Mean field kinetic particles and the Vlasov-Fokker-Planck equation

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1. Introduction
   - The model
   - Asymptotics and distances
   - Results

2. Preliminary considerations
   - Hypocoercivity without interaction
   - Interaction without hypocoercivity
   - Hamiltonian equilibrium

3. Chain of results

4. Conclusion
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   - Hypocoercivity without interaction
   - Interaction without hypocoercivity
   - Hamiltonian equilibrium

3. **Chain of results**

4. **Conclusion**
Kinetic particle

- $X(t) \in \mathbb{R}^d$ position at time $t$
- $Y(t) \in \mathbb{R}^d$ velocity at time $t$
- $U : \mathbb{R}^d \to \mathbb{R}$ external potential
- $B(t)$ Brownian motion $d$-dimensional

Newton’s law of motion:

\[
\begin{align*}
\text{d}X &= Y\text{d}t \\
\text{d}Y &= -\nabla U(X)\text{d}t - Y\text{d}t + \sqrt{2}\text{d}B
\end{align*}
\]
Kinetic particle

- \( X(t) \in \mathbb{R}^d \) position at time \( t \)
- \( Y(t) \in \mathbb{R}^d \) velocity at time \( t \)
- \( U : \mathbb{R}^d \to \mathbb{R} \) external potential
- \( B(t) \) Brownian motion \( d \)-dimensional

Newton’s law of motion:

\[
\begin{cases}
  dX &= Y dt \\
  dY &= -\nabla U(X) dt - Y dt + \sqrt{2} dB
\end{cases}
\]

The law \( m_t = \mathcal{L}(X_t, Y_t) \) is a (weak) solution of

\[
\partial_t m_t + y \cdot \nabla_x m_t = \nabla_y \cdot (\nabla_y m_t + (\nabla U(x) + y) m_t),
\]

the Langevin (or kinetic Fokker-Planck) equation.
Law of large numbers

- \( Z_i = (X_i, Y_i) \) i.i.d. particles, \( i \in [1, N] \)

- empirical measure

\[
\pi_t^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{Z_j}
\]

Morally,

\[
\pi_t^N \xrightarrow{N \to \infty} m_t.
\]
Mean field interaction

- $W : \mathbb{R}^d \to \mathbb{R}$ an even interaction potential

For $i \in [1, N]$,

$$dX_i = Y_i dt$$

$$dY_i = -\nabla U(X_i) dt - \frac{1}{N} \sum_{j=1}^{N} \nabla W(X_i - X_j) dt - Y_i dt + \sqrt{2} dB_i$$
Mean field interaction

- $W : \mathbb{R}^d \rightarrow \mathbb{R}$ an even interaction potential

For $i \in [1, N]$

\[
\begin{align*}
\mathrm{d}X_i &= Y_i \mathrm{d}t \\
\mathrm{d}Y_i &= -\nabla U(X_i) \mathrm{d}t - \frac{1}{N} \sum_{j=1}^{N} \nabla W(X_i - X_j) \mathrm{d}t - Y_i \mathrm{d}t + \sqrt{2} \mathrm{d}B_i \\
&= \int \nabla W(X_i - u) \pi_t^N(du, dv)
\end{align*}
\]
Mean field interaction

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For $i \in [1, N]$, 

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&= \int \nabla W(X_i - u) \pi_t^N(\text{d}u, \text{d}v)
\end{align*}
$$

Assuming $\pi_t^N \xrightarrow{N \to \infty} m_t$, 

$$
\partial_t m_t + y \cdot \nabla_x m_t = \nabla_y \cdot (\nabla_y m_t + (\nabla U + \nabla W \ast m_t + y) m_t)
$$

with $\nabla W \ast m_t(x) = \int \nabla W(x - u) m_t(u, v) \text{d}u \text{d}v$ (Vlasov-Fokker-Planck).
Non-linear process

For $i \in [1, N]$, 

\[
\begin{align*}
\text{d} \tilde{X}_i &= \tilde{Y}_i \text{d}t \\
\text{d} \tilde{Y}_i &= -\nabla U \left( \tilde{X}_i \right) \text{d}t - (\nabla W * m_t) \left( \tilde{X}_i \right) - \tilde{Y}_i \text{d}t + \sqrt{2} \text{d}B_i \\
\text{m}_t &= \mathcal{L} \left( \tilde{X}_i(t), \tilde{Y}_i(t) \right)
\end{align*}
\]
Non-linear process

For $i \in [1, N]$, 

\[
\begin{cases}
\quad \quad d\tilde{X}_i & = \quad \tilde{Y}_i dt \\
\quad d\tilde{Y}_i & = \quad -\nabla U (\tilde{X}_i) dt - (\nabla W \ast m_t) (\tilde{X}_i) - \tilde{Y}_i dt + \sqrt{2} dB_i \\
\quad m_t & = \quad \mathcal{L} (\tilde{X}_i(t), \tilde{Y}_i(t))
\end{cases}
\]

We are interested in :

- The law $m_t$ that solves the non-linear PDE,
- The non-independent $Z_i = (X_i, Y_i)$ with $Z = (Z_1, \ldots, Z_N)$ Markov,
- The independent $\tilde{Z}_i = (\tilde{X}_i(t), \tilde{Y}_i(t))$ with law $m_t$, $\tilde{Z}$ non Markov.
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Asymptotics

- \( N \to \infty \): propagation of chaos

- \( t \to \infty \): convergence to equilibrium

If the \( Z_i(0) = \tilde{Z}_i(0) \) are i.i.d. with law \( m_0 \), when \( N \to \infty \), \( \pi_N t = \frac{1}{N} \sum \delta_{Z_i} \) should converge to \( m_t \), \( Z_1 \) should behave like \( \tilde{Z}_1 \), \( m(N)(t) = L(Z_1(t)) \) should converge to \( m_{\infty} \).

