exists $\lambda_0 > 0$, $\varphi \in L^2(\Omega)$, $\varphi > 0$ and $\psi \in L^2(\Omega)$, $\psi > 0$ such that*

$$\mathcal{K}[\varphi](x) = \lambda_0 \varphi(x), \qquad \mathcal{K}^*[\psi](x) = \lambda_0 \psi(x).$$

This eigenvalue is simple and any other positive eigenvector of $\mathcal{K}$ is associated with λ_0.

We do not prove the theorem but indicate why it follows from the Perron-Frobenius theorem. The idea is to introduce a sequence of positive finite dimensional approximations of $K^n \to K$ in $L^2(\Omega\times\Omega)$ (for instance piecewise constant approximation based on a covering of Ω).Then, we use theorem 8.1 to obtain a sequence of normalized eigenelements, $\lambda^n > 0$, $\varphi^n > 0$, $\psi^n > 0$

$$\begin{cases} \int_\Omega K^n(x,y) \varphi^n(y) dy = \lambda^n \varphi^n(x), \qquad \int_\Omega K^n(x,y) \psi^n(x) dx = \lambda^n \psi^n(y), \\ \|\varphi^n\|_{L^2(\Omega)} = 1, \quad \|\psi^n\|_{L^2(\Omega)} = 1. \end{cases}$$

To pass to the limit, we first notice that $\lambda^n \leq \|K(\cdot,\cdot)\|_{L^2(\Omega\times\Omega)}$ because

$$\lambda^n \|\varphi^n\|_{L^2(\Omega)} \leq \int_\Omega \|K(\cdot,y)\|_{L^2(\Omega)} \varphi^n(y) dy \leq \|K(\cdot,\cdot)\|_{L^2(\Omega\times\Omega)} \|\varphi^n\|_{L^2(\Omega)},$$

Then, the compactness property allows us to consider a subsequence such that $\lambda^n \to \lambda_0$ and, in $L^2(\Omega)$, $\varphi^n \to \varphi$, $\psi^n(y) \to \psi$ with $\|\varphi\|_{L^2(\Omega)} = 1$, $\|\psi\|_{L^2(\Omega)} = 1$. From this, we pass to the limit and the positivity properties follow from the hypothesis $K > 0$. The strong compactness is used to avoid that $\varphi = 0$ for instance.

The other properties follow by duality. For example, if $\bar\varphi$ is another normalized nonnegaive eigenvector for λ_0, we write

$$\int_\Omega K(x,y)[\varphi(y) - \bar\varphi(y)] dy = \lambda_0 [\varphi(x) - \bar\varphi(x)], \quad \lambda_0 |\varphi(x) - \bar\varphi(x)| \leq \int_\Omega K(x,y) |\varphi(y) - \bar\varphi(y)| dy$$

and using the dual eigenvector

$$\lambda_0 \int_\Omega \psi(x)|\varphi(x) - \bar{\varphi}(x)|dx \leq \int_{\Omega\times\Omega} \psi(x)K(x,y)|\varphi(y) - \bar{\varphi}(y)|dydx = \lambda_0 \int_\Omega \psi(y)|\varphi(y) - \bar{\varphi}(y)|dy$$

which implies equality and thus $\mathrm{sgn}[\varphi(y) - \bar{\varphi}(y)] = \mathrm{sgn}[\varphi(x) - \bar{\varphi}(x)]$ for almost all x, y, This means that $\varphi \geq \bar{\varphi}$ (or $\leq$) which, for normalized functions, means $\varphi = \bar{\varphi}$.

