GEOMETRIC ERGODICITY OF THE BOUNCY PARTICLE SAMPLER

ALAIN DURMUS, ARNAUD GUILLIN, PIERRE MONMARCHÉ

Abstract. The Bouncy Particle Sampler (BPS) is a Monte Carlo Markov Chain algorithm to sample from a target density known up to a multiplicative constant. This method is based on a kinetic piecewise deterministic Markov process for which the target measure is invariant. This paper deals with theoretical properties of BPS. First, we establish geometric ergodicity of the associated semi-group under weaker conditions than in [10] both on the target distribution and the velocity probability distribution. This result is based on a new coupling of the process which gives a quantitative minorization condition and yields more insights on the convergence. In addition, we study on a toy model the dependency of the convergence rates on the dimension of the state space. Finally, we apply our results to the analysis of simulated annealing algorithms based on BPS.

1. INTRODUCTION

Markov chain Monte Carlo methods is a core requirement in many applications, e.g. in computational statistics [20], machine learning [1], molecular dynamics [6]. These methods are used to get approximate samples from a target distribution denoted $\pi$, with density w.r.t. the Lebesgue measure given for all $x \in \mathbb{R}^d$ by

$$\pi(x) = \exp(-U(x)),$$

for a potential $U : \mathbb{R}^d \to \mathbb{R}$, known up to an additive constant. They rely on the construction of Markov chains which are ergodic with respect to $\pi$, see [44].

While the first and best-known MCMC methods are based on reversible chains, such as many Metropolis-Hastings type algorithms [30], there has been since the last decade an increasing interest in non-reversible discrete-time processes [11, 3, 38, 34]. Indeed, consider a Markov chains $(X_k)_{k \in \mathbb{N}}$ on the state space $\{1, \ldots, n\}$. If $(X_k)_{k \in \mathbb{N}}$ is reversible, for any $n \in \mathbb{N}$, the event $X_{n+2} = X_n$ has a positive probability and therefore the process shows a diffusive behaviour, covering a distance $\sqrt{K}$ after $K$ iterations. This makes the exploration of the space slow and affects the efficiency of the algorithm. One of the first attempt to avoid this diffusive behaviour has been proposed in [36], where the author suggests to modify the transition matrix $M$ of $(X_k)_{k \in \mathbb{N}}$, reversible with respect to $\mu$, in such way that the obtained transition matrix is non-reversible but still leaves $\mu$ invariant. By definition of $\tilde{M}$, the probability of backtracking is smaller than for $M$, i.e. $\tilde{M}_{i,i}^2 \leq M_{i,i}^2$ for any $i \in \{1, \ldots, n\}$. In addition, [36] shows that the asymptotic variance of $\tilde{M}$ is always smaller than the one of $M$.

For general state space and in particular in order to sample from $\pi$ defined by (1), a now popular idea to construct non-reversible Markov chain is based on lifting, see [11] and the references therein. The idea is to extend the state space $\mathbb{R}^d$ and consider a Markov chain $(X_k, Y_k)_{k \in \mathbb{N}}$ on $\mathbb{R}^d \times \mathcal{Y}$, $\mathcal{Y} \subset \mathbb{R}^d$, which admits an invariant distribution for which the first marginal is the probability measure of interest. It turns out that, appropriately scaled, some of these lifted chains converge to continuous-time Markov processes. For instance, the persistent walk on the discrete torus introduced in [11] converges to the integrated telegraph on the continuous torus [34], while the lifted chain defined in [45] for spin models converges to the Zig-zag process [4] (see also the event-chain MC with infinitesimal steps in the physics
In these cases, the continuous-time limits belong to the class of velocity jump processes \((X_t, Y_t)_{t \geq 0}\) on \(\mathbb{R}^d \times Y, Y \subset \mathbb{R}^d\), satisfying \(X_t = X_0 + \int_0^t Y_s ds\) for all \(t \geq 0\) with \((Y_t)_{t \geq 0}\) is piecewise-constant on random time intervals. The velocity \((Y_t)_{t \geq 0}\) acts as an instantaneous memory, or inertia, so that \((X_t)_{t \geq 0}\) tends to continue in the same direction for some time instead of backtracking. In addition, these processes may be designed to target a given probability measure defined on \((\mathbb{R}^d \times Y, B(\mathbb{R}^d \times Y))\) of the form
\[
{\pi} = {\pi} \otimes \mu_v ,
\]
where \(\mu_v\) is a probability measure on \(Y\), and therefore can be used as MCMC samplers. This kind of dynamics, which are not new \([25, 19]\), have regained a particular interest in the last decade, in two separate fields: stochastic algorithms, as we presented, but also biological modelling, where they model the motion of a bacterium \([16, 8, 17]\) and are sometimes called run-\&-tumble processes.

From a numerical point of view, an advantage of these continuous-time processes is that, under appropriate conditions on the potential \(U\), an exact simulation is possible, following a thinning strategy \([28, 7, 27]\). Therefore, no discretization schemes are needed to approximate the continuous time trajectory, contrary to Langevin diffusions or Hamiltonian dynamics. As a consequence, no Metropolis filter is necessary to preserve the invariance of \(\pi\), see \([43, 13, 37, 41]\) and the reference therein.

This work deals with the velocity jump process introduced in \([38, 35]\). Following \([7]\), we refer to it as the Bouncy Particle Sampler (BPS). The aim of this paper is to establish geometric convergence to equilibrium for the BPS, in dimension larger than 1, relaxing the conditions given in \([10]\). The paper is organized as follows. Section 2.2 presents the BPS process and our main results, which are proven in Section 3. Finally, Section 4 is devoted to a discussion on our result and approach. First, in Section 4.1, we give explicit bound for a toy process and our main results, which are proven in Section 3. Finally, Section 4 is devoted to a discussion on our result and approach. First, in Section 4.1, we give explicit bound for a toy model, paying a particular attention to the dependency on the dimension of the state space in the constants we get. Second, in Section 4.2, we apply our results to study the annealing algorithm based on the BPS, extending the results of \([35]\).

Although the work is restricted to the BPS, our arguments can easily be adapted to other velocity jump processes, such as randomized variants of the BPS. In particular, the coupling argument in Section 3.3 applies as soon as the process admits a refreshment mechanism.

**Notations.** For all \(a, b \in \mathbb{R}\), we denote \(a_+ = \max(0, a)\), \(a \vee b = \max(a, b)\), \(a \wedge b = \min(a, b)\).

Id stands for the identity matrix on \(\mathbb{R}^d\).

For all \(x, y \in \mathbb{R}^d\), the scalar product between \(x\) and \(y\) is denoted by \((x, y)\) and the Euclidean norm of \(x\) by \(\|x\|\). We denote by \(S^d = \{v \in \mathbb{R}^d : \|v\| = 1\}\), the \(d\)-dimensional sphere with radius 1 and for all \(x \in \mathbb{R}^d, r > 0\), by \(B(x, r) = \{w \in \mathbb{R}^d : \|w - x\| \leq r\}\) the ball centered in \(x\) with radius \(r\). For any \(d\)-dimensional matrix \(M\), define by \(\|M\| = \sup_{w \in B(0, 1)} \|M w\|\) the operator norm associated with \(M\).

Denote by \(C(\mathbb{R}^d)\) the set of continuous function from \(\mathbb{R}^d\) to \(\mathbb{R}\) and for all \(k \in \mathbb{N}^*, C^k(\mathbb{R}^d)\) the set of \(k\)-times continuously differentiable function from \(\mathbb{R}^d \rightarrow \mathbb{R}\). Denote for all \(k \in \mathbb{N}\), \(C^k_b(\mathbb{R}^d)\) and \(C^k_b(\mathbb{R}^d)\) the set of functions belonging to \(C^k(\mathbb{R}^d)\) with compact support and the set of bounded functions belonging to \(C^k(\mathbb{R}^d)\) respectively. For all function \(f : \mathbb{R}^d \rightarrow \mathbb{R}\), we denote by \(\nabla f\) and \(\nabla^2 f\) the gradient and the Hessian of \(f\) respectively, if they exist. For all function \(F : \mathbb{R}^d \rightarrow \mathbb{R}^m\) and compact set \(K \subset \mathbb{R}^d\), denote \(\|F\|_K = \sup_{x \in K} \|F(x)\|\), \(\|F\|_{\infty, K} = \sup_{x \in K} \|F(x)\|\). We denote by \(B(\mathbb{R}^d)\) the Borel \(\sigma\)-field of and \(\mathcal{P}(\mathbb{R}^d)\) the set of probability measures on \(\mathbb{R}^d\). For \(\mu, \nu \in \mathcal{P}(\mathbb{R}^d)\), \(\xi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)\) is called a transference plan between \(\mu\) and \(\nu\) if for all \(A \in B(\mathbb{R}^d)\), \(\xi(A \times \mathbb{R}^d) = \mu(A)\) and \(\xi(\mathbb{R}^d \times A) = \nu(A)\). The set of transference plan between \(\mu\) and \(\nu\) is denoted \(\Gamma(\mu, \nu)\). The random variables \(X\) and \(Y\) on \(\mathbb{R}^d\) are a coupling between \(\mu\) and \(\nu\) if the distribution of \((X, Y)\) belongs to \(\Gamma(\mu, \nu)\). The total
Note that for all \((x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\), we call \(S\) i.e. which is rotation invariant, is continuously differentiable on \(\mathbb{R}\). Let \(\Delta\) be a closed \(C^\infty\)-submanifold \(Y \subset \mathbb{R}^d\), which is rotation invariant, i.e. for any rotation \(O \in O(d)\), \(OY = Y\). The BPS process \((X_t, Y_t)_{t \geq 0}\) associated with \(U\) evolves on \((\mathbb{R}^d \times Y, \mathcal{B}^d \times \mathcal{B}(Y))\) and is defined as follows.

Consider some initial point \((x,y) \in \mathbb{R}^d \times Y\), and a family of i.i.d. random variables \((E_1, E_2, G_i)_{i \in \mathbb{N}^*}\) on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where for all \(i \in \mathbb{N}^*, E_1, E_2\) are exponential random variables with parameter 1, \(G_i\) is a random variable with a given distribution \(\mu_i\) on \((Y, \mathcal{B}(Y))\), referred to as the refreshment distribution. In addition, for all \(i \in \mathbb{N}^*, E_1, E_2\) and \(G_i\) are independent. Let \(\lambda_t > 0\), referred to as the refreshment rate, \((X_0, Y_0) = (x, y)\) and \(S_0 = 0\). We define by recursion the jump times of the process and the process itself. Assume that \(S_n\) and \((X_t, Y_t)_{t \leq S_n}\) have been defined for \(n \geq 0\). Consider

\[
T_{n+1}^1 = E_{n+1}^1 / \lambda_t \\
T_{n+1}^2 = \inf \left\{ t \geq 0 : \int_0^t \langle Y_s, \nabla U(X_s + sY_s) \rangle ds \geq E_{n+1}^2 \right\} \\
T_{n+1} = T_{n+1}^1 \wedge T_{n+1}^2.
\]

Set \(S_{n+1} = S_n + T_{n+1}\), \((X_t, Y_t) = (X_{S_n} + tY_{S_n}, Y_{S_n})\), for all \(t \in [S_n, S_{n+1})\), \(X_{S_{n+1}} = X_{S_n} + T_{n+1}Y_{S_n}\) and

\[
Y_{S_{n+1}} = \begin{cases} 
G_{n+1} & \text{if } T_{n+1} = T_{n+1}^1 \\
R(X_{S_{n+1}}, Y_{S_n}) & \text{otherwise},
\end{cases}
\]

where \(R : \mathbb{R}^d \to \mathbb{R}^d\) is the function given for all \(x, y \in \mathbb{R}^d\) by

\[
R(x, y) = y - 2 \langle y, n(\nabla U(x)) \rangle n(\nabla U(x)),
\]

where for all \(z \in \mathbb{R}^d\), \(n(z) = \begin{cases} \frac{z}{\|z\|} & \text{if } z \neq 0 \\
0 & \text{otherwise} \end{cases}\).

Note that for all \((x, y) \in \mathbb{R}^d\) with \(\nabla U(x) \neq 0\), \(R(x, y)\) is the reflection of \(y\) orthogonal to \(\nabla U(x)\) and therefore for all \((x, y) \in \mathbb{R}^d\), \(\|R(x, y)\| = \|y\|\).