If the potential \( U \) is confining enough, \( Z \) is ergodic.

The law \( m(N)(t) = L(Z(t)) \) converges to a unique equilibrium \( m_\infty \).

Behaviour of \( m_t \)? possibly several equilibria...

Goal: quantitative estimates for the speed of these convergences.
Asymptotics

- $N \to \infty$: propagation of chaos

  If the $Z_i(0) = \tilde{Z}_i(0)$ are i.i.d. with law $m_0$, when $N \to \infty$,
  
  $\pi^N_t = \frac{1}{N} \sum \delta_{Z_i}$ should converge to $m_t$,
  
  $Z_1$ should behave like $\tilde{Z}_1$,
  
  $m^{(1,N)}_t = \mathcal{L}(Z_1(t))$ should converge to $m_t$.

- $t \to \infty$: convergence to equilibrium

  If the potential $U$ is confining enough, $Z$ is ergodic
  
  The law $m^{(N)}_t = \mathcal{L}(Z)$ converges to a unique equilibrium $m^{(N)}_{\infty}$
  
  Behaviour of $m_t$? possibly several equilibria...
Asymptotics

- $N \to \infty$ : propagation of chaos

  If the $Z_i(0) = \widetilde{Z}_i(0)$ are i.i.d. with law $m_0$, when $N \to \infty$,
  - $\pi_t^N = \frac{1}{N} \sum \delta_{Z_i}$ should converge to $m_t$,
  - $Z_1$ should behave like $\widetilde{Z}_1$,
  - $m_t^{(1,N)} = \mathcal{L}(Z_1(t))$ should converge to $m_t$.

- $t \to \infty$ : convergence to equilibrium

  If the potential $U$ is confining enough, $Z$ is ergodic
  - The law $m_t^{(N)} = \mathcal{L}(Z)$ converges to a unique equilibrium $m_\infty^{(N)}$
  - Behaviour of $m_t$? possibly several equilibria...
Asymptotics

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- \( t \to \infty \): convergence to equilibrium

  If the potential \( U \) is confining enough, \( Z \) is ergodic
  
  - The law \( m_t^{(N)} = \mathcal{L}(Z) \) converges to a unique equilibrium \( m_\infty^{(N)} \)
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Goal: quantitative estimates for the speed of these convergences.
Distances

Coupling of two laws:

\[ \Pi(\mu, \nu) = \{(Q, R) \text{ r.v. such that } \mathcal{L}(Q) = \mu, \mathcal{L}(R) = \nu\} . \]

- Total variation distance:

\[ d_{VT}(\mu, \nu) = \inf_{\Pi(\mu, \nu)} \mathbb{P}(Q \neq R) \]

\[ = \frac{1}{2} \|\mu - \nu\|_1 \quad \text{(if density)} \]

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- Wasserstein \( \mathcal{W}_2 \) distance :

\[ \mathcal{W}_2^2(\mu, \nu) = \inf_{\Pi(\mu, \nu)} \mathbb{E} \left( |Q - R|^2 \right) \]
Distances

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- Relative entropy (Kullback-Leibler divergence):

\[ \mathcal{H}(\mu \mid \nu) = \int \ln \left( \frac{d\mu}{d\nu} \right) d\mu \]
Dependency in $N$

If

- $Q = (Q_1, \ldots, Q_N)$ of law $\mu^{(N)}$ with exchangeable $Q_i$'s
- $R = (R_1, \ldots, R_N)$ of law $\nu \otimes N$ with i.i.d. $R_i$'s

Then

$$\mathbb{E} (|Q - R|^2) = \sum_{i=1}^{N} \mathbb{E} (|Q_i - R_i|^2) = N \mathbb{E} (|Q_1 - R_1|^2)$$

Hence denoting by $\mu^{(1,N)}$ the law of $Q_1$, 

$$\mathcal{W}_2^2 \left( \mu^{(1,N)}, \nu \right) \leq \frac{1}{N} \mathcal{W}_2^2 \left( \mu^{(N)}, \nu \otimes N \right).$$

If moreover the $Q_i$ are independent with law $\mu$, 

$$\mathcal{W}_2^2 \left( \mu, \nu \right) = \frac{1}{N} \mathcal{W}_2^2 \left( \mu \otimes N, \nu \otimes N \right).$$
Dependency in $N$

With again exchangeable $Q_i$’s and i.i.d. $R_i$’s (Csiszár’s Inequality):

$$\mathcal{H} \left( \mu^{(1,N)} \mid \nu \right) \leq \frac{2}{N} \mathcal{H} \left( \mu^{(N)} \mid \nu^\otimes N \right)$$

Under our assumptions (to come), there will exist $K$ independent from $N$ such that

$$\mathcal{H} \left( m_0^\otimes N \mid m^{(N)}_\infty \right) \leq KN.$$
Dependency in $N$

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Bad candidate, albeit usual quantity:

$$\text{Var}_{\nu}(\mu) = \int \left(\frac{d\mu}{d\nu} - 1\right)^2 d\nu$$

$\Rightarrow$ Hilbert norm, spectral theory, long-time convergence... but

$$\text{Var}_{\nu \otimes N}(\mu \otimes N) = (\text{Var}_{\nu}(\mu) + 1)^N - 1$$
Functional inequalities

- Pinsker’s Inequality:

\[ \|\mu - \nu\|_1^2 \leq \frac{1}{2} \mathcal{H}(\mu | \nu). \]
Functional inequalities