From the discretised version, one can also justify other estimates, for instance we can normalize φ or φ^n with unit mass and write

$$\lambda_0 \|\varphi\|_{L^\infty(\Omega)} \leq \|K\|_{L^\infty(\Omega\times\Omega)} \| |\varphi\|_{L^1(\Omega)}.$$

8.5 Eigenelements for the growth-fragmentation equation via integral operators

The eigenelements for the growth-fragmentation equation which have been used in section §3.3 can be built thanks to integral operators. We recall that the problem is to find λ_0 for which there is a solution of

$$\begin{cases} \varphi'(x) + (\lambda_0 + b(x))\varphi(x) = 4b(2x)\varphi(2x), & x > 0 \\ \varphi(0) = 0, \qquad \varphi(x) > 0 \text{ for } x > 0. \end{cases}$$

With the definition

$$B(x) = \int_0^x b(y)dy,$$

this is also written

$$\frac{d}{dx}[\varphi(x)e^{B(x)+\lambda_0 x}] = 4e^{B(x)+\lambda_0 x} b(2x)\varphi(2x),$$

$$\varphi(x) = 4e^{-B(x)-\lambda_0 x} \int_0^x e^{B(y)+\lambda_0 y} b(2y)\varphi(2y)dy. \tag{8.14}$$

We may introduce the principal eigenvalue $E(\lambda)$ of the corresponding integral operator, that is

$$E(\lambda)\varphi_\lambda(x) = 4e^{-B(x)-\lambda x} \int_0^x e^{B(y)+\lambda y} b(2y)\varphi_\lambda(2y)dy, \qquad \varphi_\lambda(x) > 0 \text{ for } x > 0. \tag{8.15}$$

We need to prove that
1. this problem can be solved; the difficulty is that the set on $(0, \infty)$ is not a bounded,
2. there is a value λ_0 such that $E(\lambda_0) = 1$.
These results follow from the

Theorem 8.3 *We assume there are two constants such that*

$$0 < b_m \leq b(x) \leq b_M < \infty.$$

Then, there is a unique solution $(E(\lambda), \varphi_\lambda)$ *of* (8.15), *normalized by* $\int_0^\infty \varphi_\lambda dx = 1$ *and* $\varphi_\lambda \in L^\infty(0,\infty)$ *and* $e^{ax}\varphi_\lambda(x) \in L^1(0,\infty)$ *for all* $a < b_m + \lambda$. *Moreover,* $E(\cdot)$ *is Lipschitz continuous, convex and one has*

$$E'(\lambda) < 0, \qquad E(0) = 2, \qquad E(+\infty) = 0,$$

Observe that since φ_λ is bounded, it is also Lipschitz continuous. If $b(\cdot)$ is continuous, φ_λ is also of class $C^1([0,\infty))$ thanks to (8.15) once it is differentiated in x.

Proof. **[Estimates on $E(\cdot)$].** We postpone to the next Proposition the existence and integrability properties of solutions and we explain the end of the theorem. We calculate $E(0)$. We write (undoing the previous reduction)

$$E(0)\frac{d}{dx}[\varphi_0(x)e^{B(x)}] = 4e^{B(x)}b(2x)\varphi_0(2x),$$

$$E(0)[\varphi_0'(x) + b(x)\varphi_0(x)] = 4b(2x)\varphi_0(2x),$$

$$E(0)\int_0^\infty b(x)\varphi_0(x)dx = 4\int_0^\infty b(2x)\varphi_0(2x) = 2\int_0^\infty b(x)\varphi_0(x)]dx.$$

Hence the value of $E(0)$.

The property $E' < 0$ follows from the argument in Exercise 8.3, once extended to the case at hand. It can formally be recovered, usingφ_λ and the dual eigenfunction ψ_λ normalised by $\int_0^\infty \varphi_\lambda(x)\psi_\lambda(x)dx = 1$. We write

$$E(\lambda) = \langle \psi_\lambda, \mathcal{L}_\lambda \varphi_\lambda \rangle.$$

Differentiating in λ, we find

$$E'(\lambda) = \langle \psi_\lambda, \frac{\partial \mathcal{L}_\lambda}{\partial \lambda}\varphi_\lambda \rangle + \langle \frac{\partial \psi_\lambda}{\partial \lambda}, \mathcal{L}_\lambda \varphi_\lambda \rangle + \langle \psi_\lambda, \mathcal{L}_\lambda \frac{\partial \varphi_\lambda}{\partial \lambda} \rangle,$$

and acting by duality for the third term,

$$E'(\lambda) = \langle \psi_\lambda, \frac{\partial \mathcal{L}_\lambda}{\partial \lambda}\varphi_\lambda \rangle + \langle \frac{\partial \psi_\lambda}{\partial \lambda}, E(\lambda)\varphi_\lambda \rangle + \langle E(\lambda)\psi_\lambda, \frac{\partial \varphi_\lambda}{\partial \lambda} \rangle.$$