If \(T_{n+1} = T^1_{n+1}\), we say that, at time \(T_{n+1}\), the velocity has been refreshed, and we call \(T_{n+1}\) a refreshment time. If \(T_{n+1} = T^2_{n+1}\), we say that, at time \(T_{n+1}\), the process has bounced, and we call \(T_{n+1}\) a bounce time.

Then, \((X_t, Y_t)\) is defined for all \(t < \sup_{n \in \mathbb{N}} S_n\) and we set for all \(t \geq \sup_{n \in \mathbb{N}} S_n\), \((X_t, Y_t) = \infty\), where \(\infty\) is a cemetery point.
In fact, it is proven in [14, Proposition 10] that almost surely, $\sup_{n \in \mathbb{N}} S_n = +\infty$. Therefore almost surely $(X_t, Y_t)_{t \geq 0}$ is a $(\mathbb{R}^d \times Y)$-valued càdlàg process. By [9, Theorem 25.5], the BPS process $(X_t, Y_t)_{t \geq 0}$ defines a strong Markov semi-group $(P_t)_{t \geq 0}$ given for all $(x, y) \in \mathbb{R}^d \times Y$ and $A \in \mathcal{B}(\mathbb{R}^d \times Y)$ by

$$P_t((x, y), A) = \mathbb{P}((X_t, Y_t) \in A) ,$$

where $(X_t, Y_t)_{t \in \mathbb{R}_+}$ is the BPS process started from $(x, y)$.

Consider the following basic assumption.

**A1.** The potential $U$ is twice continuously differentiable, $\mu_v$ is rotation invariant and $(x, y) \mapsto \|y\| \|\nabla U(x)\|$ is integrable with respect to $\tilde{\pi}$ defined by (2).

It is shown in [14, Proposition 24] that, under A1, the probability measure $\tilde{\pi}$ defined by (2), is invariant for $(P_t)_{t \geq 0}$, i.e. $\tilde{\pi}P_t = \tilde{\pi}$ for all $t \geq 0$.

2.2. **Main results.** For $V : \mathbb{R}^d \times Y \to [1, +\infty)$, the semi-group $(P_t)_{t \geq 0}$ with invariant measure $\tilde{\pi}$ is said to be $V$-uniformly geometrically ergodic if there exist $C, \rho > 0$ such that for all $t \geq 0$ and all $\mu \in \mathcal{P}(\mathbb{R}^d \times Y)$ with $\mu(V) < +\infty$, it holds

$$\|\mu P_t - \tilde{\pi}\|_V \leq Ce^{-\rho t} \mu(V) .$$

We state in this section our main results regarding the $V$-uniformly geometrically ergodicity of the BPS.

Our basic assumptions to prove geometric ergodicity are the following.

**A2.** (i) The potential $U$ satisfies $\lim_{\|x\| \to +\infty} U(x) = +\infty$ and $\int_{\mathbb{R}^d} \exp (-U(x)/2) \, dx < +\infty$. Moreover, and without loss of generality, for all $x \in \mathbb{R}^d$, $U(x) > 0$.

(ii) $\mu_v$ admits a density w.r.t. the Lebesgue measure on $\mathbb{R}^d$ or there exists $r_0 > 0$ such that $\mu_v(r_0S^d) > 0$.

Here, we establish practical conditions on the potential $U$, $\mu_v$ and $Y$ implying that $(P_t)_{t \geq 0}$ is $V$-uniformly geometrically ergodic. In fact, these conditions are derived from a more general result. However, since the assumptions and statement of the latter can seem very intricate, for sake of clarity we have decided to give this result next to its application.

Consider the following alternative conditions, which will be used in the case where $Y$ is bounded.

**A3.** The potential $U$ satisfies

$$\lim_{\|x\| \to +\infty} \|\nabla U(x)\| = \infty , \quad \sup_{x \in \mathbb{R}^2} \|\nabla^2 U(x)\| < \infty .$$

**A4.** There exists $\zeta \in (0, 1)$ such that

$$\liminf_{\|x\| \to +\infty} \left\{ \|\nabla U(x)\| / U^{1-\zeta}(x) \right\} > 0 \quad \text{and} \quad \limsup_{\|x\| \to +\infty} \left\{ \|\nabla U(x)\| / U^{1-\zeta/2}(x) \right\} < +\infty ,$$

$$\limsup_{\|x\| \to +\infty} \left\{ \|\nabla^2 U(x)\| / U^{1-\zeta}(x) \right\} < +\infty .$$

**A5.** The potential $U$ satisfies $\lim_{\|x\| \to +\infty} \left\| \frac{\nabla^2 U(x)}{\|\nabla U(x)\|} \right\| = 0$ and there exists $\zeta \in (0, 1)$ such that

$$\liminf_{\|x\| \to +\infty} \left\| \frac{\nabla U(x)}{U^{1-\zeta}(x)} \right\| > 0 \quad \text{and} \quad \lim_{\|x\| \to +\infty} \left\| \frac{\nabla U(x)}{U^{2(1-\zeta)}(x)} \right\| = 0.$$
Note that $A_5$ is similar to $A_4$ but these two conditions are different: none of them implies the other. Indeed, on $\mathbb{R}^2$, consider $U(x_1, x_2) = (1 + |x_1|^2)^{\alpha/2} + (1 + |x_2|^2)^{\beta/2}$ for some $\alpha, \beta > 1$. Then for all $(x_1, x_2) \in \mathbb{R}^2$, we have

$$\nabla U(x) = [\alpha x_1 (1 + x_1^2)^{\alpha/2 - 1}, \beta x_2 (1 + x_2^2)^{\beta/2 - 1}]^T, \quad \nabla^2 U(x) = \begin{pmatrix} F(\alpha, x_1) & 0 \\ 0 & F(\beta, x_2) \end{pmatrix}$$

where $F(\alpha, x_1) = \alpha (1 + x_1^2)^{\alpha/2 - 1} + 2\alpha x_1^2 (\alpha/2 - 1)(1 + x_1^2)^{\alpha/2 - 2}$.

In that case $A_4$ is satisfied if and only if $[(\alpha \vee \beta)/2, \alpha \wedge \beta] \neq \emptyset$, while $A_5$ is satisfied if and only if $[2(\alpha \vee \beta)/(1 + \alpha \vee \beta), \alpha \wedge \beta] \neq \emptyset$, choosing in both cases $\varsigma^{-1} > 1$ in the corresponding interval. In particular, if both $\alpha, \beta \geq 2$, then $A_5$ is satisfied, but $A_4$ may not (if $\alpha > 2\beta$ for instance). On the contrary if, say, $\alpha = 4/3$ and $\beta \in (1, 8/7)$, then $A_4$ holds while $A_5$ does not.

**Theorem 1.** Assume $A_1, A_2, Y$ is bounded and either $A_3, A_4$ or $A_5$. In the case where $A_3$ holds, set $\varsigma = 1$. Then, for any refreshment rate $\lambda_\varsigma > 0$, there exists $\kappa \in (0, 1)$ such that $(P_\varsigma)_{\varsigma \geq 0}$ is $V$-uniformly geometrically ergodic with $V : \mathbb{R}^d \times Y \to [1, +\infty)$ given for all $(x, y) \in \mathbb{R}^d \times Y$ by $V(x, y) = \exp(\kappa U^\varsigma(x))$.

**Proof.** The proof is postponed to Section 3.5. □

The geometric ergodicity of BPS was also shown in [10, Theorem 3.1] under $A_3$ and with the condition that $\lambda_\varsigma$ is sufficiently large. Note that we do not impose this last condition in Theorem 1.

Note that $A_3, A_4$ and $A_5$ all require that $\liminf_{\|x\| \to +\infty} \|\nabla U(x)\| = +\infty$. We consider now the case where $\liminf_{\|x\| \to +\infty} \|\nabla U(x)\| < +\infty$ possibly.

**A6.** The potential $U$ satisfies $\liminf_{\|x\| \to +\infty} \|\nabla U(x)\| > 0$ and $\limsup_{\|x\| \to +\infty} \|\nabla^2 U(x)\| = 0$.

The following result is a generalization of [10, Theorem 3.1] which consider $Y = S^d$ and $\mu_\nu$ is the uniform distribution on $S^d$.

**Theorem 2.** Assume $A_1, A_2, A_6$ and $Y$ is bounded. Then, there exists $\lambda_0 > 0$ such that, if $\lambda_\varsigma \in (0, \lambda_0]$, $(P_\varsigma)_{\varsigma \geq 0}$ is $V$-uniformly geometrically ergodic with $V : \mathbb{R}^d \times Y \to [1, +\infty)$ given for all $(x, y) \in \mathbb{R}^d \times Y$ by $V(x, y) = \exp(U(x)/2)$.

**Proof.** The proof is postponed to Section 3.6. □

Note that contrary to the setting of Theorem 1, the result of Theorem 2 requires that the refreshment rate $\lambda_\varsigma$ is sufficiently small for the BPS to be $V$-uniformly geometrically ergodic.

In the case where $Y$ is unbounded, $A_4$ must be strengthen as follow.

**A7.** There exists $\varsigma \in (0, 1)$ such that

$$0 < \liminf_{\|x\| \to +\infty} \{\|\nabla U(x)\| / U^{1-\varsigma}(x)\} \leq \limsup_{\|x\| \to +\infty} \{\|\nabla U(x)\| / U^{1-\varsigma}(x)\} < +\infty,$$

$$\limsup_{\|x\| \to +\infty} \{\|\nabla^2 U(x)\| / U^{1-2\varsigma}(x)\} < +\infty.$$

**A7** (and therefore **A4**) holds when $U$ is a perturbation of an $\alpha$-homogeneous function:

**Proposition 3.** Let $\alpha \in (1, +\infty)$ and assume that $U = U_1 + U_2$ with $U_1, U_2 \in C^2(\mathbb{R}^d)$ satisfying

- $U_1$ is $\alpha$-homogeneous: for all $t \geq 1$ and $x \in \mathbb{R}^d$ with $\|x\| \geq 1$,
  $$U_1(t x) = t^\alpha U_1(x)$$
  and $\lim_{\|x\| \to +\infty} U_1(x) = +\infty$. 

\[
\limsup_{\|x\| \to +\infty} \left\{ \frac{U_2(x)}{\|x\|^\alpha + \|\nabla U_2(x)\| / \|x\|^\alpha - 1} + \frac{\|\nabla^2 U_2(x)\|}{\|x\|^\alpha - 2} \right\} = 0 .
\]

Then \( A 7 \) holds with \( \zeta = 1/\alpha \).

**Proof.** The proof is postponed to Appendix A. \( \square \)

This class of potentials is considered in [24, Theorem 4.6], which shows that the Random Walk Metropolis algorithm is geometrically ergodic for target distributions \( \pi \) associated to a potential belonging to this class.

**Theorem 4.** Assume \( A 1, A 2, A 7 \) and \( \mu_\nu \) admits a Gaussian moment: there exists \( \eta > 0 \) such that \( \int_Y e^{\eta \|y\|^2} \mu_\nu(dy) < +\infty \). Then, for any refreshment rate \( \lambda_i > 0 \), there exists \( \kappa \in (0, 1] \) such that \((P_i)_{t \geq 0}\) is \( V \)-uniformly geometrically ergodic with \( V : \mathbb{R}^d \times Y \to [1, +\infty) \) given for all \( (x, y) \in \mathbb{R}^d \times Y \) by \( V(x, y) = \exp(\kappa U^*(x)) + \exp(\eta \|y\|^2) \).

**Proof.** The proof is postponed to Section 3.7. \( \square \)

As noticed before, Theorem 1, Theorem 2 and Theorem 4 ensue from a more general results, which holds under the following assumptions.

**A 8.** There exist some positive functions \( H \in C(\mathbb{R}_+) \), \( \psi \in C^2(\mathbb{R}) \), \( \ell \in C^1(\mathbb{R}^d) \), and some constants \( R, r, \delta > 0 \), \( c_i > 0 \) for \( i = 1, \ldots, 4 \) satisfying the following conditions.