- Pinsker’s Inequality:
  \[ \|\mu - \nu\|_1^2 \leq \frac{1}{2} \mathcal{H}(\mu \mid \nu). \]

- We say \( \nu \) satisfies a log-Sobolev inequality if \( \exists C \) s.t. \( \forall \mu \prec \nu \),
  \[ \mathcal{H}(\mu \mid \nu) \leq C \int \left| \nabla \ln \frac{d\mu}{d\nu} \right|^2 d\mu. \]
Functional inequalities

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\]

- A log-Sobolev inequality implies a Talagrand’s \(T_2\) one: \(\forall \mu\),

\[
\mathcal{W}_2^2(\mu, \nu) \leq C \mathcal{H}(\mu \mid \nu).
\]
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**Assumptions**

A

- The external potential $U$ is convex ($\nabla^2 U \geq c_1 > 0$) and $\nabla^2 W \geq -c_2$ with $c_2 < \frac{1}{2}c_1$. Moreover $\nabla^2 U$ and $\nabla^2 W$ are bounded.
- The law $m_0$ has a Lebesgue density, a finite 2nd moment and $\int m_0 \ln m_0 < \infty$. 

Remarks:

- Forbid the Coulomb interaction $W_c(x-y) = \pm \frac{1}{|x-y|}$, but allow $\xi^* W_c$ with a smooth kernel, provided $U$ is convex enough.
- $W_c$ small enough is not needed (contrary to [Villani 2007, Bolley-Guillin-Malrieu 2010, Hérau-Thomann 2015]).
Assumptions

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Remarks:

- Forbid the Coulomb interaction $W_c(x - y) = \pm \frac{1}{|x-y|}$, but allow $\xi * W_c$ with a smooth kernel, provided $U$ is convex enough.
- $\ll W$ small enough $\gg$ not needed (contrary to [Villani 2007, Bolley-Guillin-Malrieu 2010, Hérau-Thomann 2015]).
Results

**Theorem (M., 2016)**

Under Assumption $A$, the exist $C, \chi > 0$ which depend neither on $t$, nor $N$, nor $m_0$, and there exists $K$ that depends on $m_0$ but not on $t, N$, such that

- For the particle system, $m^{(N)}_\infty$ satisfies a log-Sobolev inequality with constant independent from $N$ and

\[
\mathcal{H} \left( m^{(N)}_t \mid m^{(N)}_\infty \right) \leq C e^{-\chi t} \mathcal{H} \left( m^{(N)}_0 \mid m^{(N)}_\infty \right).
\]

- The Vlasov-Fokker-Planck PDE admits a unique equilibrium $m_\infty$ and

\[
\| m_t - m_\infty \|_1 \leq K e^{-\chi t}, \quad \mathcal{W}_2 \left( m_t, m_\infty \right) \leq K e^{-\chi t}.
\]
Results

Theorem (M., 2016)

Under Assumption $A$, there exist $b, \alpha, > 0$ that depend neither on $t$, nor $N$, nor $m_0$, and there exists $K$ that depends on $m_0$ but not on $t, N$, such that

- **Uniform in time propagation of chaos**: 

  $$W_2 \left( m_t^{(1,N)}, m_t \right) \leq K \min \left( \frac{e^{bt}}{N}, \frac{1}{N^\alpha} \right)$$

  and

  $$\| m_t^{(1,N)} - m_t \|_1 \leq \frac{K}{N^\alpha}.$$

- **Numerical error bound** (cf. Bolley-Guillin-Villani 2006) :

  $$\mathbb{P} \left( W_2 \left( \pi^N_t, m_\infty \right) \geq \varepsilon \right) \leq \frac{K}{\varepsilon^2} \left( e^{-\chi t} + \frac{1}{N} \right)$$
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Coercivity without interaction

\[
\begin{aligned}
\begin{cases}
    dX &= Y dt \\
    MdY &= -\nabla U(X) dt - Y dt + \sqrt{2} dB
\end{cases}
\end{aligned}
\]

When the mass \( M \to 0 \), overdamped Langevin (or Fokker-Planck) diffusion:

\[
\begin{aligned}
dX &= -\nabla U(X) dt + \sqrt{2} dB
\end{aligned}
\]

with equilibrium \( \rho_\infty(dx) = e^{-U(x)} dx \) and whose law \( \rho_t \) satisfies

\[
\begin{aligned}
\partial_t \rho_t &= \nabla \cdot (\rho_t \nabla U + \nabla \rho_t).
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Coercivity without interaction

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\begin{align*}
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dX = -\nabla U(X) dt + \sqrt{2} dB
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with equilibrium $\rho_\infty(dx) = e^{-U(x)} dx$ and whose law $\rho_t$ satisfies

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\partial_t \rho_t = \nabla \cdot (\rho_t \nabla U + \nabla \rho_t).
\]

Large-time convergence, assuming a log-Sobolev inequality holds:

\[
\partial_t \left( \mathcal{H} \left( \rho_t \mid \rho_\infty \right) \right) = -\int \left| \nabla \ln \frac{d\rho_t}{d\rho_\infty} \right|^2 d\rho_t \leq -\frac{1}{C} \mathcal{H} \left( \rho_t \mid \rho_\infty \right)
\]

\[
\Rightarrow \quad \mathcal{H} \left( \rho_t \mid \rho_\infty \right) \leq e^{-\frac{t}{C}} \mathcal{H} \left( \rho_0 \mid \rho_\infty \right).
\]
Hypoercivity without interaction

\[\begin{align*}
\{ & \quad \text{d}X = Y \text{d}t \\
& \quad \text{d}Y = -\nabla U(X) \text{d}t - Y \text{d}t + \sqrt{2} \text{d}B
\end{align*}\]

