Boundary layers for Navier-Stokes equations with slip boundary conditions

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1. Stability and instability in the 2D incompressible model
   - The inviscid limit problem and boundary layers
   - Navier boundary conditions and $L^2$ stability
   - $L^\infty$ instability of shear flows

2. Convergence of strong solutions in the 3D isentropic model
   - The equations and conormal derivatives
   - Uniform existence and convergence towards Euler
   - Improvements and possible extensions
1. Stability and instability in the 2D incompressible model
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2D incompressible Navier-Stokes on the half-space \((x, y) \in \Omega = \mathbb{R} \times ]0, +\infty[:\)

\[
\begin{aligned}
\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \varepsilon \Delta u^\varepsilon + \nabla p^\varepsilon &= 0 \\
\text{div } u^\varepsilon &= 0 \\
u^\varepsilon|_{t=0} &= u_0^\varepsilon,
\end{aligned}
\]  

modelling the motion of a viscous, homogeneous fluid.

Usual boundary conditions (BC) for macroscopic flows:

- impermeability of the boundary
  
  \[ u_2|_{y=0} = 0, \]

- and the fluid does not slip along the boundary - Dirichlet BC
  
  \[ u|_{y=0} = 0. \]
Nondimensional viscosity $\varepsilon = 1/\text{Re}$, $\text{Re}$ the Reynolds number.

Inviscid limit problem: behaviour of the solutions $u^\varepsilon$ relative to $\nu$, solution of the incompressible Euler equation

$$
\begin{align*}
\partial_t \nu + \nu \cdot \nabla \nu + \nabla q &= 0 \\
\text{div } \nu &= 0 \\
\nu |_{t=0} &= \nu_0 
\end{align*}
$$

(2)

when $\varepsilon$ goes to 0 $\Leftrightarrow$ high-Reynolds approximation.

The Euler equation is of order one, so only one BC is required:

$$
\nu_2 |_{y=0} = 0.
$$
As Navier-Stokes solutions can converge to Euler solutions in the whole space, a Prandtl **boundary layer** expansion\(^1\) is proposed:

\[
\begin{align*}
    u^\varepsilon(t, x, y) & \sim v(t, x, y) + V\left(t, x, \frac{y}{\sqrt{\varepsilon}}\right) \\
    \varepsilon \to 0
\end{align*}
\]

\(^1\)Prandtl, *Verhand. III Intern. Math. Kongresses Heidelberg*, 1904
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\[
u^\varepsilon(t, x, y) \sim v(t, x, y) + V \left(t, x, \frac{y}{\sqrt{\varepsilon}}\right)
\]  

(3)

Boundary layer stability in a normed function space \((E, \|\cdot\|)\) if

\[
\sup_{t \in [0, T]} \|u^\varepsilon(t) - v(t)\| \xrightarrow[\varepsilon \to 0]{} 0.
\]

\(^1\)Prandtl, *Verhand. III Intern. Math. Kongresses Heidelberg*, 1904
Issues with the inviscid limit problem with Dirichlet BCs:

- If \( u = (u_1, u_2)(t, x, y) = (\tilde{u}, \sqrt{\varepsilon}\tilde{v})(t, x, y' = \varepsilon^{-1/2}y) \), the boundary layer is governed by the ill-posed\(^2\) Prandtl equation:

\[
\begin{align*}
\partial_t \tilde{u} + \tilde{u} \partial_x \tilde{u} + \tilde{v} \partial_{y'} \tilde{u} - \partial_{y'y'} \tilde{u} &= (\partial_t v + v \cdot \nabla v)_1|_{y=0} \\
\partial_x \tilde{u} + \partial_{y'} \tilde{v} &= 0 \\
(\tilde{u}, \tilde{v})|_{y=0} &= 0 \\
\lim_{y \to +\infty} \tilde{u} &= v_1|_{y=0};
\end{align*}
\]

\(^3\)Grenier, *Comm. Pure Appl. Math.*, 2000  
\(^4\)Kato, *Seminar on nonlinear PDEs*, 1984
Issues with the inviscid limit problem with Dirichlet BCs:

- if \( u = (u_1, u_2)(t, x, y) = (\tilde{u}, \sqrt{\varepsilon}\tilde{v})(t, x, y' = \varepsilon^{-1/2}y) \), the boundary layer is governed by the ill-posed\(^2\) Prandtl equation:

\[
\begin{align*}
\partial_t\tilde{u} + \tilde{u}\partial_x\tilde{u} + \tilde{v}\partial_y'\tilde{u} - \partial_y'y'\tilde{u} &= (\partial_t v + v \cdot \nabla v)_{1|y=0} \\
\partial_x\tilde{u} + \partial_y'\tilde{v} &= 0 \\
(\tilde{u}, \tilde{v})_{|y=0} &= 0 \\
\lim_{y \to +\infty} \tilde{u} &= v_1_{|y=0};
\end{align*}
\]

- even if Prandtl is well-posed, ansatz (3) can be invalid due to creation of vorticity inside the boundary layer\(^3\), and control of the vorticity (turbulence) near the boundary is crucial\(^4\).

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\(^3\)Grenier, *Comm. Pure Appl. Math.*, 2000
\(^4\)Kato, *Seminar on nonlinear PDEs*, 1984
Our boundary condition: the Navier, or slip, boundary condition\(^5\), a mixed BC that allows the fluid to slip along the boundary at a speed proportional to the normal stress:

\[
    u_2\big|_{y=0} = 0 \quad \text{and} \quad \partial_y u_1^{\epsilon}\big|_{y=0} = 2a^{\epsilon} u_1^{\epsilon}\big|_{y=0}
\]  

(4)

The slip phenomenon is observed at microscopic levels, for instance in capillary blood vessels.

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Let $a^\varepsilon = a/\varepsilon^\beta$ for $a, \beta \in \mathbb{R}$. $\beta > 0$: approximation of Dirichlet BCs as $\varepsilon \to 0$, transition between the stable $\beta = 0$ case and the unstable vorticity in the Dirichlet ($a^\varepsilon = \infty$) case.

With $a > 0$ and $\beta = 0$: stability in $L^2$, using energy estimates\textsuperscript{6}.

\textsuperscript{6}Iftimie & Planas, *Nonlinearity*, 2006
Let \( a^\varepsilon = a / \varepsilon^\beta \) for \( a, \beta \in \mathbb{R} \). \( \beta > 0 \): approximation of Dirichlet BCs as \( \varepsilon \to 0 \), transition between the stable \( \beta = 0 \) case and the unstable vorticity in the Dirichlet \((a^\varepsilon = \infty)\) case.

With \( a > 0 \) and \( \beta = 0 \): stability in \( L^2 \), using energy estimates\(^6\). We extend this result:

**Theorem 1.1 (Diff. Integral Eqns. 2014)**

Let \( u_0^\varepsilon \) be a converging sequence in \( L^2(\Omega) \), whose limit \( v_0 \) is in \( H^s(\Omega) \) for some \( s > 2 \). Then, for every time \( T > 0 \), the sequence of global weak solutions to the Navier-Stokes equation \( u^\varepsilon \) with initial condition \( u_0^\varepsilon \) and Navier boundary condition as above, converges in \( L^\infty([0, T], L^2(\Omega)) \) towards the unique strong solution \( v \) of the Euler equation with initial condition \( v_0 \) if

- either \( a > 0 \) and \( \beta < 1 \),
- or \( a < 0 \) and \( \beta \leq 1/2 \).

\(^6\)Iftimie & Planas, *Nonlinearity*, 2006
Further extensions.

- The theorem can be extended to higher dimensions by starting with the Leray energy inequality.

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Further extensions.

- The theorem can be extended to higher dimensions by starting with the Leray energy inequality.
- The first point of the theorem (with $a \geq 0$) has been extended to the isentropic compressible model by Sueur\textsuperscript{7}.