(i) **Conditions on \( U \).** The function \( \bar{U} \), defined by \( \bar{U} = \psi \circ U \), satisfies

\[
\lim_{\|x\| \to +\infty} \bar{U}(x) = +\infty , \quad \int_{\mathbb{R}^d} \exp \left( \bar{U}(x) - U(x) \right) dx < +\infty
\]

(6)

and for all \( x \in \mathbb{R}^d \) with \( \|x\| > R \),

\[
\|\nabla \bar{U}(x)\| \ell(x) \geq c_1 , \quad \ell(x) \leq c_2 , \quad \|\nabla U(x)\| \ell(x)/\|\nabla \bar{U}(x)\| \geq c_3 .
\]

(7)

(ii) **Conditions on \( \mu_\nu \).**

\[
\int_Y e^{H(\|y\|)} \mu_\nu(dy) < \infty , \quad \sup_{y \in Y} \left\{ e^{-H(\|y\|)/2} \|y\|^2 \right\} < \infty , \quad \int_Y \mathbb{1}_{[r, +\infty)}(y) \mu_\nu(dy) \geq \frac{\delta}{2} .
\]

(8)

(iii) **Conditions on \( U \) and \( \mu_\nu \).** For \( x \in \mathbb{R}^d \), define

\[
A_x = \{ y \in Y : H(\|y\|) \leq 3\bar{U}(x) \} .
\]

Assume that

\[
\lim_{\|x\| \to +\infty} \left[ \|\nabla \ell(x)\| \left\{ 1 \lor \sup_{y \in A_x} \|y\| \right\} \right] = 0 ,
\]

(9)

and for all \( x \in \mathbb{R}^d \) with \( \|x\| > R \),

\[
\|\nabla^2 \bar{U}(x)\| \ell(x) \left\{ \sup_{y \in A_x} \|y\|^2 \right\} \leq c_4 .
\]

(10)

**Theorem 5.** Assume \( A 1 - A 2 - A 8 \). Assume in addition that the following inequalities hold

\[
[16\lambda,c_2/(rc_1)] \land [64c_4c_2/(rc_1)^2] \leq [(1/3) \land \{\lambda_\nu \delta rc_1/(16c_4)\}] \left\{ c_3/(4c_2) \land \{\lambda_\nu \delta c_3/(100rc_1)\}^{1/2} \right\} .
\]

(11)
Then there exists $\kappa \in (0, 1]$ given below by (32), such that $(P_t)_{t \geq 0}$ is $V$-uniformly geometrically ergodic with $V$ given for all $(x, y) \in \mathbb{R}^d \times Y$ by $V(x, y) = \exp(\kappa U(x)) + \exp(H(\|y\|))$.

Proof. The proof is postponed to Section 3.4. \qed

Remark 6. Note that, under $A_8$, (12) is implied by either one of the two following additional assumptions:

(a) $\lim_{|x| \to +\infty} \| \nabla U(x) \| = +\infty$;
(b) $\lim_{|x| \to +\infty} \ell(x) = 0$;
(c) $\lim_{|x| \to +\infty} \| \nabla U(x) \| \ell(x) / \| \nabla U(x) \| = +\infty$.

Indeed, if (a) holds, then $c_1$ can be chosen as large as necessary while $c_2, c_4, c_3$ can be held fixed so that (12) is satisfied. If (b) holds, then $c_2$ can be chosen as small as necessary while $c_1, c_3, c_4$ can be held fixed. Finally if (c) holds, then $c_3$ can be chosen as large as necessary while $c_1 c_2, c_4$ can be held fixed.

Note that if $(P_t)_{t \geq 0}$ is $V$-uniformly geometrically ergodic then by [18, Theorem 4.4], a functional Central Limit Theorem (FCLT) holds. Let $g : \mathbb{R}^d \times Y \to \mathbb{R}$ satisfying for all $(x, y) \in \mathbb{R}^d \times Y$, $|g|^2 \leq CV$ for some $C > 0$. Let $(X_t, Y_t)_{t \geq 0}$ be a BPS process with initial distribution $\mu_0 \in \mathcal{P}(\mathbb{R}^d \times Y)$, satisfying $\mu_0(V) < +\infty$. For $t \geq 0$ and $n \in \mathbb{N}$, define

$$G^m_t = \frac{1}{\sqrt{n}} \int_0^t (g(X_s, Y_s) - \tilde{\pi}(g)) \, ds.$$  

Then, there exists $\sigma_g \geq 0$ such that the sequence of processes $\{(G^m_t)_{t \geq 0}, n \in \mathbb{N}\}$ converges as $n \to \infty$ toward $(\sigma_g B_t)_{t \geq 0}$ in the Skorokhod space, where $(B_t)_{t \geq 0}$ is a standard Brownian motion. It is also possible to consider moderate deviation [21, 12] or large deviation principle [46, 26]

3. Proofs of the main results

For the proof Theorem 5, we follow the Meyn and Tweedie approach, based upon two ingredients: a Foster-Lyapunov drift and a local Doeblin condition on compact sets. This section is organized as follows. Before showing the Foster-Lyapunov drift in Section 3.2, we introduce the generator of the BPS in Section 3.1. Then in Section 3.3, we show that under appropriate conditions, the BPS satisfies a local Doeblin condition on compact sets. Contrary to the previous works [35, 10, 5], this result is obtained in the case where $\mu_\ast$ has a density with respect to the Lebesgue measure by a direct coupling. With these two elements in hand, Theorem 5 is proven in 3.4. The proofs of Theorem 1, Theorem 2 and Theorem 4 are given in Section 3.5, Section 3.6 and Section 3.7.

3.1. Generator of the BPS. The BPS process belongs to the class of Piecewise Deterministic Markov Processes (PDMP). Indeed, consider the ordinary differential equation on $\mathbb{R}^{2d}$

$$\frac{d}{dt} \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} y_t \\ 0 \end{pmatrix},$$

and define for all $t \geq 0$, the map $\phi_t : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ given for all $(x, y) \in \mathbb{R}^{2d}$ by

$$\phi_t(x, y) = (x + ty, y).$$

The family $(\phi_t)_{t \in \mathbb{R}_+}$ is referred to as the flow of diffeomorphisms associated with (13) i.e. for all $(x, y) \in \mathbb{R}^{2d}$, $t \mapsto \phi_t(x, y)$ is solution of (13) started at $(x, y)$ and for all $t \geq 0$, $(x, y) \mapsto \phi_t(x, y)$ is a $C^\infty$-diffeomorphism. In addition to the deterministic flow $(\phi_t)_{t \in \mathbb{R}_+}$, the BPS, as a PDMP, is characterized by a function $\lambda : \mathbb{R}^d \times Y \to \mathbb{R}_+$, referred to as the jump rate, and
a Markov kernel $Q$ on $\mathbb{R}^d \times \mathcal{Y} \times \mathcal{B}(\mathbb{R}^d \times \mathcal{Y})$, defined for all $(x,y) \in \mathbb{R}^d \times \mathcal{Y}$ and $A \in \mathcal{B}(\mathbb{R}^d \times \mathcal{Y})$ by

$$
\begin{align*}
\lambda(x,y) &= (y, \nabla U(x))_+ + \bar{\lambda}, \\
Q((x,y), A) &= \left[ \delta_x \otimes \left\{ \frac{\langle y, \nabla U(x) \rangle}{\lambda(x,y)} + \delta_{R(x,y)} + \frac{\bar{\lambda}}{\lambda(x,y)} \right\} \right](A),
\end{align*}
$$

where $\delta_x$ is the Dirac measure at $x \in \mathbb{R}^d$. With these definitions in mind, we can define a PDMP (in the sense of [9]) $(X_t, Y_t)_{t \geq 0}$ which has the same distribution as $(X_t, Y_t)_{t \geq 0}$ on the space $D(\mathbb{R}^d, \mathbb{R}^d)$ of càdlàg functions $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}^d$, endowed with the Skorokhod topology, see [23, Chapter 6].

Consider some initial data $(x,y) \in \mathbb{R}^{2d}$, a family of i.i.d. random variables $(\bar{E}_i, \bar{G}_i, \bar{W}_i)_{i \geq 1}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ introduced in Section 2.1, where for all $i \geq 1$, $\bar{E}_i$ is an exponential random variable with parameter $1$, $\bar{G}_i$ is a random variable with distribution $\mu$, $\bar{W}_i$ is a uniform random variable and $\bar{E}_i, \bar{G}_i$ and $\bar{W}_i$ are independent. Set $(\bar{X}_0, \bar{Y}_0) = (x,y)$ and $\bar{S}_0 = 0$. We define by recursion the jump times of the process and the process itself. For all $n \geq 0$, let

$$
\bar{T}_{n+1} = \inf \left\{ t > 0 : \int_0^t \lambda \left\{ \phi_s(\bar{X}_{S_n}, \bar{Y}_{S_n}) \right\} ds \geq \bar{E}_{n+1} \right\}.
$$

Set $\bar{S}_{n+1} = \bar{S}_n + \bar{T}_{n+1}$, $(\bar{X}_t, \bar{Y}_t) = \phi_t(\bar{X}_{\bar{S}_n}, \bar{Y}_{\bar{S}_n})$ for all $t \in [\bar{S}_n, \bar{S}_{n+1}]$, $\bar{X}_{\bar{S}_{n+1}} = \bar{X}_{\bar{S}_n} + \bar{T}_{n+1} \bar{Y}_{\bar{S}_n}$ and

$$
\bar{Y}_{\bar{S}_{n+1}} = \begin{cases} 
\bar{G}_{n+1} & \text{if } \bar{W}_{n+1} \leq \bar{\lambda}/\lambda(\bar{X}_{\bar{S}_{n+1}}, \bar{Y}_{\bar{S}_{n+1}}) \\
R(\bar{X}_{\bar{S}_{n+1}}, \bar{Y}_{\bar{S}_{n+1}}) & \text{otherwise}
\end{cases}
$$

where $R$ is defined by (4). Thus, $(\bar{X}_t, \bar{Y}_t)$ is defined for all $t < \sup_{n \in \mathbb{N}} \bar{S}_n$ and we set for all $t \geq \sup_{n \in \mathbb{N}} \bar{S}_n$, $(\bar{X}_t, \bar{Y}_t) = \infty$, where $\infty$ is a cemetery point. Note that for all $n \in \mathbb{N}^*$, $(\bar{X}_{\bar{S}_n}, \bar{Y}_{\bar{S}_n})$ is distributed according to $Q((\bar{X}_{\bar{S}_n}, \bar{Y}_{\bar{S}_{n-1}}), \cdot)$.

From [14, Lemma 7], $(\bar{X}_t, \bar{Y}_t)_{t \geq 0}$ and $(X_t, Y_t)_{t \geq 0}$ have the same distribution (in particular, almost surely $\sup_{n \in \mathbb{N}} \bar{S}_n = \infty$ and $(\bar{X}_t, \bar{Y}_t)_{t \geq 0}$ is a $(\mathbb{R}^d \times \mathcal{Y})$-valued càdlàg process).