The entropy dissipation may vanish outside of equilibrium:

\[\partial_t (\mathcal{H} (m_t | m_\infty)) = - \int \left| \nabla_y \ln \frac{d m_t}{d m_\infty} \right|^2 d m_t.\]
Hypoercivity without interaction

\[
\begin{aligned}
\begin{cases}
\frac{dX}{dt} &= Y \\
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\partial_t (H(m_t | m_\infty)) = - \int \left| \nabla_y \ln \frac{d m_t}{d m_\infty} \right|^2 d m_t.
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Modified entropy (Hérau 2006, Villani 2007): set \( h_t = \frac{d m_t}{d m_\infty} \) and

\[
\mathcal{N}(h) := \int h \ln h d m_\infty + \int |P \nabla \ln h|^2 h d m_\infty.
\]

With a well-chosen \( P \) and log-Sobolev inequality,

\[
\partial_t (\mathcal{N}(h_t)) \leq -c \int |\nabla \log h_t|^2 d m_t \leq -c' \mathcal{N}(h_t)
\]
Hypoercivity without interaction

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\Rightarrow \quad \mathcal{N}(h_t) \leq e^{-\frac{(t-t_0)}{c'}} \mathcal{N}(h_{t_0})
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Hypoercivity without interaction

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With a well-chosen \( P \) and log-Sobolev inequality,

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\partial_t \left( \mathcal{N}(h_t) \right) \leq -c \int \left| \nabla \log h_t \right|^2 dmt \leq -c' \mathcal{N}(h_t)
\]

\[
\Rightarrow \mathcal{H}(m_t \mid m_\infty) \leq \mathcal{N}(h_t) \leq e^{-\frac{(t-t_0)}{c'}} \mathcal{N}(h_{t_0}) \overset{\text{regul.}}{\leq} Ce^{-\frac{t}{c'}} \mathcal{H}(m_0 \mid m_\infty)
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The McKean-Vlasov equation

Overdamped mean-field particles: for $i \in [1, N]$,

$$dX_i = -\nabla U(X_i) dt - \frac{1}{N} \sum_{j=1}^{N} \nabla W(X_i - X_j) dt + \sqrt{2} dB_i$$

$$d\tilde{X}_i = -\nabla U(\tilde{X}_i) dt - \int \nabla W(\tilde{X}_i - u) \rho_t(u) du dt + \sqrt{2} dB_i$$

$$\rho_t = L(\tilde{X}_1) .$$

Elliptic but non-linear EDP:

$$\partial_t \rho_t = \nabla \cdot (\nabla \rho_t + (\nabla U + \nabla W * \rho_t) \rho_t) .$$

Parallel coupling: same Brownian motions $B_i$ for both $X_i$ and $\tilde{X}_i$. 
Propagation of chaos

Denote by

\[ \alpha(t) = \mathbb{E} \left( \left| X_1(t) - \tilde{X}_1(t) \right|^2 \right) = \frac{1}{N} \mathbb{E} \left( \left| X(t) - \tilde{X}(t) \right|^2 \right). \]

Parallel coupling and convexity assumption (Malrieu 2001):

\[ \alpha'(t) \leq -c\alpha(t) + \frac{K}{\sqrt{N}} \sqrt{\alpha(t)} \]
Propagation of chaos

Denote by

\[ \alpha(t) = \mathbb{E}\left(\left| X_1(t) - \tilde{X}_1(t) \right|^2 \right) = \frac{1}{N} \mathbb{E}\left(\left| X(t) - \tilde{X}(t) \right|^2 \right). \]

Parallel coupling and convexity assumption (Malrieu 2001):

\[ \alpha'(t) \leq -c\alpha(t) + \frac{K}{\sqrt{N}} \sqrt{\alpha(t)} \quad \Rightarrow \quad \sqrt{\alpha(t)} \leq \frac{K}{c\sqrt{N}} \]

Consequences:

\[ W_2^2(\rho(1,N)^t,\rho^t) \leq K N W_2^2(\rho(1,\infty)^\infty,\rho^\infty) \leq K N \quad \text{for a unique } \rho^\infty; \text{ moreover a log-Sobolev inequality for } \rho(\infty) \text{ independently from } N, \text{ and for } \rho^\infty \]
Propagation of chaos

Denote by

$$\alpha(t) = \mathbb{E} \left( |X_1(t) - \tilde{X}_1(t)|^2 \right) = \frac{1}{N} \mathbb{E} \left( |X(t) - \tilde{X}(t)|^2 \right).$$

Parallel coupling and convexity assumption (Malrieu 2001):

$$\alpha'(t) \leq -c\alpha(t) + \frac{K}{\sqrt{N}} \sqrt{\alpha(t)} \quad \Rightarrow \quad \sqrt{\alpha(t)} \leq \frac{K}{c\sqrt{N}}$$

Consequences:

$$\mathcal{W}_2^2 \left( \rho_t^{(1,N)}, \rho_t \right) \leq \frac{K}{N}$$

$$\mathcal{W}_2^2 \left( \rho_\infty^{(1,N)}, \rho_\infty \right) \leq \frac{K}{N}$$

(for a unique $\rho_\infty$; moreover a log-Sobolev inequality for $\rho_\infty^{(N)}$ independently from $N$, and for $\rho_\infty$)
1 Introduction
   • The model
   • Asymptotics and distances
   • Results

2 Preliminary considerations
   • Hypocoercivity without interaction
   • Interaction without hypocoercivity
   • Hamiltonian equilibrium