\textsuperscript{7}Sueur, *J. Math. Fluid Mech.*, 2014
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Set $a > 0$ and $\beta = 1/2$: (4) is written

$$
\partial_y u_1|_{y=0} = \frac{2a}{\sqrt{\varepsilon}} u_1|_{y=0}.
$$

(5)

---

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**Shear flow**: vector field of the form $u_{sh}(y) = (u_{sh}(y), 0)$; stationary solution to the incompressible Euler equation (with constant pressure). It is **linearly unstable** if there exist $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$, $k \in \mathbb{R}$ and $\psi \in H^1_0(\mathbb{R}^+)$ such that $u(t, x, y) = e^{\lambda t} \nabla \perp (e^{ikx} \psi(y))$ solves the Euler equation linearised around $u_{sh}$:

$$\begin{aligned}
\partial_t u + u \cdot \nabla u_{sh} + u_{sh} \cdot \nabla u + \nabla p &= 0 \\
\text{div } u &= 0 \\
|y=0 &= 0,
\end{aligned} \quad (6)$$

For the Euler equation, linear instability implies nonlinear instability$^8$.

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Theorem 1.2 (*Diff. Integral Eqns.* 2014)

Let \( Y = y/\sqrt{\varepsilon} \). We consider a family of boundary-layer scale initial conditions written as \( u_{sh}(Y) \), with \( u_{sh} \) a linearly unstable shear profile for the Euler equation. We generate a reference solution to the Navier-Stokes equation as follows:

\[
\begin{aligned}
\frac{\partial}{\partial t} u_{sh}(t, Y) - \frac{\partial^2}{\partial Y^2} u_{sh}(t, Y) &= 0 \\
\frac{u_{sh}}{u_{sh}(0, Y)} &= u_{sh}(Y) \\
\frac{\partial}{\partial Y} u_{sh}(t, 0) &= 2au_{sh}(t, 0)
\end{aligned}
\]
Theorem 1.2 (*Diff. Integral Eqns.* 2014)

Let $Y = y / \sqrt{\varepsilon}$. We consider a family of boundary-layer scale initial conditions written as $u_{sh}(Y)$, with $u_{sh}$ a linearly unstable shear profile for the Euler equation. We generate a reference solution to the Navier-Stokes equation as follows:

$$
\begin{align*}
\partial_t u_{sh}(t, Y) - \partial_{YY} u_{sh}(t, Y) &= 0 \\
\frac{u_{sh}(0, Y)}{u_{sh}(0, Y)} &= u_{sh}(Y) \\
\partial_Y u_{sh}(t, 0) &= 2a u_{sh}(t, 0)
\end{align*}
$$

Then, for any $n \in \mathbb{N}^*$, there exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, there exist initial data $u_0^\varepsilon$ such that

$$
\|u_0^\varepsilon(x, y) - u_{sh}(Y)\|_{\mathcal{H}^s(\Omega)} \leq C\varepsilon^n
$$

for some $s > 0$, which generates a solution to the Navier-Stokes equation with boundary condition (5) that satisfies, at a time $T^\varepsilon \sim -n \sqrt{\varepsilon} \ln(\varepsilon)$,

$$
\|u^\varepsilon(T^\varepsilon, x, y) - u_{sh}(T^\varepsilon, Y)\|_{L^\infty(\Omega)} \geq \delta_0.
$$
Main ideas of the proof. After rescaling the equation \((t, x, y) = \varepsilon^{-1/2}(t, x, y)\), construct an approximate solution to the Navier-Stokes equation written as:

\[
u^{ap}(t, x, y) = \overline{u_{sh}}(\sqrt{\varepsilon}t, y) + \sum_{j=1}^{N} \varepsilon^{jn} U^i_j(t, x, y) + \varepsilon^{j+1/4} U^b_j(t, x, y/\varepsilon^{1/4}).\]

(7)

Note the \(\varepsilon^{1/4}\) amplitude of the boundary layer terms, hinted by the asymptotic expansion for fixed Navier BCs\(^9\). It will allow to bound \(\nabla u^{ap}\) independently of \(\varepsilon\).