Consider the canonical process associated with the BPS process $(X_t, Y_t)_{t \geq 0}$, still denoted by $(X_t, Y_t)_{t \geq 0}$ on the Skorokhod space $(D(\mathbb{R}_+, \mathbb{R}^d \times \mathcal{Y}), \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_{x,y})_{(x,y) \in \mathbb{R}^d \times \mathcal{Y}})$, where $\mathcal{F}$ is the Borel $\sigma$-field associated with the Skorokhod topology, $(\mathcal{F}_t)_{t \geq 0}$ is the completed natural filtration, and for all $(x,y) \in \mathbb{R}^d \times \mathcal{Y}$, $\mathbb{P}_{x,y}$ is the distribution of the BPS process starting from $(x,y) \in \mathbb{R}^d \times \mathcal{Y}$. For all $t \geq 0$ and Borel measurable functions $f, g : \mathbb{R}^d \times \mathcal{Y} \rightarrow \mathbb{R}$ such that, for all $(x,y) \in \mathbb{R}^d \times \mathcal{Y}$, $s \mapsto g((X_s, Y_s))$ is integrable $\mathbb{P}_{(x,y)}$-almost surely, denote

$$
M^{f,g}_t = f(X_t, Y_t) - f(X_0, Y_0) - \int_0^t g(X_s, Y_s)ds.
$$

The (extended) generator and its domain $(A, D(A))$ associated with the semi-group $(P_t)_{t \geq 0}$ are defined as follows: $f \in D(A)$ if there exists a Borel measurable function $g : \mathbb{R}^d \times \mathcal{Y} \rightarrow \mathbb{R}$ such that $(M^{f,g}_t)_{t \geq 0}$ is a local martingale under $\mathbb{P}_{(x,y)}$ for all $(x,y) \in \mathbb{R}^d \times \mathcal{Y}$ and, for such a function, $Af = g$. Despite its very formal definition, $(A, D(A))$ associated with $(P_t)_{t \geq 0}$ can be easily described. Indeed, [9, Theorem 26.14] shows that $D(A) = E_1 \cap E_2$ where

$$
E_1 = \left\{ f \in \mathcal{M}(\mathbb{R}^d \times \mathcal{Y}) : t \mapsto f(\varphi_t(x,y)) \text{ is absolutely continuous on } \mathbb{R}_+ \text{ for all } (x,y) \in \mathbb{R}^{2d} \right\},
$$

and $E_2$ is the set of Borel measurable functions $f : \mathbb{R}^d \times \mathcal{Y} \rightarrow \mathbb{R}$ such that there exists an increasing sequence of $(\mathcal{F}_t)_{t \geq 0}$-stopping time $(\sigma_n)_{n \geq 0}$, such that for all $(x,y) \in \mathbb{R}^{2d}$,
\[
\lim_{n \to +\infty} \sigma_n = +\infty \mathbb{P}_{(x,y)}\text{-almost surely, and for all } n \in \mathbb{N}^*,
\]

\[
\mathbb{E}(x,y) \left[ \sum_{k=1}^{+\infty} 1_{\{S_k \leq \sigma_n\}} |f(X_{S_k}, Y_{S_k}) - f(X_{S_k^-}, Y_{S_k^-})| \right] < +\infty.
\]

Taking for all \( n \in \mathbb{N}^* \), \( \sigma_n = S_n \wedge n \wedge \nu_n \), where \( \nu_n = \inf\{t \geq 0 : \|X_t\| \geq n\} \), (16) is satisfied for any continuous \( f \). As a consequence, \( C(\mathbb{R}^d \times \mathcal{Y}) \subset D(A) \).

Then, for all \( f \in D(A) \) and \( x, y \in \mathbb{R}^d \times \mathcal{Y} \),

\[
Af(x, y) = D_y f(x, y) + (\langle y, \nabla U(x) \rangle)_+ \{ f(x, R(x, y)) - f(x, y) \}
+ \lambda_f \left\{ \int_{\mathcal{Y}} f(x, w) d\mu_\nu(w) - f(x, y) \right\},
\]

where

\[
D_y f(x, y) = \begin{cases} \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}, & \text{if this limit exists} \\ 0, & \text{otherwise}. \end{cases}
\]

In particular, if \( x \mapsto f(x, y) \) is \( C^1 \) for all \( y \in \mathcal{Y} \), then

\[
Af(x, y) = \langle y, \nabla f(x, y) \rangle + (\langle y, \nabla U(x) \rangle)_+ \{ f(x, R(x, y)) - f(x, y) \}
+ \lambda_f \left\{ \int_{\mathcal{Y}} f(x, w) d\mu_\nu(w) - f(x, y) \right\}.
\]

### 3.2. Foster-Lyapunov drift condition

For \( a, b, c \in \mathbb{R}_+, a \leq b \leq c, c - b \leq b - a \leq a \) and \( \varepsilon \in (0, 1] \) consider a non-decreasing continuously differentiable function \( \varphi : \mathbb{R}_+ \to [1, +\infty) \) satisfying

\[
\begin{align*}
\varphi(s) &= 1 & \text{if } s \in (-\infty, -2] \\
1 + a(s + 2) - \varepsilon &\leq \varphi(s) \leq 1 + a(s + 2) + \varepsilon & \text{if } s \in (-2, -1) \\
\varphi(s) &= 1 + b + s(b - a) & \text{if } s \in [-1, 0] \\
1 + b + s(c - b) - \varepsilon &\leq \varphi(s) \leq 1 + b + s(c - b) + \varepsilon & \text{if } s \in (0, 1) \\
\varphi(s) &= 1 + c & \text{if } s \in [1, +\infty],
\end{align*}
\]

and

\[
\sup_{s \in [-2, -1]} \varphi'(s) \leq a + \varepsilon, \quad \sup_{s \in [0, 1]} \varphi'(s) \leq c - b + \varepsilon.
\]

In addition for \( \kappa \in (0, 1] \), under A8, define the Lyapunov function \( V : \mathbb{R}^d \times \mathcal{Y} \to [1, +\infty) \) by

\[
V(x, y) = \exp(\kappa U(x)) \varphi \left\{ (2\ell(x)/(rc_1)) \langle y, \nabla U(x) \rangle \right\} + \exp(H(\|y\|)).
\]

This section is devoted to the proof of a Foster-Lyapunov drift condition for the generator \( A \) given by (18) and the function \( V \) defined in (21).

**Lemma 7.** Assume A1-A2-A8 and (12) hold. There exist \( a, b, c \in \mathbb{R}_+, a \leq b \leq c, c - b \leq b - a \leq a, \varepsilon \in (0, 1] \) and \( \kappa \in (0, 1] \) such that \( A \) given by (18) satisfies a Foster-Lyapunov drift condition with the Lyapunov function \( V \), i.e. there exist \( A_1, A_2 > 0 \) such that, for all \( (x, y) \in \mathbb{R}^d \times \mathcal{Y} \),

\[
AV(x, y) \leq A_1 \left( A_2 - V(x, y) \right).
\]
Proof. For ease of notation, we denote in the following \( \theta(x, y) = \langle \nabla U(x), y \rangle \) for any \((x, y) \in \mathbb{R}^d \times Y \). From (18) and the facts that \( \nabla \tilde{U}(x) = \psi'(U(x)) \nabla U(x) \) and \( \| R(x, y) \| = \| y \| \), for any \((x, y) \in \mathbb{R}^d \times Y \),

\[(23) \quad AV(x, y) = \exp (\kappa \tilde{U}(x)) J(x, y) + \lambda_r \left\{ \int_Y \exp (\| w \|) \mu_\ast (dw) - \exp (\| H(\| y \|) \|) \right\} , \]

where

\[(24) \quad J(x, y) = \kappa \theta(x, y) \varphi \left\{ 2 \ell(x) \theta(x, y) / (rc_1) \right\} \\
+ (2/(rc_1)) \varphi' \left\{ 2 \ell(x) \theta(x, y) / (rc_1) \right\} \left[ \ell(x) \langle \nabla^2 \tilde{U}(y), y \rangle + \theta(x, y) \langle \nabla \tilde{U}(x), y \rangle \right] \\
+ \left\{ \| \nabla U(x) \| / \| \nabla \tilde{U}(x) \| \right\} \left\{ \theta(x, y) \right\} \left[ \varphi \left\{ -2 \ell(x) \theta(x, y) / (rc_1) \right\} - \varphi \left\{ 2 \ell(x) \theta(x, y) / (rc_1) \right\} \right] \\
+ \lambda_r \left\{ \int_Y \varphi \left\{ (2 \ell(x) / (rc_1)) \langle \nabla \tilde{U}(x), w \rangle \right\} d\mu_\ast (w) - \varphi \left\{ 2 \ell(x) \theta(x, y) / (rc_1) \right\} \right\} . \]

The first step of the proof is to show that there exist \( A_{1,1}, A_{1,2} > 0 \) such that

\[(25) \quad AV(x, y) \leq -A_{1,1} V(x, y) + A_{1,2} \text{ for any } (x, y) \in \mathbb{R}^d \times Y, \ y \notin A_x, \]

where \( A_x \subset Y \) is defined by (9). In a second step, we show that there exist \( A_{2,1}, A_{2,2} > 0 \) such that

\[(26) \quad AV(x, y) \leq -A_{2,1} V(x, y) + A_{2,2} \text{ for any } (x, y) \in \mathbb{R}^d \times Y, \ y \in A_x \]

Note that if (25) and (26) hold, then the proof is concluded.

Proof of (25). Let \((x, y) \in \mathbb{R}^d \times Y, \ y \notin A_x \). From (24) and the facts that \( \varphi \) is bounded by \( 1 + c \), that \( \varphi(-s) - \varphi(s) \leq 0 \) for any \( s \in \mathbb{R}_+ \) since \( \varphi \) is non-decreasing, and that \( \sup_{s \in \mathbb{R}} \varphi'(s) \leq (a + \varepsilon) \lor b \lor ((c - b) + \varepsilon) \leq 1 + c \) since \( \varepsilon \leq 1 \), we have

\[(27) \quad J(x, y) \\
\leq (1 + c) \left[ \kappa \| \nabla \tilde{U}(x) \| \| y \| + (2/(rc_1)) \left\{ \| y \| \| \nabla \ell(x) \| + \ell(x) \| y \| ^2 \| \nabla^2 \tilde{U}(x) \| \right\} + \lambda_r \right] . \]

By (8) and (10) and since \( \ell \in C^1(\mathbb{R}^d), \| \nabla \ell \|_\infty + \| \ell \|_\infty < \infty \). Therefore plugging (27) in (24) and using (7) and A8-(ii), we get

\[AV(x, y) \leq C_1 (1 \lor \| y \| ^2) \exp(5 \tilde{U}(x)/4) + C_2 - \lambda_r \exp(\| H(\| y \|) \|) , \]

\[C_1 = (1 + c) \left\{ (\kappa \| \nabla \tilde{U} e^{-\ell/4} \|_\infty) \lor 2 \| \nabla \ell \|_\infty \lor \lambda_r \lor 2 \| \nabla^2 \tilde{U} e^{\ell/4} \|_\infty \| \ell \|_\infty \right\} < +\infty , \]

\[C_2 = \lambda_r \int_Y \exp(\| H(\| y \|) \|) d\mu_\ast (w) < +\infty . \]

Using now A8-(ii) and the continuity of \( H \), we get that \( C_3 = C_1 \sup_{y \in Y} (1 \lor \| y \| ^2) e^{H(\| y \|)/2} \) is finite. Since \( y \notin A_x \), \( 3 \tilde{U}(x) \leq H(\| y \|) \) and we obtain

\[AV(x, y) \leq C_3 \exp(11H(\| y \|)/12) + C_2 - \lambda_r \exp(\| H(\| y \|) \|) , \]

\[\leq -\lambda_r / 2 \exp(\| H(\| y \|) \|) + C_4 , \]

\[C_4 = C_2 + \sup_{s \in \mathbb{R}_+} \{ C_3 e^{11s/12} - \lambda_r e^s \} , \]

The proof of (25) follows upon noting that \( \kappa \leq 1 \) and that \( \varphi \) is bounded by \( 1 + c \), so that \( V(x, y) \leq (2 + c) \exp(H(\| y \|)) \) if \( y \notin A_x \).

Proof of (26). We show in Lemma 8 that there exist \( a, b, c \in \mathbb{R}_+, \ a \leq b \leq c, \ v \in (0,1], \ \kappa \in (0,1), \ R_1 \in \mathbb{R}_+ \) and \( \eta \in \mathbb{R}_+^* \) such that for all \((x, y) \in \mathbb{R}^d \times Y, \ y \in A_x \) and \( \| x \| \geq R_1 \), \( J(x, y) < -\eta \). Note that if this result holds, then for all \((x, y) \in \mathbb{R}^d \times Y, \ y \in A_x \) and \( \| x \| \geq R_1 \) by (23),

\[(29) \quad AV(x, y) \leq -\eta \exp(\kappa \tilde{U}(x)) + C_2 - \lambda_r \exp(\| H(\| y \|) \|) \leq -((\eta/(1 + c)) \lor \lambda_r) V(x, y) + C_2 , \]
where $C_2$ is given by (28) and we have used for the last inequality that $\varphi$ is bounded by $1 + c$. This result concludes the proof of (26) for $\|x\| \geq R_1$. It remains to consider the case $\|x\| \leq R_1$.

Since $\psi$ and $U$ are continuous, so is $\bar{U}$, so that there exists $M_1$ such that for all $x \in B(0, R_1)$ and $y \in A_x$, $H(\|y\|) \leq M_1$. Since $\sup_{w \in Y} \|w\|^2 e^{-H(\|w\|)} < +\infty$ by A 8-(ii), it follows that there exists $M_2$ such that for all $x \in B(0, R_1)$, $A_x \subset B(0, M_2)$. Then, using that $\bar{U} \in C^2(\mathbb{R}^d)$, $\ell \in C^1(\mathbb{R}^d)$, $H \in C(\mathbb{R}^+)$ and $\varphi \in C^1(\mathbb{R})$ we get that there exists $C_5, C_6$ such that for all $x \in B(0, R_1)$ and $y \in A_x$, $AV(x, y) \leq C_5$ and $V(x, y) \leq C_6$. Combining this result and (29) concludes the proof of (26).