3 Chain of results

4 Conclusion
Without interaction

\[
\begin{aligned}
    \left\{ \begin{array}{ll}
        \frac{dX}{dt} &= Y dt \\
        \frac{dY}{dt} &= -\nabla U(X) dt - Y dt + \sqrt{2} dB.
    \end{array} \right.
\end{aligned}
\]

Denoting by \( \gamma \) the standard Gaussian density, the equilibrium is

\[
m_\infty(dx, dy) = \left( e^{-U(x)} dx \right) \otimes \gamma(dy)
= \rho_\infty(dx) \otimes \gamma(dy)
\]

where \( \rho_\infty \) is the invariant measure of

\[
\frac{dX}{dt} = -\nabla U(X) dt + \sqrt{2} dB.
\]
With interaction

\[
\begin{aligned}
\text{d}\tilde{X} &= \tilde{Y}\,dt \\
\text{d}\tilde{Y} &= -\nabla U(\tilde{X})\,dt - (\nabla W \ast m_t)(\tilde{X}) - \tilde{Y}\,dt + \sqrt{2}\,dB \\
m_t &= \mathcal{L}(\tilde{X}(t),\tilde{Y}(t)).
\end{aligned}
\]

Then (Duong-Tugaut 2016)

\[
m_\infty(dx,dy) = \rho_\infty(dx) \otimes \gamma(dy)
\]

is a bijection between the equilibria \(m_\infty\) and the equilibria \(\rho_\infty\) of

\[
\text{d}\tilde{X} = -\nabla U(\tilde{X})\,dt - \int \nabla W(\tilde{X} - u)\,\rho_t(u)\,du\,dt + \sqrt{2}\,dB.
\]
1 Introduction
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4 Conclusion
Large time for the particle system

- The system $Z = (X, Y) \in \mathbb{R}^{2dN}$ satisfies a Langevin SDE

\[
\begin{align*}
    dX &= Y \, dt \\
    dY &= -\nabla U_N(X) dt - Y \, dt + \sqrt{2} \, dB
\end{align*}
\]

with $U_N(x) = \frac{1}{2N} \sum_{i,j=1}^{N} (U(x_i) + U(x_j) + W(x_i - x_j))$.
Large time for the particle system

- The system \( Z = (X, Y) \in \mathbb{R}^{2dN} \) satisfies a Langevin SDE
  \[
  \begin{align*}
  \text{d}X &= Y \text{d}t \\
  \text{d}Y &= -\nabla U_N(X) \text{d}t - Y \text{d}t + \sqrt{2} \text{d}B
  \end{align*}
  \]
  with \( U_N(x) = \frac{1}{2N} \sum_{i,j=1}^{N} (U(x_i) + U(x_j) + W(x_i - x_j)) \),

- Convexity \( \Rightarrow \) log-Sobolev independent from \( N \) for \( \rho^{(N)} = e^{-U_N} \)
Large time for the particle system

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- Convexity \( \Rightarrow \) log-Sobolev independent from \( N \) for \( \rho_\infty^{(N)} = e^{-U_N} \)

- \( \Rightarrow \) log-Sobolev independent from \( N \) for \( m_\infty^{(N)} = \rho_\infty^{(N)} \otimes \gamma \) (\& unique)
Large time for the particle system

- The system $Z = (X, Y) \in \mathbb{R}^{2dN}$ satisfies a Langevin SDE
  
  \[
  \begin{aligned}
  dX &= Y \, dt \\
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  \end{aligned}
  \]

  with $U_N(x) = \frac{1}{2N} \sum_{i,j=1}^{N} (U(x_i) + U(x_j) + W(x_i - x_j))$,

- Convexity $\Rightarrow$ log-Sobolev independent from $N$ for $\rho^{(N)}_\infty = e^{-U_N}$

- $\Rightarrow$ log-Sobolev independent from $N$ for $m^{(N)}_\infty = \rho^{(N)}_\infty \otimes \gamma$ (& unique)

- + modified entropy + mean field,

  $\Rightarrow$ \[
  \mathcal{H} \left( m^{(N)}_t \mid m^{(N)}_\infty \right) \leq C e^{-\chi t} \mathcal{H} \left( m^{\otimes N}_0 \mid m^{(N)}_\infty \right)
  \]
Large time for the particle system

- The system \( Z = (X, Y) \in \mathbb{R}^{2dN} \) satisfies a Langevin SDE

\[
\begin{align*}
    dX &= Y \, dt \\
    dY &= -\nabla U_N(X) \, dt - Y \, dt + \sqrt{2} \, dB
\end{align*}
\]

with \( U_N(x) = \frac{1}{2N} \sum_{i,j=1}^{N} (U(x_i) + U(x_j) + W(x_i - x_j)) \),

- Convexity \( \Rightarrow \) log-Sobolev independent from \( N \) for \( \rho_{\infty}^{(N)} = e^{-U_N} \)
- \( \Rightarrow \) log-Sobolev independent from \( N \) for \( m_{\infty}^{(N)} = \rho_{\infty}^{(N)} \otimes \gamma \) (\& unique)
- + modified entropy + mean field,

\[
\mathcal{H} \left( m_t^{(N)} \mid m_{\infty}^{(N)} \right) \leq C e^{-\chi t} \mathcal{H} \left( m_0^{\otimes N} \mid m_{\infty}^{(N)} \right) \leq K Ne^{-\chi t}
\]

with \( C, \chi, K \) independent from \( t \) and \( N \).
Large time for the particle system