But the last two terms combine, to give

$$E(\lambda)\big[\langle \frac{\partial \psi_\lambda}{\partial \lambda}, \varphi_\lambda \rangle + \langle \psi_\lambda, \frac{\partial \varphi_\lambda}{\partial \lambda} \rangle\big] = E(\lambda)\frac{\partial}{\partial \lambda}\langle \psi_\lambda, \varphi_\lambda \rangle = 0.$$

Therefore we find

$$E'(\lambda) = -4\int_0^\infty e^{-B(x)-\lambda x}\psi_\lambda(x)\int_0^x e^{B(y)+\lambda y}(x-y)b(2y)\varphi_\lambda(2y)dydx < 0.$$

The Lipschitz continuity follows also because

$$-E'(\lambda) \leq 4 \int_0^\infty x e^{-B(x)-\lambda x} \psi_\lambda(x) \int_0^x e^{B(y)+\lambda y} b(2y) \varphi_\lambda(2y) dy dx = \int_0^\infty x \psi_\lambda(x) \frac{\varphi_\lambda(x)}{E(\lambda)} dx$$

which is controled because of the properties of φ_λ and $\psi_\lambda(x)$.

The property $E''(\lambda) > 0$ follows in the same way. □

Proof. **[Existence of φ_λ].** We now proceed to the existence of a solution of (8.15). It can be obtained in passing to the limit in a problem relevant from the theory in Section §8.4. We take two paramaters $\varepsilon > 0$ (small) and $R > 0$ (large) and consider the eigenproblem (in which we use $z = 2y$) to find $(E_{\varepsilon,R}, \varphi)$ solution of

$$E_{\varepsilon,R}\, \varphi(x) = 2e^{-B(x)-\lambda x} \int_0^R \left[\varepsilon + \mathbb{1}_{\{0<z<2x\wedge R\}} e^{B(\frac{z}{2})+\lambda \frac{z}{2}} b(z)\right] \varphi(z) dz, \quad \varphi(x) > 0 \text{ for } x \in (0, R], \tag{8.16}$$

From Theorem 8.2, there is a normalized solution

$$\varphi(x) > 0 \quad \forall x > 0, \qquad \int_0^R \varphi(x) dx = 1.$$

We prove uniform estimates in ε, R

Proposition 8.5 *With the assumption of theorem 8.3, for the eigenproblem (8.16), with λ fixed, we have*

$$2\frac{b_m}{b_M + \lambda}[1 - e^{-R(b_M+\lambda)/2}] \leq E_{\varepsilon,R} \leq 2[\frac{\varepsilon}{b_m + \lambda} + \frac{b_M}{b_m + \lambda}],$$

$$\varphi(x) \leq \frac{2(b_M + \varepsilon)}{E_{\varepsilon,R}}, \quad \forall x \in (0, R),$$

$$E_{\varepsilon,R} \left(\int_0^R e^{ax} \varphi(x) dx \right)^{1/2} \leq \frac{2(\varepsilon + b_M)}{b_m + \lambda - a}, \qquad \textit{for } a < b_m + \lambda.$$

With this result, the existence of a solution of (8.15), as stated in Theorem 8.3, follows by
1. extracting a subsequence φ_n which converges weakly in $L^p(0, \infty)$ as $\varepsilon_n \to 0$, $R_n \to \infty$,
2. passing to the limit $\varepsilon_n \to 0$ and $R_n \to \infty$ in the linear integral operator,
3. noticing that $\int_0^\infty \lim_{n\to\infty} \varphi_n(x) dx = 1$ thanks to the exponential control of decay at infinity.