Let us now precise the parameters we chose in the definition of $V$. Set
\begin{align*}
a &= 1 \wedge \left( \left( 1/3 \right) \wedge \left( \{c_3/(4c_2)\} \wedge \{\lambda_2\} \wedge \{\lambda_3\}\right) \right)^{-1} \\
b &= a = a \left( 1/3 \right) \wedge \left( \{c_3/(4c_2)\} \wedge \{\lambda_2\} \wedge \{\lambda_3\}\right) \\
\kappa &= (b - a) \left( \{c_3/(4c_2)\} \wedge \{\lambda_2\} \wedge \{\lambda_3\}\right) \\
c &= b - \left( \delta \lambda_3 a/(4(4c_4/c_2) + 2\lambda_4) \right) \wedge \left( b - a \right) \wedge \left( \lambda_3 - \lambda_2 \right) \\
\varepsilon &= (1/2) \wedge (c - b) \wedge (\kappa r c_1/4) \wedge (\lambda_3 c_2).
\end{align*}

Note that $\kappa \leq 1$ and
\begin{equation}
0 \leq c - b \leq b - a \leq a \leq 1.
\end{equation}

**Lemma 8.** Assume A 1-A 2-A 8 and (12) hold. Then for $a, b, c, \kappa, \varepsilon \in (0, 1]$, given in (30)-(31)-(33)-(32)-(34) respectively, there exist $\bar{R}, \eta > 0$ such that for all $x \in \mathbb{R}^d$ with $\|x\| \geq \bar{R}$ and all $y \in A_x$, $J(x, y) < -\eta$, where $J$ and $\varphi$ are defined by (24) and (19) respectively.

**Proof.** In the proof, we first give a bound on $J$ for any $(x, y) \in \mathbb{R}^d$, $y \in A_x$. Second, we distinguish five cases depending on the value of $2\ell(x)\theta(x)/(r c_1)$ which determines the contribution of $\varphi$ and $\varphi'$ in $J$. For ease of notation, we denote for any $(x, y) \in \mathbb{R}^d \times Y$, $\theta(x, y) = \langle \nabla U(x), y \rangle$ again.

By (10), there exists $R_1 \in \mathbb{R}_+$ such that for any $(x, y) \in \mathbb{R}^d$, $y \in A_x$, $\|x\| \geq R_1$,
\begin{equation}
\|\nabla \ell(x)\| \leq \varepsilon.
\end{equation}

From (8), $\|\nabla U(x)\| \leq c_1$ for all $x \in \mathbb{R}^d$ with $\|x\| \geq R$. Using A 8-(ii) and the facts that $\mu_x$ is rotation invariant and that $\varphi$ is non-decreasing, bounded by $1 + c$ and equal to 1 on $(-\infty, 2)$, we then have for any $x \in \mathbb{R}^d$ with $\|x\| \geq R$
\begin{align*}
\int \varphi \left\{ (2\ell(x)/(rc_1)) \langle \nabla U(x), w \rangle \right\} d\mu_x(w) &= \int \varphi \left\{ 2\ell(x) \langle \nabla U(x), w \rangle/w/(rc_1) \right\} d\mu_x(w) \\
&\leq \int \mathbf{1}_{(-\infty, -\eta]}(w) d\mu_x(w) + (1 + c) \int \mathbf{1}_{(-\eta, +\infty)}(w) d\mu_x(w) \leq 1 + (1 - \delta/2)c.
\end{align*}

Therefore, combining this result, (36), (11) and the fact that $\varphi$ is non-decreasing so that $\varphi'(s) \geq 0$ for any $s \in \mathbb{R}$, we get, for any $x \in \mathbb{R}^d$ with $\|x\| \geq R_2 = R \land R_1$ and all $y \in A_x$,
\begin{align*}
J(x, y) &\leq \kappa \theta(x, y) \varphi \left\{ 2\ell(x)\theta(x)/(rc_1) \right\} + (2/(rc_1))\varphi' \left\{ 2\ell(x)\theta(x)/(rc_1) \right\} [c_4 + |\theta| (x, y)]c \\
&+ \left\{ \|\nabla U(x)\| \|\nabla U(x)\| \theta(x, y) \right\} + \varphi \left\{ -2\ell(x)\theta(x)/(rc_1) \right\} - \varphi \left\{ 2\ell(x)\theta(x)/(rc_1) \right\} \\
&+ \lambda_\varepsilon \left\{ 1 + (1 - \delta/2)c - \varphi \left\{ 2\ell(x)\theta(x)/(rc_1) \right\} \right\}.
\end{align*}

Let $(x, y) \in \mathbb{R}^d \times Y$, $y \in Y$, $\|x\| \geq R_2$. We consider now five cases.
where we have used that $J(x,y)/(rc_1) \in (-\infty,-2]$. Since for $s \in (-2,\infty)$, $\varphi(s) = 1$, (37) reads
\begin{equation}
J(x,y) \leq \kappa \theta(x,y) + (1 - \delta/2) \lambda c.
\end{equation}
Using the facts that $2\ell(x)\theta(x,y)/(rc_1) \in (-\infty,-2]$, that $\ell(z) \leq c_2$ for all $z \in \mathbb{R}^d$ by (8), that $(b-a)\vee (c-b) \leq a$ by (35), that $a \leq rc_1\kappa/(6\lambda c_2)$ by (32) and that (12) holds, we get
\begin{equation}
rc_1\kappa/(2\ell(x)) \geq rc_1\kappa/(2c_2) \geq 3\lambda c a \geq (1 - \delta/2) \lambda c.
\end{equation}
By this result and (38), we obtain
\begin{equation}
J(x,y) \leq -rc_1\kappa/(2c_2).
\end{equation}

Case 2: $2\ell(x)\theta(x,y)/(rc_1) \in (-2,-1)$. By (19)-(20), $1 + 2a + sa - \varepsilon \leq \varphi(s) \leq 1 + 2a + sa + \varepsilon$ and $\varphi'(s) \leq a + \varepsilon$ for $s \in (-2,-1)$, so that (37) reads
\begin{equation}
J(x,y) \leq \kappa \theta(x,y)\{1 + 2a + 2a\ell(x)\theta(x,y)/(rc_1) - \varepsilon\} + (2(a + \varepsilon)/(rc_1))\{c_4 - \varepsilon \theta(x,y)\}
+ \lambda_1\{(1 - \delta/2)c - 2a - 2a\ell(x)\theta(x,y)/(rc_1) + \varepsilon\}
\leq B_0 + B_1\theta(x,y) + 2\ell(x)B_2\theta(x,y)^2/(rc_1) \leq B_0 + (B_1 - 2B_2)\theta(x,y),
\end{equation}
where we have used that $2\ell(x)\theta(x,y)/(rc_1) \in [-2,-1]$ and that $\ell(x) \leq c_2$ by (8), and defined
\begin{align*}
B_0 &= 2(a + \varepsilon)c_4/(rc_1) + \lambda_1\{(1 - \delta/2)c - 2a + \varepsilon\}
B_1 &= \kappa(1 + 2a - \varepsilon) - 2\lambda_1ac_2/(rc_1) - 2\varepsilon(a + \varepsilon)/(rc_1)
B_2 &= \kappa a.
\end{align*}
First, (34) and (35) ensures that $\varepsilon \leq (1/2) \wedge a \wedge (\lambda c_2)$, and therefore $B_1 - 2B_2 \geq \kappa/2 - 4\lambda_1ac_2/(rc_1) \geq \kappa/4$, where we have used that $a \leq rc_1\kappa/(16\lambda c_2)$ for the last inequality, which is a consequence of (32) and (12). In particular, $B_1 \geq 2B_2$ and using again that $2\ell(x)\theta(x,y)/(rc_1) \in (-2,-1)$ and $\ell(x) \leq c_2$ from (8), then
\begin{equation}
J(x,y) \leq B_0 + (rc_1/(2c_2))(2B_2 - B_1) \leq B_0 - rc_1\kappa/(8c_2).
\end{equation}
Since $\varepsilon \leq a \wedge (c - b)$ by (34), $c - b \leq b - a$ by (33) and $b - a \leq a/3$ by (31), we have $B_0 \leq 4ac_4/(rc_1)$. Hence, (40) becomes
\begin{equation}
J(x,y) \leq 4ac_4/(rc_1) - rc_1\kappa/(8c_2) \leq -rc_1\kappa/(16c_2),
\end{equation}
where we have used (32) and (12) for the last inequality.

Case 3: $2\ell(x)\theta(x,y)/(rc_1) \in [-1,0]$. Using the expression of $\varphi$ on $[-1,0]$ given by (19), (37) reads
\begin{equation}
J(x,y) \leq \kappa \theta(x,y)\{1 + b + (b - a)2\ell(x)\theta(x,y)/(rc_1)\} + (2(b - a)/(rc_1))\{c_4 - \theta(x,y)\varepsilon\}
+ \lambda_1\{(1 - \delta/2)c - b - 2\ell(x)\theta(x,y)\}b - a\}/(rc_1)\}
\leq B_0 + B_1\theta(x,y) + B_2\ell(x)/(rc_1)\theta(x,y)^2 \leq B_0 + (B_1 - 2B_2)\theta(x,y),
\end{equation}
where we have used that $2\ell(x)\theta(x,y)/(rc_1) \in [-1,0]$ and $\ell(x) \leq c_2$ by (8), and defined
\begin{align*}
B_0 &= 2(b - a)c_4/(rc_1) + \lambda_1\{(1 - \delta/2)c - b\}
B_1 &= \kappa(1 + b - 2\varepsilon + \lambda_1c_2)(b - a)/(rc_1)
B_2 &= \kappa(b - a).
\end{align*}
First, since $c - b \leq \delta b/4 \leq \delta c/4$ and $a \leq c$ by (33) and (35), we have
\begin{equation}
B_0 \leq 2(b - a)c_4/(rc_1) - \lambda_1\delta c/4 \leq 2(b - a)c_4/(rc_1) - \lambda_1\delta a/4 \leq -a\lambda_1\delta/8,
\end{equation}