- The system $Z = (X, Y) \in \mathbb{R}^{2dN}$ satisfies a Langevin SDE
  \[
  \begin{cases}
  dX = Y \, dt \\
  dY = -\nabla U_N(X) \, dt - Y \, dt + \sqrt{2} \, dB
  \end{cases}
  \]
  with $U_N(x) = \frac{1}{2N} \sum_{i,j=1}^{N} (U(x_i) + U(x_j) + W(x_i - x_j))$,

- Convexity $\Rightarrow$ log-Sobolev independent from $N$ for $\rho_{\infty}^{(N)} = e^{-U_N}$

- $\Rightarrow$ log-Sobolev independent from $N$ for $m_{\infty}^{(N)} = \rho_{\infty}^{(N)} \otimes \gamma$ (\& unique)

- + modified entropy + mean field,

  \[
  \Rightarrow \quad \mathcal{H} \left( m_t^{(N)} \mid m_{\infty}^{(N)} \right) \leq C e^{-\chi t} \mathcal{H} \left( m_0^{\otimes N} \mid m_{\infty}^{(N)} \right) \leq K Ne^{-\chi t}
  \]

  with $C, \chi, K$ independent from $t$ and $N$.

- + Talagrand Inequality independent from $N$,

  \[
  \mathcal{W}_2^2 \left( m_t^{(N)}, m_{\infty}^{(N)} \right) \leq K Ne^{-\chi t}.
  \]
Crude propagation of chaos

Parallel coupling between

\[
\begin{align*}
\text{d}X_i &= Y_i \text{d}t \\
\text{d}Y_i &= -\nabla U(X_i) \text{d}t - \frac{1}{N} \sum_{j=1}^{N} \nabla W(X_i - X_j) \text{d}t - Y_i \text{d}t + \sqrt{2} \text{d}B_i
\end{align*}
\]

and

\[
\begin{align*}
\text{d}\tilde{X}_i &= \tilde{Y}_i \text{d}t \\
\text{d}\tilde{Y}_i &= -\nabla U(\tilde{X}_i) \text{d}t - (\nabla W \ast m_t)(\tilde{X}_i) - \tilde{Y}_i \text{d}t + \sqrt{2} \text{d}B_i.
\end{align*}
\]

If the forces were close to be linear (Bolley-Guillin-Malrieu 2010) we could have a coercive drift.
Crude propagation of chaos

Parallel coupling between

\[
\begin{align*}
\frac{dX_i}{dt} &= Y_i dt \\
\frac{dY_i}{dt} &= -\nabla U(X_i) dt - \frac{1}{N} \sum_{j=1}^{N} \nabla W(X_i - X_j) dt - Y_i dt + \sqrt{2} dB_i
\end{align*}
\]

and

\[
\begin{align*}
\frac{d\tilde{X}_i}{dt} &= \tilde{Y}_i dt \\
\frac{d\tilde{Y}_i}{dt} &= -\nabla U(\tilde{X}_i) dt - (\nabla W \ast m_t)(\tilde{X}_i) - \tilde{Y}_i dt + \sqrt{2} dB_i.
\end{align*}
\]

If the forces were close to be linear (Bolley-Guillin-Malrieu 2010) we could have a coercive drift. Here we simply say

\[
\alpha'(t) \leq b\alpha(t) + \frac{K}{\sqrt{N}} \sqrt{\alpha(t)}
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Crude propagation of chaos

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\begin{align*}
    dX_i &= Y_i dt \\
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\end{align*}
\]

and

\[
\begin{align*}
    d\tilde{X}_i &= \tilde{Y}_i dt \\
    d\tilde{Y}_i &= -\nabla U(\tilde{X}_i) dt - (\nabla W \ast m_t)(\tilde{X}_i) - \tilde{Y}_i dt + \sqrt{2} dB_i.
\end{align*}
\]

If the forces were close to be linear (Bolley-Guillin-Malrieu 2010) we could have a coercive drift. Here we simply say

\[
\alpha'(t) \leq b\alpha(t) + \frac{K}{\sqrt{N}} \sqrt{\alpha(t)}
\]

\[
\Rightarrow \quad \mathbb{E} \left( \left| Z_1(t) - \tilde{Z}_1(t) \right|^2 \right) \leq \frac{K e^{bt}}{N}.
\]
Large-time in $\mathcal{W}_2$ for the non-linear process

At fixed $t$ and for all $N$,

$$\mathcal{W}_2 (m_t, m_\infty)$$

$$\leq \mathcal{W}_2 (m_t, m_t^{(1,N)}) + \mathcal{W}_2 (m_t^{(1,N)}, m_\infty^{(1,N)}) + \mathcal{W}_2 (m_\infty^{(1,N)}, m_\infty)$$
At fixed $t$ and for all $N$,

\[ \mathcal{W}_2(m_t, m_\infty) \]

\[ \leq \mathcal{W}_2\left(m_t, m_t^{(1,N)}\right) + \mathcal{W}_2\left(m_t^{(1,N)}, m_\infty^{(1,N)}\right) + \mathcal{W}_2\left(m_\infty^{(1,N)}, m_\infty\right) \]

- crude prop chaos
- large time Markov
- prop chaos equilibrium
Large-time in $\mathcal{W}_2$ for the non-linear process