Each \(U^i_j\) and \(U^b_j\) are built using another asymptotic expansion in powers of \(\varepsilon^{1/8}\), e.g. \(U^i_1 = \sum_{m=0}^{8n-1} \varepsilon^{m/8} U^i_m\).
The first term $u_0^i$ is chosen as a wavepacket of unstable solutions of (6):

$$u_0^i(t, x, y) = \int_{\mathbb{R}} \varphi_0(k) e^{\lambda(k)t + ikx} v_0(k, y) \, dk$$

(8)

with $\varphi_0 \in C^\infty_0(\mathbb{R})$ localising the eigenmodes around the most unstable wavenumber.
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Denote $\sigma_0 = \max\{\Re\lambda(k) \mid k \in \mathbb{R}\}$ the most unstable eigenvalue, and let $g^\varepsilon(t) = \varepsilon^n(1 + t)^{-1/2}e^{\sigma_0 t}$.

Time of study: $t \leq T_0^\varepsilon - \tau$ such that $g^\varepsilon(T_0^\varepsilon) = 1$, and $\tau > 0$ independent of $\varepsilon$. 
The first term of $u^{ap}$ carries the growth we are after:

$$\|u_0^i(t)\|_{L^2(\Omega_A(t))} \geq C_1' (1 + t)^{1/4} g^\varepsilon(t),$$

for $\Omega_A(t) \subset \Omega$ of measure $\sqrt{1 + t}$. 
The first term of $u^{ap}$ carries the growth we are after:

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for $\Omega_A(t) \subset \Omega$ of measure $\sqrt{1 + t}$.

By induction, we then show that

$$\|\varepsilon^{jn}(U_j^i(t) + \varepsilon^{1/4} U_j^b(t))\|_{L^2(\Omega)} \leq C_j(1 + t)^{1/4}(g^\varepsilon(t))^j.$$

This is done by noticing that the interior terms $U_j^i$ solve (6) with a source term $F_j$ satisfying

$$\|F_j(t)\|_{H^s(\Omega)} \leq C'_j \frac{e^{\gamma t}}{(1 + t)^\alpha}$$

with $\gamma > \sigma_0$ and $\alpha > 0$. A resolvent estimate for the linearised equation then yields that $U_j^i$ also satisfies this inequality.
Now we examine $w = u^\varepsilon - u^{ap}$. Choosing $N$ so that

$$N\sigma_0 > \|\nabla u^{ap}\|_{L^\infty([0,T_0^\varepsilon-\tau] \times \Omega)} + 1/2,$$

a quick energy estimate on the equation satisfied by $w$ yields

$$\|w(t)\|_{L^2(\Omega)} \leq C_{N+1}(1 + t)^{1/4}(g^\varepsilon(t))^{N+1}.$$
Now we examine $w = u^\varepsilon - u^{ap}$. Choosing $N$ so that
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a quick energy estimate on the equation satisfied by $w$ yields
\[ \|w(t)\|_{L^2(\Omega)} \leq C_{N+1}(1 + t)^{1/4}(g^\varepsilon(t))^{N+1}. \]

As a result,
\[ \|u^\varepsilon(t) - \overline{u}_{sh}(\sqrt{\varepsilon}t)\|_{L^2(\Omega_A(t))} \geq (1 + t)^{1/4} \left[ C_1 g^\varepsilon(t) - \sum_{j=2}^{N+1} C_j (g^\varepsilon(t))^j \right] \]
Now we examine $w = u^\varepsilon - u^{ap}$. Choosing $N$ so that
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\[ \|u^\varepsilon(t) - u_{sh}(\sqrt{\varepsilon}t)\|_{L^2(\Omega_A(t))} \geq (1 + t)^{1/4} \left[ C_1g^\varepsilon(t) - \sum_{j=2}^{N+1} C_j(g^\varepsilon(t))^j \right] \]
\[ \geq (1 + t)^{1/4}(C_1/2)g^\varepsilon(t) \]
for $t \leq T_0^\varepsilon - \tau'$, with $\tau' > \tau$ independent of $\varepsilon$. 
Now we examine $w = u^\varepsilon - u^\text{ap}$. Choosing $N$ so that

$$N\sigma_0 > \| \nabla u^\text{ap} \|_{L^\infty([0, T_0^\varepsilon - \tau] \times \Omega)} + 1/2,$$

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$$\| w(t) \|_{L^2(\Omega)} \leq C_{N+1}(1 + t)^{1/4}(g^\varepsilon(t))^{N+1}.$$

As a result,

$$\| u^\varepsilon(t) - u^\text{sh}(\sqrt{\varepsilon} t) \|_{L^2(\Omega_A(t))} \geq (1 + t)^{1/4} \left[ C_1 g^\varepsilon(t) - \sum_{j=2}^{N+1} C_j (g^\varepsilon(t))^j \right]$$

$$\geq (1 + t)^{1/4} \delta_0$$

for $t = T_0^\varepsilon - T$, with $T > \tau'$ chosen independently of $\varepsilon$. □
Comparison with the Dirichlet boundary condition.