\begin{equation}
\end{equation}
where we have used that \( b - a \leq \lambda \delta \arccos(1/16c_4) \) by (31) for the last inequality. Second, from the facts that \( \varepsilon \leq \lambda_c c_2 \) by (34) and \( (b - a) \leq a/3 \leq 1/3 \) by (31)-(30), we have
\[
B_2 - B_1 \leq \kappa(b - a) + 4\lambda_c c_2(b - a)/(rc_1) - \kappa(1 + b) \leq 4\lambda_c c_2/(rc_1) - \kappa \leq 0 ,
\]
where we used the definition of \( \kappa \) (32) and the condition (12) for the last inequality. Combining (43) and (44) in (42), we get
\[
J(x, y) \leq -a\lambda_\varepsilon \delta /8
\]
Case 4 : \( 2\ell(x)\theta(x, y)/(rc_1) \in (0, 1) \). First, note that since \( \varphi(s) = 1 + b + s(b - a) \) for \( s \in [-1, 0] \), \( \varphi \) is non-decreasing, we have for any \( s \in [0, 1] \),
\[
\varphi(-s) - \varphi(s) \leq \varphi(-s) - \varphi(0) \leq -(b - a)s .
\]
From this result and the fact by (19)-(20) that \( 1 + b + s(c - b) - \varepsilon \leq \varphi(s) \leq 1 + b + s(c - b) + \varepsilon \) and \( \varphi'(s) \leq c - b + \varepsilon \) for \( s \in (0, 1) \) we get that (37) reads
\[
J(x, y) \leq \kappa\theta(x, y) \{ 1 + b + 2\ell(x)\theta(x, y)(c - b + \varepsilon)/(rc_1) + \varepsilon \}
+ (2(c - b + \varepsilon)/(rc_1)) \{ c_4 + \theta(x, y)\varepsilon \} - (\|\nabla U(x)\| / \|\nabla U(x)\|) 2\ell(x)(b - a)\theta(x, y)^2/(rc_1)
+ \lambda_r \{ 1 + (1 - \delta/2)c - 1 - b - 2\ell(x)\theta(x, y)(c - b)/(rc_1) + \varepsilon \}
\leq B_0 + B_1\theta(x, y) + 2\ell(x)B_2\theta(x, y)^2/(rc_1) ,
\]
where we have used that \( (\|\nabla U(x)\| / \|\nabla U(x)\|)\ell(x) \geq c_3 \) by (8), \( \theta(x, y) \geq 0 \) and defined
\[
B_0 = 2c_4(c - b + \varepsilon)/(rc_1) + \lambda_r \{ (1 - \delta/2)c - b + \varepsilon \}
B_1 = \kappa(1 + b + \varepsilon) + 2\varepsilon(c - b + \varepsilon)/(rc_1)
B_2 = \kappa(c - b + \varepsilon) - c_3(b - a)/\ell(x) .
\]
Since \( \varepsilon \leq c - b \) by (34), \( \ell(x) \leq c_2 \) by (8) and \( 2\kappa c_2(c - b) \leq c_3(b - a)/2 \) by (33), we get
\[
B_2 \leq -\tilde{B}_2 = -c_3(b - a)/2\ell(x) ,
\]
and therefore
\[
J(x, y) \leq B_0 + B_1\theta(x, y) - 2\ell(x)\tilde{B}_2\theta(x, y)^2/(rc_1) .
\]
Then, using that \( s \mapsto C_1s - C_2s^2 \) is bounded by \( C_1^2/(2C_2) \) on \( \mathbb{R} \), we obtain
\[
J(x, y) \leq B_0 + \theta(x, y)rc_1B_2^2/(4\ell(x)\tilde{B}_2)
\]
Therefore, since \( \theta(x, y) \in (0, 1) \), to show that
\[
J(x, y) \leq -\lambda_\varepsilon \delta c/16 ,
\]
it is sufficient to prove that
\[
B_0 \leq -\lambda_\varepsilon \delta c/4
\]
\[
rc_1B_1^2/(4\ell(x)\tilde{B}_2) \leq \lambda_\varepsilon \delta c/8 .
\]
First (48) holds since using that \( \varepsilon \leq (c - b) \) by (34) and that \( a \leq c \), we have
\[
B_0 - \delta/4 = 2c_4(c - b + \varepsilon)/(rc_1) + \lambda_r \{ (1 - \delta/4)c - b + \varepsilon \}
\leq (4c_4/(rc_2) + 2\lambda_r)(c - b) - \delta a\lambda_r/4 \leq 0 ,
\]
using \( (c - b) \leq \delta a\lambda_r/(4(4c_4/(rc_2) + 2\lambda_r)) \) by (33) for the last inequality. It remains to establish (49) which is equivalent by definition of \( B_1 \) and \( \tilde{B}_2 \) (46) to
\[
\kappa(1 + b + \varepsilon) + 2\varepsilon(c - b + \varepsilon)/(rc_1) \leq \{ \lambda_\varepsilon c\delta c_3(b - a)/(4rc_1) \}^{1/2} .
\]
Since \( \varepsilon \leq 1 \wedge (\kappa rc_1/4) \) by (34), \( c - b \leq 1 \) and \( b \leq 2 \) by (35) and (30), we get
\[
\kappa(1 + b + \varepsilon) + 2\varepsilon(c - b + \varepsilon)/(rc_1) \leq 5\kappa .
\]
This result, the inequality \( b - a \leq c \) and the definition of \( \kappa \) (32) implies that (50) holds.

**Case 5 :** \( 2\ell(x)\theta(x, y)/(r c_1) \geq 1 \). Since by (19), \( \varphi(s) = 1 + c, \varphi'(s) = 0 \) and \( \varphi(-s) - \varphi(s) \leq a - c \) for \( s \geq 1 \), (37) reads

\[
\begin{align*}
J(x, y) & \leq \kappa \theta(x, y)(1 + c) - \{\|\nabla U(x)\|/\|\nabla U(x)\|\}\theta(x, y)(c - a) - \lambda_r \delta c/2 \\
& \leq \kappa \theta(x, y)(1 + c) - \{\|\nabla U(x)\|/c_2 \|\nabla U(x)\|\}\theta(x, y)(c - a) - \lambda_r \delta c/2 \\
& \leq \{\kappa(1 + c) - c_3(c - a)/c_2\} \theta(x, y) - \lambda_r \delta c/2 ,
\end{align*}
\]

where we used successively that by (8) \( \ell(x) \leq c_2 \) and \( \|\nabla U(x)\|/\|\nabla U(x)\| \geq c_3 \). Since \( c \leq 3 \) by (35) we have

\[
\begin{align*}
J(x, y) & \leq \{\kappa(1 + c) - c_3(c - a)/c_2\} \theta(x, y) - \lambda_r \delta c/2 \\
& \leq \{4\kappa - c_3(b - a)/c_2\} \theta(x, y) - \lambda_r \delta c/2 \leq -\lambda_r \delta c/2 ,
\end{align*}
\]

where we have used the definition of \( \kappa \) given by (32) and \( \theta(x, y) \geq 0 \) for the last inequality.

The proof follows from combining (39)-(41)-(46)-(47)-(51). \( \square \)

**Corollary 9.** Under A 8, for all \((x, y) \in \mathbb{R} \times Y\) and \( t \geq 0 \),

\[ P_t V(x, y) \leq V(x, y) e^{-A_1 t} + A_2 (1 - e^{-A_1 t}) \]

where \( V \) is given by (21) and \( A_1, A_2 \) are given by Lemma 7.

**Proof.** By [9, Section 31.5], since \( V \in D(A) \), the process \((M_t)_{t \geq 0}\), defined for any \( t \in \mathbb{R}_+ \) by

\[ M_t = e^{A_1 t} V(X_t, Y_t) - V(x, y) - \int_0^t \{ A_1 e^{A_1 s} V(X_s, Y_s) + A^{*}A V(X_s, Y_s) \} ds , \]

is a local martingale. Therefore \((M_{t \wedge \tau_n})_{t \geq 0}\) is a martingale where for all \( n \in \mathbb{N}^* \), \( \tau_n = \inf\{t \geq 0 : \|X_t\| + \|Y_t\| \geq n\} \) and

\[
\begin{align*}
\mathbb{E} \left[ e^{A_1(t \wedge \tau_n)} V(X_{t \wedge \tau_n}, Y_{t \wedge \tau_n}) \right] - V(x, y) &= \mathbb{E} \left[ \int_0^{t \wedge \tau_n} e^{A_1 s} \{ A_1 V(X_s, Y_s) + A V(X_s, Y_s) \} ds \right] \\
& \leq \mathbb{E} \left[ \int_0^{t \wedge \tau_n} e^{A_1 s} A_2 ds \right] \leq A_2 \left( e^{A_1 t} - 1 \right).
\end{align*}
\]

Letting \( n \) go to infinity concludes the proof since it yields

\[ e^{A_1 t} \mathbb{E} [V(X_t, Y_t)] \leq V(x, y) + A_2 (e^{A_1 t} - 1) . \]

\( \square \)

### 3.3. Mirror Coupling.

To obtain geometric ergodicity, the classical Meyn and Tweedie approach is, once a Lyapunov drift condition holds, to show that some sets \( C \subset \mathbb{R}^d \times Y \) are small sets: there exist \( t > 0, \varepsilon > 0 \) and \( \nu \in \mathcal{P}(\mathbb{R}^d \times Y) \), \( \nu(C) = 1 \), such that

\[ P_t((x, y), A) \geq \varepsilon \nu(A) , \text{ for all } A \subset C . \]

It is commonly known that this condition is equivalent to: there exist \( t > 0, \varepsilon > 0 \) such that for all \((x, y), (\tilde{x}, \tilde{y}) \in C , \)

\[ \|P_t((x, y), \cdot) - P_t((\tilde{x}, \tilde{y}), \cdot)\|_{TV} \leq 2(1 - \varepsilon) . \]

This section is devoted to the proof of the following result:

**Lemma 10.** Assume A 1 and A 2-(ii). Then, any compact set \( K \subset \mathbb{R}^d \times Y \) is a small set.
Previous works [35, 10] show that Lemma 10 hold in case where $\mathcal{Y} = S^d$. The proof consists in establishing that the occurrence of more than two refreshment events suffices for the distribution of $X_t, t \geq 0$, to have some density w.r.t. the Lebesgue density on a ball with a radius proportional to $t$. Nevertheless, the latter strategy usually yields a non-explicit rate of convergence. In particular the dependence of the obtained rate in the dimension of the space is either intractable or very rough.

For this reason, we will present a different argument, based on an explicit coupling of two BPS processes. However, this will only work under the assumption that $\mu_v$ is not singular with respect to the Lebesgue measure on $\mathbb{R}^d$, which rules out, for example, the case of the uniform measure on $S^d$. A general proof of Lemma 10, with no additional assumption on $\mu_v$, may be obtained by a straightforward adaptation of [35, Lemma 5.2] or [10, Lemma 2]. We will only treat the non-singular case, with a particular emphasis on the case where $\mu_v$ is a $d$-dimensional non-degenerate Gaussian distribution with zero-mean and covariance matrix $\Sigma$.

Before stating our main result, we need the following lemma concerning the reflexion coupling (see [29], [15] and references therein) between two $d$ standard Gaussian random variables with different means.

**Lemma 11.** Let $x^{(1)}, x^{(2)} \in \mathbb{R}^d$, $\Sigma$ be a positive definite matrix and $(W_t^{(1)})_{t \geq 0}$ be a standard one dimensional Brownian motion. Define $T_c = \inf\{t \geq 0 : W_t^{(1)} \geq \|\Sigma^{-1/2}(x^{(2)} - x^{(1)})\|/2\}$, the stochastic process $(W_t^{(2)})_{t \geq 0}$ by

\[
W_t^{(2)} = \begin{cases} -W_t^{(1)} - \|\Sigma^{-1}(x^{(2)} - x^{(1)})\| + W_t^{(1)} & \text{if } t \leq T_c \\ \text{otherwise} \end{cases},
\]

and the $d$-dimensional random variables

\[
G^{(1)} = W_1^{(1)} n \left\{ \Sigma^{-1/2}(x^{(2)} - x^{(1)}) \right\} + \bar{G} , \quad G^{(2)} = W_1^{(2)} n \left\{ \Sigma^{-1/2}(x^{(2)} - x^{(1)}) \right\} + \bar{G}
\]

\[
\bar{G} = \left( \text{Id} - n \left\{ \Sigma^{-1/2}(x^{(2)} - x^{(1)}) \right\} n \left\{ \Sigma^{-1/2}(x^{(2)} - x^{(1)}) \right\}^T \right) G ,
\]

where $G$ is a standard $d$-dimensional Gaussian random variable independent of $(W_t^{(1)})_{t \geq 0}$ and $n$ is given by (4). Then $G^{(1)}$ and $G^{(2)}$ are $d$-dimensional standard Gaussian random variables and for all $M \geq 0$,

\[
\mathbb{P} \left( x^{(1)} + \Sigma^{1/2} G^{(1)} = x^{(2)} + \Sigma^{1/2} G^{(2)} , \left\| G^{(1)} - \Sigma^{-1/2}(x^{(2)} - x^{(1)})/2 \right\| \leq M \right) = \tilde{\alpha}(\|\Sigma^{-1/2}(x^{(2)} - x^{(1)})\|, M)
\]

where for all $r \geq 0$,

\[
\tilde{\alpha}(r, M) = \frac{r}{2(2\pi)^{(d+1)/2}} \int_0^1 \left\{ s^{-3/2} \exp \left( -r^2/(8s) \right) \right\} \int_{\mathbb{R}^d} \mathbb{1}_{[0,M]} \left( \left( (1-s)w_1^2 + \cdots + w_d^2 \right)^{1/2} \right) e^{-\|x\|^2/2} \, dw \, ds .
\]

**Proof.** Without loss of generalities, we can assume that $\Sigma = \text{Id}$. By the Markov property of the Brownian motion $(W_t^{(1)})_{t \geq 0}$, since $T_c$ is a $(\mathcal{F}^W_t)_{t \geq 0}$-stopping time, where $\mathcal{F}^W_t = \sigma(W_s^{(1)}, s \leq t)$, $W_t^{(2)}$ is a Brownian motion. Therefore, $G^{(1)}$ and $G^{(2)}$ are $d$-dimensional standard Gaussian random variables.