At fixed $t$ and for all $N$,

$$\mathcal{W}_2 (m_t, m_\infty)$$

\[
\leq \quad \mathcal{W}_2 \left( m_t, m_t^{(1,N)} \right) + \mathcal{W}_2 \left( m_t^{(1,N)}, m_\infty^{(1,N)} \right) + \mathcal{W}_2 \left( m_\infty^{(1,N)}, m_\infty \right)
\]

\[
\leq \quad Ke^{bt} + Ke^{-\chi t} + \frac{K}{N}
\]
At fixed $t$ and for all $N$,

$$\mathcal{W}_2 (m_t, m_\infty)$$

\[
\leq \mathcal{W}_2 \left( m_t, m_t^{(1,N)} \right) + \mathcal{W}_2 \left( m_t^{(1,N)}, m_\infty^{(1,N)} \right) + \mathcal{W}_2 \left( m_\infty^{(1,N)}, m_\infty \right)
\]

- crude prop chaos
- large time Markov
- prop chaos equilibrium

\[
\leq \frac{K e^{bt}}{N} + K e^{-\chi t} + \frac{K}{N}
\]

$$N \rightarrow \infty \quad K e^{-\chi t}.$$
Uniform propagation of chaos in $\mathcal{W}_2$

For $t \leq \epsilon \ln N$,

For $t \geq \epsilon \ln N$,
Uniform propagation of chaos in $\mathcal{W}_2$

For $t \leq \varepsilon \ln N$, parallel coupling:

$$\mathcal{W}_2^2 \left( m_t^{(1,N)}, m_t \right) \leq \frac{K e^{bt}}{N} \leq \frac{K}{N^{1-b\varepsilon}}.$$  

For $t \geq \varepsilon \ln N$, 

Conclusion, for all time,

$$\mathcal{W}_2^2 \left( m_t^{(1,N)}, m_t \right) \leq K N^{\alpha}.$$
Uniform propagation of chaos in $\mathcal{W}_2$

For $t \leq \varepsilon \ln N$, parallel coupling:

$$\mathcal{W}_2^2 \left( m_t^{(1,N)}, m_t \right) \leq \frac{Ke^{bt}}{N} \leq \frac{K}{N^{1-b\varepsilon}}.$$ 

For $t \geq \varepsilon \ln N$, coupling through the equilibria:

$$\mathcal{W}_2 \left( m_t^{(1,N)}, m_t \right) \leq \mathcal{W}_2 \left( m_t^{(1,N)}, m_\infty \right) + \mathcal{W}_2 \left( m_\infty^{(1,N)}, m_\infty \right) + \mathcal{W}_2 \left( m_\infty, m_t \right)$$
Uniform propagation of chaos in $\mathcal{W}_2$

For $t \leq \varepsilon \ln N$, parallel coupling:

$$\mathcal{W}_2^2 \left( m_t^{(1,N)}, m_t \right) \leq \frac{Ke^{bt}}{N} \leq \frac{K}{N^{1-b\varepsilon}}.$$

For $t \geq \varepsilon \ln N$, coupling through the equilibria:

$$\mathcal{W}_2 \left( m_t^{(1,N)}, m_t \right)$$

$$\leq \mathcal{W}_2 \left( m_t^{(1,N)}, m_\infty^{(1,N)} \right) + \mathcal{W}_2 \left( m_\infty^{(1,N)}, m_\infty \right) + \mathcal{W}_2 \left( m_\infty, m_t \right)$$

$$\leq Ke^{-\chi t} + \frac{K}{N} + Ke^{-\chi t}$$
Uniform propagation of chaos in $\mathcal{W}_2$

For $t \leq \varepsilon \ln N$, parallel coupling:

\[
\mathcal{W}_2^2 \left( m_t^{(1,N)}, m_t \right) \leq \frac{Ke^{bt}}{N} \leq \frac{K}{N^{1-b\varepsilon}}.
\]

For $t \geq \varepsilon \ln N$, coupling through the equilibria:

\[
\mathcal{W}_2 \left( m_t^{(1,N)}, m_t \right) \\
\leq \mathcal{W}_2 \left( m_t^{(1,N)}, m_\infty^{(1,N)} \right) + \mathcal{W}_2 \left( m_\infty^{(1,N)}, m_\infty \right) + \mathcal{W}_2 \left( m_\infty, m_t \right) \\
\leq Ke^{-\chi t} + \frac{K}{N} + Ke^{-\chi t} \\
\leq \frac{3K}{N^{\varepsilon\chi}}
\]
Uniform propagation of chaos in $\mathcal{W}_2$

For $t \leq \epsilon \ln N$, parallel coupling:

$$\mathcal{W}_2^2 \left( m_t^{(1,N)}, m_t \right) \leq \frac{Ke^{bt}}{N} \leq \frac{K}{N^{1-b\epsilon}}.$$

For $t \geq \epsilon \ln N$, coupling through the equilibria:

$$\mathcal{W}_2 \left( m_t^{(1,N)}, m_t \right) \leq \mathcal{W}_2 \left( m_t^{(1,N)}, m_\infty^{(1,N)} \right) + \mathcal{W}_2 \left( m_\infty^{(1,N)}, m_\infty \right) + \mathcal{W}_2 \left( m_\infty, m_t \right)$$

$$\leq Ke^{-\chi t} + \frac{K}{N} + Ke^{-\chi t}$$

$$\leq \frac{3K}{N^{\epsilon\chi}}$$

Conclusion, for all time, $\mathcal{W}_2 \left( m_t^{(1,N)}, m_t \right) \leq \frac{K}{N^\alpha}$. 
Total variation

Based on Malrieu’s 2001 guideline,

\[
\partial_t \left( \mathcal{H} \left( m_t^{(N)} \mid m_t \otimes N \right) \right) \leq KNW_2 \left( m_t^{(1,N)}, m_t \right) \leq KN^{1-\alpha}
\]
Total variation