- WKB expansion based on the Prandtl ansatz (3): $O(1)$ boundary layers.
- The study time $T_0^\varepsilon$ is shorter. One misses instability in $L^\infty$, $\varepsilon^{1/4}\delta_0$ replaces $\delta_0$ in the final equality, but we do get instability of the vorticity.
Robustness of the method.
This ‘linear instability implies nonlinear instability’ argument requires:

- a ‘most unstable’ eigenmode for the linearised system,
- a resolvent estimate for the linearised system,
- and good energy estimates for the equation solved by the difference between the exact and approximate solutions.

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10 Rousset & Tzvetkov, multiple references
11 arxiv 1502.05647
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- a resolvent estimate for the linearised system,
- and good energy estimates for the equation solved by the difference between the exact and approximate solutions.

Thus it has also been used to obtain transverse instability of solitary waves in numerous settings: NLS, KP-I, water-waves\textsuperscript{10}, Euler-Korteweg\textsuperscript{11}, ...

\textsuperscript{10}Rousset & Tzvetkov, multiple references
\textsuperscript{11}arxiv 1502.05647
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2 Convergence of strong solutions in the 3D isentropic model
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3D isentropic, compressible Navier-Stokes equation on the half space \( \Omega = \{ X = (x, y, z) \in \mathbb{R}^2 \times ]0, +\infty[ \} \):

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0 \\
\rho \partial_t u + \rho u \cdot \nabla u &= \text{div} \Sigma + \rho F,
\end{align*}
\]

(9)

modelling the motion of a viscous, non-heat-conducting fluid with typically supersonic speeds.
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\end{align*}
\] (9)

modelling the motion of a viscous, non-heat-conducting fluid with typically supersonic speeds.

- \( \Sigma \) is the stress tensor:

\[
\Sigma = \varepsilon \mu(\rho)(\nabla + \nabla^t)u + \varepsilon \lambda_0(\rho)\text{div} \; ul_3 - P(\rho)l_3. \] (10)

- The viscosity coefficients \( \mu > 0 \) and \( \mu + \lambda_0 > 0 \) can depend smoothly on the density.

- The pressure \( P(\rho) \) is given by a barotropic law: \( P(\rho) = k \rho^\gamma \) with \( k > 0 \) and \( \gamma > 1 \).
- \( F \) is a smooth force term.
  We solve for \( t \in \mathbb{R} \) with the fluid at rest for negative times:
  \[
  (\rho, u, F) = (1, 0, 0) \quad \text{for} \quad t < 0.
  \]

- Navier boundary condition:
  \[
  u_3 |_{z=0} = 0 \quad \text{and} \quad (\varepsilon^{-1} \Sigma \vec{n} + au)_\tau |_{z=0} = 0,
  \]
  where \( v_\tau = (v_1, v_2) \) is the tangential part of a 3D vector \( v \), and \( a \in \mathbb{R} \).
\( F \) is a smooth force term.

We solve for \( t \in \mathbb{R} \) with the fluid at rest for negative times:

\[
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\]

Navier boundary condition:

\[
u_3|_{z=0} = 0 \quad \text{and} \quad [\mu(\rho) \partial_z u_\tau]|_{z=0} = 2a u_\tau|_{z=0},
\]

(11)

where \( \nu_\tau = (\nu_1, \nu_2) \) is the tangential part of a 3D vector \( \nu \), and \( a \in \mathbb{R} \).
Global existence is assured for weak solutions with constant viscosity coefficients, and only under certain conditions for variable coefficients\textsuperscript{12}.

Strong solutions are only local in time\textsuperscript{13}.