Using again the Markov property of $(W_t^{(1)})_{t \geq 0}$, given $T_c < 1$, $W_1^{(1)} - W_1^{(2)}$ is independent of $\mathcal{F}^W_{T_c}$. Therefore, since $\{x^{(1)} + G^{(1)} = x^{(2)} + G^{(2)} \} = \{T_c \leq 1\}$ and $G$ is independent of
\((W_t^{(1)})_{t \geq 0}\), we get for all \(M \geq 0\),

\[
\mathbb{P}\left(x^{(1)} + \Sigma^{1/2} G^{(1)} = x^{(2)} + \Sigma^{1/2} G^{(2)}, \left\| G^{(1)} - \Sigma^{-1/2} (x^{(2)} - x^{(1)})/2 \right\| \leq M\right)
\]

\[= \mathbb{E} \left[ \mathbb{1}_{[0,1]}(T_c) \mathbb{P}\left( ((W_1^{(1)} - W_{T_c}^{(1)})^2 + \|G\|^2)^{1/2} \leq M \right) \mathcal{F}_{T_c}^{\mathbb{W}} \right] \]

\[= (2\pi)^{-d/2} \mathbb{E} \left[ \mathbb{1}_{[0,1]}(T_c) \int_{\mathbb{R}^d} \mathbb{1}_{[0,M]} \left\{ ((1 - T_c)w_1^2 + \cdots + w_d^2)^{1/2} \right\} e^{-\|x\|^2/2} dw \right]. \]

The proof then follows from the explicit expression of the density of \(T_c\) w.r.t. the Lebesgue measure (see e.g. [40, p. 107]).

**Lemma 12.** Assume \(A 1\), \(Y = \mathbb{R}^d\) and \(\mu_v\) is the Gaussian measure with zero-mean and covariance matrix \(\Sigma\). Then, for all \(t > 0\) and all compact set \(K \subset B(0, R_K)\) of \(\mathbb{R}^d \times Y\), \(R_K \geq 0\), for all \((x, y), (\tilde{x}, \tilde{y}) \in K\) and for all \(M \geq 0\),

\[
\Vert P_t((x, y), \cdot) - P_t((\tilde{x}, \tilde{y}), \cdot) \Vert_{TV} \leq 2 \left\{ 1 - \mathbb{E} \left[ \mathbb{1}_{[0,t]}(\lambda^{-1}M_1 + E_2) \right] \alpha \left( M, 2(1 + E_1/\lambda_t)R_K \left\| \Sigma^{1/2} \right\| \sqrt{\lambda_t/E_2} \right) g^2(E_2) \right\} ,
\]

where \(\alpha\) is given by (52), for all \(r \geq 0\),

\[
g(r) = \mathbb{P} \left( rM \sup_{z \in \mathbb{B}(0, (1 + E_1/\lambda_t)R_K + (r/\lambda_t)M)} \|\nabla U(z)\| \geq E_3 \right) ,
\]

(53)

\[
\bar{M} = M + \|\Sigma^{1/2}\| (1 + E_1/\lambda_t)R_K ,
\]

and \(E_1, E_2, E_3\) are three independent exponential random variables with parameter 1.

**Proof.** Let \(K\) be a compact set of \(\mathbb{R}^{2d}\). Let \((x, y), (\tilde{x}, \tilde{y}) \in K\), \((x, y) \neq (\tilde{x}, \tilde{y})\). We construct a non Markovian coupling \((X_t, Y_t, \tilde{X}_t, \tilde{Y}_t)\) between \(P_t((x, y), \cdot)\) and \(P_t((\tilde{x}, \tilde{y}), \cdot)\) for all \(t > 0\), and lower bound the quantity \(\mathbb{P}((X_t, Y_t) = (\tilde{X}_t, \tilde{Y}_t))\), which will conclude the proof using the characterization of the total variation distance by coupling.

Before proceeding to its precise definition, let us give a brief and informal description of this coupling (see Fig. 1, 2 and 3). We couple both processes to have the same two first refreshment times \(S_1\) and \(S_2\). At time \(S_1\), the Gaussian velocities are chosen according to Lemma 11 so that, in the absence of bounces in the meanwhile, with positive probability, the processes will reach the same position at time \(S_2\). At time \(S_2\), both velocities are refreshed with the same Gaussian variable. Hence, with positive probability, at time \(S_2\), the processes have the same position and same velocity, in which case we can keep them equal for all times \(t \geq S_2\).

More precisely, the coupling we consider is defined as follows. Let \((\tilde{E}_t^1, \tilde{E}_t^2, \tilde{E}_t^3, \tilde{G}_t)_{t \in \mathbb{N}^*}\) be i.i.d. random variables, where for all \(i \in \mathbb{N}^*\), \(\tilde{E}_t^1, \tilde{E}_t^2, \tilde{E}_t^3\) are independent exponential random variables with parameter 1 and \(\tilde{G}_t\) is independent of \(\tilde{E}_t^j\), \(j \in \{1, 2, 3\}\).

Set \((X_0, Y_0) = (x, y), (\tilde{X}_0, \tilde{Y}_0) = (\tilde{x}, \tilde{y})\), \(S_0 = S_0 = \tilde{S}_0 = 0\), \(N_0 = 0\), \(H_1 = \tilde{E}_1^1/\lambda_t\) and \(N_1 = 1\). We define then by recursion the jump times of the process and the process between these jump times. For \(n \geq 0\), we consider \(A_n = \{(X_{S_n}, Y_{S_n}) = (\tilde{X}_{S_n}, \tilde{Y}_{S_n})\}\) and distinguish two cases.
Figure 1. Before the first refreshment at time $S_1$, both processes may bounce freely. At time $S_1$, the Gaussian velocities are coupled so that, at time $S_2$ (which is the next refreshment time), provided this Gaussian coupling of the velocities succeeds, and provided they haven’t bounced in the meanwhile (i.e. $S_2 < S_b \land \bar{S}_b$), both processes reach the same position. At time $S_2$, both processes take the same velocity: they have merged, the coupling is a success.

Figure 2. If one (at least) of the processes bounces between times $S_1$ and $S_2$, then the coupling fails. There may be other bounces after the first one.

Figure 3. Even if none of the process bounces between time $S_1$ and $S_2$, the coupling may also fail if the Gaussian coupling of the velocities at time $S_1$ fails.
(A) If $N_{n+1} = 1$. Define

$$T^1_{n+1} = \inf \left\{ t \geq 0 : \int_0^t \left\{ \langle Y_{S^a_n}, \nabla U(X_{S^a_n} + sY_{S^a_n}) \rangle \right\} ds \geq E^1_{n+1} \right\},$$

$$\tilde{T}^1_{n+1} = \inf \left\{ t \geq 0 : \int_0^t \left\{ \langle \tilde{Y}_{S^a_n}, \nabla U(\tilde{X}_{S^a_n} + s\tilde{Y}_{S^a_n}) \rangle \right\} ds \geq E^2_{n+1} I_{A^*_n} + E^1_{n+1} I_{A_n} \right\},$$

$$T_{n+1} = \tilde{H}_{n+1} \land T^1_{n+1} \land \tilde{T}^1_{n+1}.$$ 

Set $S^a_n = S^a_n + T_{n+1}$, for all $t \in [S_n, \tilde{S}_n)$, $(X_t, Y_t) = \phi_t(X_{S^a_n}, Y_{S^a_n})$, $X_{S^a_n} = X_{S^a_n} + T_{n+1}Y_{S^a_n}$, $(\tilde{X}_t, \tilde{Y}_t) = \phi_t(\tilde{X}_{S^a_n}, \tilde{Y}_{S^a_n})$, $\tilde{X}_{S^a_n} = \tilde{X}_{S^a_n} + T_{n+1}\tilde{Y}_{S^a_n}$. If $\tilde{T}_{n+1} = \tilde{H}_{n+1}$, consider the two random variables $G^{(1)}$, $G^{(2)}$ defined by Lemma 11, associated with a Brownian motion independent of $(\tilde{E}^1_i, \tilde{E}^2_i, \tilde{E}^3_i, (\tilde{G}_{i,j})_{j\in \mathbb{N}^*})_{i\in \mathbb{N}^*}$, and for $x^{(1)} = X_{S^a_n}$, $x^{(2)} = \tilde{X}_{S^a_n}$, $\Sigma$ the co-variance matrix associated with $\mu_\nu$ multiplied by $E^3_2/\lambda$, and $M \geq 0$.

Still if $\tilde{T}_{n+1} = \tilde{H}_{n+1}$, set

$$\begin{cases} Y_{S^a_n} = \Sigma^{1/2}G^{(1)}, & \tilde{Y}_{S^a_n} = \Sigma^{1/2}G^{(2)} \\ N_{n+2} = 2, & H_{n+2} = E^3_3 N_{n+2}/\lambda \end{cases}$$

Otherwise set $\tilde{N}_{n+2} = \tilde{N}_{n+1}$, $\tilde{H}_{n+2} = \tilde{H}_{n+1} - T_{n+1}$ and if $T_{n+1} = T^1_{n+1} = \tilde{T}^1_{n+1}$, $Y_{S^a_n} = R(\tilde{X}_{S^a_n} + T_{n+1}\tilde{Y}_{S^a_n}, \tilde{Y}_{S^a_n})$, $Y^R_{S^a_n} = R(\tilde{X}_{S^a_n} + T_{n+1}\tilde{Y}_{S^a_n}, \tilde{Y}^R_{S^a_n})$,

$$\begin{cases} T_{n+1} = T^1_{n+1} < \tilde{T}^1_{n+1}, & Y_{S^a_n} = R(X_{S^a_n} + T_{n+1}Y_{S^a_n}, Y_{S^a_n}), \tilde{Y}_{S^a_n} = \tilde{Y}_{S^a_n} \\ T_{n+1} = \tilde{T}^1_{n+1} < T^1_{n+1}, & Y_{S^a_n} = R(\tilde{X}_{S^a_n} + T_{n+1}\tilde{Y}_{S^a_n}, \tilde{Y}_{S^a_n}), Y_{S^a_n} = Y_{S^a_n} \end{cases}$$

(B) If $N_{n+1} \geq 2$. Define

$$T^1_{n+1} = \inf \left\{ t \geq 0 : \int_0^t \left\{ \langle Y_{S^a_n}, \nabla U(X_{S^a_n} + sY_{S^a_n}) \rangle \right\} ds \geq E^1_{n+1} \right\},$$

$$\tilde{T}^1_{n+1} = \inf \left\{ t \geq 0 : \int_0^t \left\{ \langle \tilde{Y}_{S^a_n}, \nabla U(\tilde{X}_{S^a_n} + s\tilde{Y}_{S^a_n}) \rangle \right\} ds \geq E^2_{n+1} I_{A^*_n} + E^1_{n+1} I_{A_n} \right\},$$

$$T_{n+1} = \tilde{H}_{n+1} \land T^1_{n+1} \land \tilde{T}^1_{n+1}.$$ 

Set $S^a_n = S^a_n + T_{n+1}$, for all $t \in [S_n, \tilde{S}_n)$, $(X_t, Y_t) = \phi_t(X_{S^a_n}, Y_{S^a_n})$, $X_{S^a_n} = X_{S^a_n} + T_{n+1}Y_{S^a_n}$, $(\tilde{X}_t, \tilde{Y}_t) = \phi_t(\tilde{X}_{S^a_n}, \tilde{Y}_{S^a_n})$, $\tilde{X}_{S^a_n} = \tilde{X}_{S^a_n} + T_{n+1}\tilde{Y}_{S^a_n}$ and

$$\begin{cases} Y_{S^a_n} = \tilde{G}_{n+1}, & \tilde{Y}_{S^a_n} = \tilde{G}_{n+1} \\ N_{n+2} = N_{n+1} + 1, & H_{n+2} = E^3_3 N_{n+2}/\lambda \end{cases}$$