Based on Malrieu’s 2001 guideline,

\[
\frac{\partial}{\partial t} \left( \mathcal{H} \left( m_t^{(N)} \mid m_t \otimes N \right) \right) \leq KN \mathcal{W}_2 \left( m_t^{(1,N)}, m_t \right) \leq KN^{1-\alpha}
\]

hence

\[
\left\| m_t^{(1,N)} - m_t \right\|_1^2 \leq \frac{1}{N} \mathcal{H} \left( m_t^{(N)} \mid m_t \otimes N \right) \leq \frac{Kt}{N^{\alpha}}
\]
Total variation

Based on Malrieu’s 2001 guideline,

$$\partial_t \left( \mathcal{H} \left( m_t^{(N)} \mid m_t \otimes N \right) \right) \leq KN \mathcal{W}_2 \left( m_t^{(1,N)}, m_t \right) \leq KN^{1-\alpha}$$

hence

$$\|m_t^{(1,N)} - m_t\|_1^2 \leq \frac{1}{N} \mathcal{H} \left( m_t^{(N)} \mid m_t \otimes N \right) \leq \frac{Kt}{N^\alpha}$$

Not uniform, but sufficient:

$$\|m_t - m_\infty\|_1 \leq \|m_t - m_t^{(1,N)}\|_1 + \|m_t^{(1,N)} - m^{(1,N)}_\infty\|_1 + \|m^{(1,N)}_\infty - m_\infty\|_1$$
Total variation

Based on Malrieu’s 2001 guideline,

\[ \partial_t \left( \mathcal{H} \left( m_t^{(N)} \mid m_t^{\otimes N} \right) \right) \leq K N W_2 \left( m_t^{(1,N)}, m_t \right) \leq K N^{1-\alpha} \]

hence

\[ \| m_t^{(1,N)} - m_t \|_2^2 \leq \frac{1}{N} \mathcal{H} \left( m_t^{(N)} \mid m_t^{\otimes N} \right) \leq \frac{K t}{N^\alpha} \]

Not uniform, but sufficient:

\[ \| m_t - m_\infty \|_1 \]
\[ \leq \| m_t - m_t^{(1,N)} \|_1 + \| m_t^{(1,N)} - m_\infty^{(1,N)} \|_1 + \| m_\infty^{(1,N)} - m_\infty \|_1 \]
\[ \leq \frac{\sqrt{K t}}{N^{\frac{\alpha}{2}}} + \sqrt{K N e^{-\frac{1}{2} \chi t}} + \frac{K}{\sqrt{N}} \]
Total variation

Based on Malrieu’s 2001 guideline,

\[
\partial_t \left( \mathcal{H} \left( m_t^{(N)} \mid m_t \otimes N \right) \right) \leq K N W_2 \left( m_t^{(1,N)}, m_t \right) \leq K N^{1-\alpha}
\]

hence

\[
\| m_t^{(1,N)} - m_t \|_1^2 \leq \frac{1}{N} \mathcal{H} \left( m_t^{(N)} \mid m_t \otimes N \right) \leq \frac{K t}{N^\alpha}
\]

Not uniform, but sufficient: for \( N \) of order \( e^{\frac{\chi t}{\alpha+1}} \).

\[
\| m_t - m_\infty \|_1 \leq \| m_t - m_t^{(1,N)} \|_1 + \| m_t^{(1,N)} - m_\infty^{(1,N)} \|_1 + \| m_\infty^{(1,N)} - m_\infty \|_1 \leq \sqrt{K t} \leq N^{\alpha/2} + \sqrt{K N} e^{-\frac{1}{2} \chi t} + \frac{K}{\sqrt{N}} \leq K e^{-\chi' t}.
\]
Total variation

Based on Malrieu’s 2001 guideline,

\[ \partial_t \left( \mathcal{H} \left( m_t^{(N)} \mid m_t \otimes N \right) \right) \leq K N \mathcal{W}_2 \left( m_t^{(1,N)}, m_t \right) \leq K N^{1-\alpha} \]

hence

\[ \| m_t^{(1,N)} - m_t \|_1^2 \leq \frac{1}{N} \mathcal{H} \left( m_t^{(N)} \mid m_t \otimes N \right) \leq \frac{K t}{N^\alpha} \]

Not uniform, but sufficient : for \( N \) of order \( e^{\frac{\chi t}{\alpha + 1}} \).

\[
\| m_t - m_\infty \|_1 \\
\leq \| m_t - m_t^{(1,N)} \|_1 + \| m_t^{(1,N)} - m_\infty^{(1,N)} \|_1 + \| m_\infty^{(1,N)} - m_\infty \|_1 \\
\leq \frac{\sqrt{K t}}{N^\frac{\alpha}{2}} + \sqrt{K N} e^{-\frac{1}{2} \chi t} + \frac{K}{\sqrt{N}} \leq K e^{-\chi' t}.
\]

(\( \Rightarrow \) uniform in time propagation of chaos in the total variation sense...)
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4 Conclusion
Without convexity

If $U$ has several minima and the interaction is attractive, in the small noise regime, the non-linear PDE has several distinct equilibria, but there is unicity for a large enough noise

- If uniqueness, uniform estimates, with respect to $t$ or $N$?
- Without uniqueness, replace THE invariant measure by quasi-stationary ones? Are there two regimes

$$t \ll e^{aN} \Rightarrow \mathcal{W}_2 \left( m_{t}^{(1,N)}, m_t \right) \leq \frac{K}{N}$$

$$t \gg e^{aN} \Rightarrow \mathcal{W}_2 \left( m_{t}^{(1,N)}, m_t \right) \geq K$$

and convergence of the QSD towards the equilibria of the PDE?

- toy model (Curie-Weiss).
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