In the constant coefficient case, convergence of weak solutions to a strong solution of the Euler equations has been obtained by Sueur.

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\textsuperscript{12}Bresch, Desjardins & Gérard-Varet, \textit{J. Math. Pures Appl.}, 2007

\textsuperscript{13}Solonnikov, \textit{Zap. Naučn. Sem. LOMI Steklov.}, 1976
Method: *a priori* estimates in conormal Sobolev spaces, similar to works on the incompressible Navier-Stokes system with Navier BCs (fixed boundary and free surface\(^{14}\)). This avoids constructing approximate solutions.

Method: *a priori* estimates in conormal Sobolev spaces, similar to works on the incompressible Navier-Stokes system with Navier BCs (fixed boundary and free surface\textsuperscript{14}). This avoids constructing approximate solutions.

**Conormal regularity** is necessary to get uniform estimates for boundary layers. Indeed, we expect the ansatz

$$(\rho, u)(t, y, z) \sim (r, v)(t, y, z) + \sqrt{\varepsilon}(R, V)(t, y, z/\sqrt{\varepsilon}),$$

with $(r, v)$ solving the isentropic Euler equation: Lipschitz control at best.

Consider this set of derivatives that are tangent to $\partial \Omega$:

\[ Z_0 = \partial_t, \quad Z_{1,2} = \partial_{y_{1,2}}, \quad Z_3 = \phi(z) \partial_z = \frac{z}{1 + z} \partial_z, \]
Consider this set of derivatives that are tangent to $\partial \Omega$:

$$Z_0 = \partial_t, \quad Z_{1,2} = \partial_{y_{1,2}}, \quad Z_3 = \phi(z) \partial_z = \frac{z}{1 + z} \partial_z,$$

For $m \in \mathbb{N}$ and $p \in \{2, \infty\}$, we introduce the notations

$$\|f(t)\|_{m,p}^2 = \sum_{\alpha \in \mathbb{N}^4, \ |\alpha| \leq m} \|(Z^\alpha f)(t)\|_{L^p(\Omega)}^2,$$

and

$$\|\|f\|\|_{m,p, T} = \sup_{t \in [0, T]} \|f(t)\|_{m,p}$$

for $T \geq 0$. 
The solutions $U = (\rho - 1, u)$ we consider will satisfy

$$
\mathcal{E}_m(T, U) = \|U\|^2_{m,2,T} + \|\partial_z u_\tau\|^2_{m-1,2,T} + \int_0^T \|\partial_z (\rho, u_3)(s)\|^2_{m-1,2} \, ds
$$

$$
+ \|\partial_z u_\tau\|^2_{1,\infty,T} + \int_0^T \| (\partial_z \rho, \partial_{tz} \rho)(s)\|^2_{1,\infty} \, ds < +\infty.
$$
The solutions \( U = (\rho - 1, u) \) we consider will satisfy

\[
\mathcal{E}_m(T, U) = \| U \|_{m, 2, T}^2 + \| \partial_z u_T \|_{m-1, 2, T}^2 + \int_0^T \| \partial_z (\rho, u_3)(s) \|_{m-1, 2}^2 \, ds
\]

\[
+ \| \partial_z u_T \|_{1, \infty, T}^2 + \int_0^T \| (\partial_z \rho, \partial_t \rho)(s) \|_{1, \infty}^2 \, ds < +\infty.
\]

The force term will satisfy, for every \( T \geq 0 \),

\[
\mathcal{N}_m(T, F) = \sup_{t \in [-T, T]} \left( \| F(t) \|_{m, 2}^2 + \| \partial_z F(t) \|_{m-1, 2}^2 + \| F(t) \|_{2, \infty}^2 \right) < +\infty
\]

and \( F(t, x) = 0 \) for \( t < 0 \).
Theorem 2.1 (submitted, 2014\textsuperscript{15})

- Under the conditions stated previously, for $\varepsilon_0 > 0$ and $m \geq 7$, there exists $T^* > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, there exists a unique solution $U$ to the isentropic Navier-Stokes system (9) with Navier boundary condition (11) in the class of functions satisfying $E_m(T^*, U) < +\infty$. Moreover, there is uniformly no vacuum on this time interval.