□

Proof. **[of Propositions 8.5].**
1. Estimate on $E_{\varepsilon,R}$. We integrate equation (8.16) and get

$$\frac{E_{\varepsilon,R}}{2} = \varepsilon \int_0^R e^{-B(x)-\lambda x} + \int_{z=0}^R \int_{x=\frac{z}{2}}^R e^{-B(x)-\lambda x} e^{B(\frac{z}{2})+\lambda \frac{z}{2}} b(z) \varphi(z) dx\, dz$$

and we estimate separately the two terms on the right hand side. Firstly, we have

$$0 \leq \varepsilon \int_0^R e^{-B(x)-\lambda x} \leq \frac{\varepsilon}{b_m+\lambda}.$$

Secondly, we have, with $\bar{x} = x - \frac{z}{2}$,

$$\int_{x=\frac{z}{2}}^{R} e^{-B(x)-\lambda x} e^{B(\frac{z}{2})+\lambda\frac{z}{2}} b(z)dx = \int_{\bar{x}=0}^{R-\frac{z}{2}} e^{-[B(\bar{x}+\frac{z}{2})-B(\frac{z}{2})]-\lambda\bar{x}} b(z)d\bar{x}$$

and this term is controled from above and below as

$$b_m \int_{\bar{x}=0}^{\frac{R}{2}} e^{-(b_M+\lambda)x} d\bar{x} \leq \int_{\bar{x}=0}^{R-\frac{z}{2}} e^{-[B(\bar{x}+\frac{z}{2})-B(\frac{z}{2})]-\lambda\bar{x}} b(z)d\bar{x} \leq b_M \int_0^\infty e^{-(b_m+\lambda)x} d\bar{x}$$

$$\frac{b_m}{b_M+\lambda}\big[1-e^{-(b_M+\lambda)\frac{R}{2}}\big] \leq \int_{\bar{x}=0}^{R-\frac{z}{2}} e^{-[B(\bar{x}+\frac{z}{2})-B(\frac{z}{2})]-\lambda\bar{x}} b(z)d\bar{x} \leq \frac{b_M}{b_m+\lambda}.$$

Altogether, these estimates generate the two bounds for $E_{\varepsilon,R}$.

2. L^∞ bound. It comes from

$$2e^{-B(x)-\lambda x}[\varepsilon + e^{B(\frac{z}{2})+\lambda\frac{z}{2}} b(z)] \leq 2[\varepsilon + b_M], \quad \text{for } x \geq \frac{z}{2}.$$

3. Exponential control. As for the first control, we check that

$$\varepsilon \int_0^R e^{-B(x)-\lambda x+ax} \leq \frac{\varepsilon}{b_m+\lambda-a},$$

and

$$\int_{x=\frac{z}{2}}^{R} e^{-B(x)-\lambda x+ax} e^{B(\frac{z}{2})+\lambda\frac{z}{2}} b(z)dx \leq \int_{\bar{x}=0}^{R-\frac{z}{2}} e^{-b_m\bar{x}-\lambda\bar{x}+a(\bar{x}+\frac{z}{2})} b(z)d\bar{x} \leq \frac{b_M e^{a\frac{z}{2}}}{b_m+\lambda-a}.$$

As a consequence, we find

$$\begin{aligned} \tfrac{E_{\varepsilon,R}}{2} \int_0^R e^{ax}\varphi(x)dx &\leq \tfrac{\varepsilon}{b_m+\lambda-a} + \tfrac{b_M}{b_m+\lambda-a}. \int_0^R e^{a\frac{z}{2}}\varphi(z)dz \\ &\leq \tfrac{\varepsilon}{b_m+\lambda-a} + \tfrac{b_M}{b_m+\lambda-a}. \Big(\int_0^R e^{az}\varphi(z)dz\Big)^{1/2} \end{aligned}$$

and the bound follows. $\square$

Exercise 8.6 *Compare the dual integral operator of equation (8.15) and the integral form of the dual equation (3.11).*

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