Parabolic equations in biology

Missprints, errors and improvments

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- Page 8. Irreversible reactions, in the right hand side, it should be $n_j^{a_j}$, not $n_j^{b_j}$.
- Page 14. In Proposition 1.6, it is better to recalll: With initial data $u^\varepsilon_S(0) > 0$ and $u^\varepsilon_C(0) = 0$.
- Page 16. Caption of Figure 1.1 should be: The solution n_B is the top (red) curve, n_A is in blue and n_C in green.
- Page 23. The formula in the middle should be (no 'dot') $\int_\Omega \nabla w_j.\nabla w_k = \lambda_k \int_\Omega w_j w_k = 0$.
- Page 33. The problem 2.7 is badly written. See next page for a better version.
- Page 55, Corrolary 3.16. The proof is not correct because the nonlinearity is not of the form $Q(u)$ but a nonlocal functional. See a complete existence proof in
Desvillettes, L., Jabin, P.-E., Mischler, S., Raoul, G. On selection dynamics for continuous structured populations. Comm. Math. Sci. 6(3), 729–747 (2008).
- Page 127. In system (7.15), the boundary condition is Neumann, not Dirichlet.

Correction to the problem page 33

2.7 Problem

The goal of this problem is to show that for rank-1 nonlinearities, the long term behaviour is determinded by simple states in x without a size condition.

Let Ω a smooth bounded domain. Consider a smooth positive solution of the Lotka-Volterra system with diffusion and Neumann boundary condition

$$\begin{cases} \frac{\partial}{\partial t} n_i - D_i \Delta n_i + a_i(x) n_i = n_i \, r_i(t), & t \geq 0, \; x \in \Omega, \quad i = 1, 2, ..., I, \\ \frac{\partial n_i}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \quad i = 1, 2, ..., I, \\ n_i(t = 0, x) = n_i^0(x), \end{cases}$$

with the nonlinearity defined, for some given positive and smooth functions $\big(\psi_i(x)\big)_{i=1,...,I}$, by

$$r_i(t) = \mathcal{R}_i\big(\int_\Omega \psi_1(x) n_1(t,x) dx, ..., \int_\Omega \psi_I(x) n_I(t,x) dx\big).$$

1. We also define the first eigenfunctions $N_i(x) > 0$ for the eigenvalue λ_i, defined by

$$\begin{cases} -D_i \Delta N_i + a_i(x) N_i = \lambda_i N_i, & x \in \Omega, \\ \frac{\partial N_i}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, & \int_\Omega N_i(x)^2 dx = 1, \qquad i = 1, 2, ..., I. \end{cases}$$

Explain why the pair (N_i, λ_i) exists and give the corresponding Poincaré-Wirtinger inequality.
2. We consider $\widetilde{n}_i$ the solution of

$$\begin{cases} \frac{\partial}{\partial t} \widetilde{n}_i - D_i \Delta \widetilde{n}_i + a_i(x) \widetilde{n}_i = \lambda_i \widetilde{n}_i, & t \geq 0, \; x \in \Omega, \quad i = 1, 2, ..., I, \\ \frac{\partial \widetilde{n}_i}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \quad i = 1, 2, ..., I, \\ \widetilde{n}_i(t = 0, x) = n_i^0(x). \end{cases}$$

Prove that $\int_\Omega \widetilde{n}_i(t,x) N_i(x) dx = \int_\Omega \widetilde{n}_i^0(x) N_i(x) dx$. Find the constant α_i such that, as $t \to \infty$,

$$\|\widetilde{n}_i(t,x) - \alpha_i N_i(x)\|_{L^2(\Omega)} \leq \|\widetilde{n}_i^0(x) - \alpha_i N_i(x)\|_{L^2(\Omega)} e^{-\mu_i t}$$

and identify μ_i.
3. We write the solution of () as $n_i(t,x) = \rho_i(t) \widetilde{n}_i(t,x)$. Identify the evolution equation giving $\frac{d}{dt}\rho_i(t)$ in terms of the ρ_j and $\widetilde{n}_j$.
4. Assume that for some $M > 0$ we have $\mathcal{R}_i\big(Y_1, ... Y_I\big) < \lambda_i$ whenever $Y_i > M$, $i = 1, ..., I$. Show that the $\rho_i(t)$ are bounded and that for some $\mu > 0$ we have

$$\|n_i(t,x) - \rho_i(t) \alpha_i N_i(x)\|_{L^2(\Omega)} \leq C^0 e^{-\mu t}.$$

5. In dimension 1, assuming $\mathcal{R}(+\infty) < \lambda$, $\mathcal{R}(-\infty) > \lambda$ and $\mathcal{R}' < 0$. Show that the long term dynamic is given by that of the equation

$$\dot{\varrho}(t) = \varrho(t) \; \mathcal{R}\big(\varrho(t) \alpha \int \psi(x) N(x) dx\big),$$

and that $n(t,x)$ converges generically to a steady state $\bar{\rho} N(x)$ as $t \to \infty$.