- The family of solutions $(U_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ constructed above converges in $L^2([0, T^*] \times \Omega)$ and in $L^\infty([0, T^*] \times \Omega)$ towards the unique solution of the isentropic Euler equation ((9) with $\varepsilon = 0$ and the non-penetration boundary condition) in the same class.

\textsuperscript{15}arxiv 1410.2811
Remarks.

- The results extend to less regular forces or different initial conditions, providing the compatibility conditions yield uniform bounds on $\mathcal{N}_m(0, U)$. 

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- The results extend to less regular forces or different initial conditions, providing the compatibility conditions yield uniform bounds on $\mathcal{N}_m(0, U)$.

- Wang and Williams\textsuperscript{16} proved similar results in the constant coefficient case by constructing approximate solutions, and needs prior existence of the solution to the Euler system.

Idea of the proof: \textit{a priori} estimates + compactness argument.
Idea of the proof: *a priori* estimates + compactness argument.

**Theorem 2.2**

Under the conditions of Theorem 2.1, for $\varepsilon_0 > 0$, there exist $T^* > 0$ and a positive, increasing function $Q : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that, for every $0 < \varepsilon < \varepsilon_0$ and $t \leq T^*$,

$$
\mathcal{E}_m(t, U) + \||\partial_z u_3||^2_{1,\infty,t} + \varepsilon \int_0^t \|\nabla u(s)\|^2_{m,2} + \|\nabla^2 u_\tau(s)\|^2_{m,2} \, ds \\
\leq (t + \varepsilon)Q(\mathcal{E}_m(t, U) + \mathcal{N}_m(t, F)).
$$

Throughout, $\rho$ is assumed to be uniformly bounded from below - this is checked by a bootstrap argument using the fact that, along the flow of $u$, $\partial_t \rho = -\rho \text{div} u$, and $\text{div} u$ is bounded.
Idea of the proof: *a priori* estimates + compactness argument.

**Theorem 2.2**

Under the conditions of Theorem 2.1, for \( \varepsilon_0 > 0 \), there exist \( T^* > 0 \) and a positive, increasing function \( Q : \mathbb{R}^+ \to \mathbb{R}^+ \) such that, for every \( 0 < \varepsilon < \varepsilon_0 \) and \( t \leq T^* \),

\[
\mathcal{E}_m(t, U) + \| \partial_z u_3 \|_{1, \infty, t}^2 + \varepsilon \int_0^t \| \nabla u(s) \|_{m, 2}^2 + \| \nabla^2 u_\tau(s) \|_{m, 2}^2 \, ds \\
\leq (t + \varepsilon)Q(\mathcal{E}_m(t, U) + \mathcal{N}_m(t, F)).
\]

Throughout, \( \rho \) is assumed to be uniformly bounded from below - this is checked by a bootstrap argument using the fact that, along the flow of \( u \), \( \partial_t \rho = -\rho \text{div } u \), and \( \text{div } u \) is bounded.
The energy estimates on $U$ use the symmetrisable hyperbolic-parabolic structure of the equation. The system on $U^\alpha = Z^\alpha U$ is

$$(DA_0)(\rho) \partial_t U^\alpha + \sum_{j=1}^{3} (DA_j)(U) \partial_{x_j} U^\alpha - \varepsilon(0, \text{div} (\mu \nabla u)^\alpha + (\mu + \lambda_0) \nabla \text{div} u^\alpha)$$

$$= (0, (\rho F + \varepsilon \sigma(\nabla U))^\alpha) - C^\alpha,$$

with the $DA_j(U)$ symmetric matrices, $\sigma(\nabla U)$ the remainder of the order-two terms and $C^\alpha$ containing commutators between $Z^\alpha$ and the operators involved in the equation (e.g. $[Z^\alpha, DA_j \partial_{x_j}] U$).

Basic inequalities are used in the energy estimate.
We now need estimates on $\partial_z U$: these are obtained component by component.
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The norms of $\partial_z u_3$ are easily controlled by using the equation of conservation of mass:

$$
\rho \partial_z u_3 = -\partial_t \rho - u \cdot \nabla \rho - \rho (\partial_x u_1 + \partial_y u_2)
$$
To deal with $\partial_z u_\tau$, we use a modified vorticity

$$W = (\text{curl } u)_\tau - \frac{2a}{\mu(\rho)} u_\tau^\perp,$$

so that we have $\rho \partial_t W + \rho u \cdot \nabla W - \varepsilon \mu \Delta W = H$ and $W = 0$ on the boundary.
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The commutator estimates are standard, except for $[Z^\alpha, \rho u_3 \partial_z]W$, in which a Hardy-type inequality is used to avoid the appearance of $\nabla^2 u$. 
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$L^\infty$ bounds are obtained using a variant of the maximum principle, on $W$ in the constant coefficient case, on $(\text{curl } u)_\tau$ otherwise.
The mass equation is used to control $R = \partial_z \rho$:

$$l \varepsilon (\partial_t R + u \cdot \nabla R) + \gamma P(\rho) R = h,$$

with $l = 2\mu + \lambda_0$.

The conormal energy estimates are $L^2$ in time.
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with $l = 2\mu + \lambda_0$.

The conormal energy estimates are $L^2$ in time.

By following the flow of $u$, we reduce the equation to the ODE $l \varepsilon \partial_t R + \gamma PR = h$, and the Duhamel formula yields the $L^\infty$ estimates. □
1. Stability and instability in the 2D incompressible model
   - The inviscid limit problem and boundary layers
   - Navier boundary conditions and $L^2$ stability
   - $L^\infty$ instability of shear flows

2. Convergence of strong solutions in the 3D isentropic model
   - The equations and conormal derivatives
   - Uniform existence and convergence towards Euler
   - Improvements and possible extensions
A recent preprint obtains a similar result\(^{17}\)

- using the natural energy structure of the Navier-Stokes system to get estimates on \((u, P)\),
- using \(H^1_{co}\) bounds on \(\Delta P\) to control \(\|\|\rho\|\|_{1,\infty}\).

\(^{17}\)Wang, Xin & Yong, arxiv 1501.01718
A recent preprint obtains a similar result\textsuperscript{17}

- using the natural energy structure of the Navier-Stokes system to get estimates on \((u, P)\),
- using \(H^1\) bounds on \(\Delta P\) to control \(\|\rho\|_{1,\infty}\).

They thus exhibit weaker boundary layers on \(\rho\) compared to \(u\), and, on the domain \(\mathbb{T} \times (0, 1)\), they obtain the rates of convergence:

\[
\|U^\epsilon(t) - U^E(t)\|_{L^2} \leq C \epsilon^2 \quad \text{and} \quad \|U^\epsilon(t) - U^E(t)\|_{L^\infty} \leq C \epsilon^{2/5}.
\]

\textsuperscript{17}Wang, Xin & Yong, arxiv 1501.01718
Try the same approach as ours on the full Navier-Stokes equation:

\[
\begin{aligned}
\partial_t \rho + \text{div} (\rho u) &= 0 \\
\rho \partial_t u + \rho u \cdot \nabla u &= \text{div} \Sigma + \rho F \\
\rho \partial_t \theta + \rho u \cdot \nabla \theta + P \text{div} u &= \varepsilon \kappa \Delta \theta + \Sigma : \nabla u + \rho u \cdot F.
\end{aligned}
\]  

- Internal energy proportional to the temperature $\theta$
- Neumann boundary condition for $\theta$
- Ideal gas pressure law $P(\rho, \theta) = k \rho \theta$
Disaster strikes! The component-by-component examination of the normal derivatives has worked for the incompressible and isentropic models, but fails in this case!

The problem: the terms involving $\text{div } u$ in the equations on $\rho$ and $\theta$.

The field $\partial_z V = (\partial_z \rho, \partial_z u_3, \partial_z \theta)$ cannot be decoupled.
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The field $\partial_z V = (\partial_z \rho, \partial_z u_3, \partial_z \theta)$ cannot be decoupled.

Work in progress: we can get $H^{m-1}_{co}$ estimates on $\partial_z V$, but we still need $W^{1,\infty}_{co}$ bounds: Green function study?
Thank you for your attention.