Otherwise set $\tilde{N}_{n+2} = \tilde{N}_{n+1}$, $\tilde{H}_{n+2} = \tilde{H}_{n+1} - T_{n+1}$ and if $T_{n+1} = T^1_{n+1} = \tilde{T}^1_{n+1}$, $Y_{S^a_n} = R(\tilde{X}_{S^a_n} + T_{n+1}\tilde{Y}_{S^a_n}, \tilde{Y}_{S^a_n})$, $Y^R_{S^a_n} = R(\tilde{X}_{S^a_n} + T_{n+1}\tilde{Y}_{S^a_n}, \tilde{Y}^R_{S^a_n})$,

$$\begin{cases} T_{n+1} = T^1_{n+1} < \tilde{T}^1_{n+1}, & Y_{S^a_n} = R(X_{S^a_n} + T_{n+1}Y_{S^a_n}, Y_{S^a_n}), \tilde{Y}_{S^a_n} = \tilde{Y}_{S^a_n} \\ T_{n+1} = \tilde{T}^1_{n+1} < T^1_{n+1}, & Y_{S^a_n} = R(\tilde{X}_{S^a_n} + T_{n+1}\tilde{Y}_{S^a_n}, \tilde{Y}_{S^a_n}), Y_{S^a_n} = Y_{S^a_n} \end{cases}$$

For $t \geq \sup_{n\in \mathbb{N}^*} S^a_n$ set $(X_t, Y_t) = (\tilde{X}_t, \tilde{Y}_t)_{t \geq 0} = \infty$. Denote by $(\mathcal{F}_n)_{n \geq 1}$, the filtration associated with $(\tilde{E}_i^1, \tilde{E}_i^2, \tilde{E}_i^3, (\tilde{G}_{i,j})_{j \in \mathbb{N}^*})_{i \in \mathbb{N}^*}$. In the construction of the two processes $(X_t, Y_t)_{t \geq 0}$, $(\tilde{X}_t, \tilde{Y}_t)_{t \geq 0}$, note that $(W_t)_{t \geq 0}$ is independent of $(\tilde{E}_i^1, \tilde{E}_i^2, \tilde{E}_i^3, (\tilde{G}_{i,j})_{j \in \mathbb{N}^*})_{i \in \mathbb{N}^*}$. By Lemma 11, we have that conditionally to $(\tilde{E}_i^1, \tilde{E}_i^2, \tilde{E}_i^3, (\tilde{G}_{i,j})_{j \in \mathbb{N}^*})_{i \in \mathbb{N}^*}$, $G^{(2)}$ is a standard $d$-dimensional Gaussian random variable. Therefore from their definitions and [14, Proposition 5], marginally, $(X_t, Y_t)_{t \geq 0}$ and $(\tilde{X}_t, \tilde{Y}_t)_{t \geq 0}$ are two BPS processes starting from $(x, y)$ and $(\tilde{x}, \tilde{y})$. However,
since the conditional distribution of \((G^{(1)}, G^{(2)})\) given \((\tilde{E}^1, \tilde{E}^2, \tilde{E}^3, (G_{i,j})_{j \in \mathbb{N}^*})_{i \in \mathbb{N}^*}\) depends on \(\tilde{E}^2_i, (X_i, Y_i, \tilde{X}_i, \tilde{Y}_i)_{i \geq 0}\) is not Markovian.

Furthermore, from the construction of the two processes, for all \(n \in \mathbb{N}\) if \((X_{S_n}^\tau, Y_{S_n}^\tau) = (\tilde{X}_{S_n}^\tau, \tilde{Y}_{S_n}^\tau)\), then \((X_t, Y_t) = (\tilde{X}_t, \tilde{Y}_t)\) for all \(t > S_n^\tau\). Besides, consider \(\tau = \inf(n \in \mathbb{N} : N_{n+2} = 2)\). Then by definition, if \(T_{\tau+2} = \bar{T}_{\tau+2}\) and \(X_{S_{\tau+1}^\tau} + \tilde{E}^3 G^{(1)}(\lambda_\tau) = \tilde{X}_{S_{\tau+1}^\tau} + \tilde{E}^3 G^{(2)}(\lambda_\tau)\), then \((X_{S_{\tau+2}^\tau}^\tau, Y_{S_{\tau+2}^\tau}^\tau) = (\tilde{X}_{S_{\tau+2}^\tau}^\tau, \tilde{Y}_{S_{\tau+2}^\tau}^\tau)\). Finally, by definition of \(\tau\), \(T_{\tau+1} = \bar{T}_{\tau+1}\) implies \(S_{\tau+1} = \tilde{E}^3_1 / \lambda_\tau\) and if \(T_{\tau+2} = \bar{T}_{\tau+2}\), \(S_{\tau+2} = S = (\tilde{E}^1_1 + \tilde{E}^3_2) / \lambda_\tau\). Based on these three observations, we get for all \(t > 0\),

\[
P \left( (X_t, Y_t) = (\tilde{X}_t, \tilde{Y}_t) \right) \\
\geq P \left( t \geq \sup \{ T_{\tau+2} = T_{\tau+1} \} \right) + \left\{ X_{S_{\tau+1}^\tau} + \tilde{E}^3_2 \Sigma^{1/2} G^{(1)}(\lambda_\tau) = \tilde{X}_{S_{\tau+1}^\tau} + \tilde{E}^3_2 \Sigma^{1/2} G^{(2)}(\lambda_\tau) \right\} \\
\geq P \left( A \cap \{ t \geq S \} \right) \cap \left\{ X_{E_1^1 / \lambda_\tau} + \tilde{E}^3_2 \Sigma^{1/2} G^{(1)}(\lambda_\tau) = \tilde{X}_{E_1^1 / \lambda_\tau} + \tilde{E}^3_2 \Sigma^{1/2} G^{(2)}(\lambda_\tau) \right\} .
\]

where \(A = A_1 \cap A_2\),

\[
A_1 = \left\{ \int_0^{E_1^3 / \lambda_\tau} \left\{ \left( \frac{Y_{E_1^1 / \lambda_\tau}}{E_1^1 / \lambda_\tau}, \nabla U(X_{E_1^1 / \lambda_\tau} + sY_{E_1^1 / \lambda_\tau}) \right)_s \right\} ds \geq \tilde{E}^1_{\tau+2} \right\} ,
\]

\[
A_2 = \left\{ \int_0^{E_1^2 / \lambda_\tau} \left\{ \left( \frac{\tilde{Y}_{E_1^1 / \lambda_\tau}}{E_1^1 / \lambda_\tau}, \nabla U(\tilde{X}_{E_1^1 / \lambda_\tau} + s\tilde{Y}_{E_1^1 / \lambda_\tau}) \right)_s \right\} ds \geq \tilde{E}^2_{\tau+2} \right\} .
\]

Since for all \(n \in \{ 1, \ldots, \tau \}, T_{n+1} = T_{n+1}^1 \land \tilde{T}_{n+1}^1, \| Y_{S_n}^\tau \| = \| y \|, \| \tilde{Y}_{S_n}^\tau \| = \| \tilde{y} \|, \) so for all \(s \in [0, E_1^3 / \lambda_\tau]\),

\[
\| X_s \| \leq \| x \| + (\tilde{E}^1_1 / \lambda_\tau) \| y \| \leq (1 + \tilde{E}^1_1 / \lambda_\tau) R_K, \| \tilde{X}_s \| \leq (1 + \tilde{E}^1_1 / \lambda_\tau) R_K
\]

Therefore, for \(i = 1, 2\), by the definition (53) of \(\hat{M}\),

\[
B = \left\{ \| G^{(i)} - (\Sigma^{1/2} / 2)(X_{E_1^1 / \lambda_\tau} - \tilde{X}_{E_1^1 / \lambda_\tau}) \| \leq M \right\} \subset \{ \| G^{(i)} \| \leq \hat{M} \} ,
\]

and

\[
A_1 \cap B \subset \tilde{A}_1 = \left\{ \left( \frac{E_1^1 / \lambda_\tau}{E_1^3 / \lambda_\tau} M \right) \sup_{z \in B(0, (1+\tilde{E}^1_1 / \lambda_\tau) R_K + (E_1^3 / \lambda_\tau) M)} \| \nabla U(z) \| \geq \tilde{E}^1_{\tau+2} \right\} ,
\]

\[
A_2 \cap B \subset \tilde{A}_2 = \left\{ \left( \frac{E_1^2 / \lambda_\tau}{E_1^3 / \lambda_\tau} \tilde{M} \right) \sup_{z \in B(0, (1+\tilde{E}^1_1 / \lambda_\tau) R_K + (E_1^3 / \lambda_\tau) \tilde{M})} \| \nabla U(z) \| \geq \tilde{E}^2_{\tau+2} \right\} .
\]

Then, we get by (54) setting \(\tilde{A} = \tilde{A}_1 \cap \tilde{A}_2\),

\[
P \left( (X_t, Y_t) = (\tilde{X}_t, \tilde{Y}_t) \right) \\
\geq P \left( \tilde{A} \cap B \cap \{ t \geq S \} \cap \{ X_{E_1^1 / \lambda_\tau} + \tilde{E}^3_2 \Sigma^{1/2} G^{(1)}(\lambda_\tau) = \tilde{X}_{E_1^1 / \lambda_\tau} + \tilde{E}^3_2 \Sigma^{1/2} G^{(2)}(\lambda_\tau) \} \right) .
\]

Conditioning on \(F_{\tau+1}\) and \(\tilde{E}^1_1, \tilde{E}^2_1, \tilde{E}^2_{\tau+2}\) are independent and independent of \(G^{(1)}, G^{(2)}, \tilde{E}^3_1\), and \(F_{\tau+1}\), the definition of \(G^{(1)}, G^{(2)}\) conditionnaly to \(\tilde{E}^3_1\) and \(F_{\tau+1}\), and Lemma 11 we have

\[
P \left( (X_t, Y_t) = (\tilde{X}_t, \tilde{Y}_t) \right) \\
\geq E \left[ \int_{[0,1]} \frac{1}{\lambda_\tau} \{ E_1 + E_2 \} \hat{d} \left( \sqrt{\frac{\Sigma^{1/2}(X_{E_1^1 / \lambda_\tau} - \tilde{X}_{E_1^1 / \lambda_\tau})}{\lambda_\tau / E_2} \right) g^2(\tilde{E}_2) \right]
\]

Combining this result with (55) concludes the proof.
Consider the more general case where $\mu_x$ is rotation invariant and not singular with respect to the Lebesgue measure on $\mathbb{R}^d$. The previous proof may be adapted to this case but the result is less explicit.

**Lemma 13.** Assume for all $A \in B(\mathbb{R}^d)$,

$$
\mu_{x}(A) \geq c_{\nu_{r,\delta}}(A),
$$

for some $r, \delta, c > 0$, where $\nu_{r,\delta}$ the uniform law on $\{y \in \mathbb{R}^d, r < \|y\| < r + \delta\}$. Let $K \subset \mathbb{R}^d$, be a compact set. Then there exists two random variables $G^{(1)}, G^{(2)}$ with distribution $\mu_x$, $t_0 \geq 0$, $\varepsilon > 0$ such that for all $s \geq t_0$, there exists $M \geq 0$ satisfying for all $x, \tilde{x} \in K$,

$$
P \left( x + sG^{(1)} = \tilde{x} + sG^{(2)}, \left\| G^{(1)} - (x - \tilde{x})/2 \right\| \leq M \right) \geq 1 - \varepsilon.
$$

**Proof.** Let $s \geq \|x - \tilde{x}\|/(2(r + \delta))$ and $M \geq R_{K} + s(r + \delta)$, then $I(x, \tilde{x}, s) = \{w \in \mathbb{R}^d, \|w\| \leq M\} \cap \{w \in \mathbb{R}^d : sr < \|w - x\| < s(r + \delta)\} \cap \{w \in \mathbb{R}^d : sr < \|w - \tilde{x}\| < s(r + \delta)\} \neq \emptyset$. Writing $\tilde{\nu}_{x, s}$ the law of $x + sG$ where $G$ has law $\mu_x$, then for all $A \in B(\mathbb{R}^d)$, by (56), there exists $\tilde{c} > 0$ such that

$$
\tilde{\nu}_{x, s}(A) \wedge \tilde{\nu}_{x, s}(A) \geq c_{\text{Leb}}(A \cap I(x, \tilde{x}, s)).
$$

Besides, (see e.g. [39] or [42]), we can construct a pair $(G_1, G_2)$ of random variables with both $G_1$ and $G_2$ distributed according to $\mu_x$, and such that $P(x + sG = \tilde{x} + sG) = \tilde{\nu}_{x, s}(A) \wedge \tilde{\nu}_{x, s}(A)$. Combining this result with (57), the fact the function in the right hand side of (57) is positive and depends continuously of $x$ and $\tilde{x}$, hence is lower bounded on $K$, concludes. \hfill \Box

**Lemma 14.** Assume $A$ and (56) for some $r, \delta, c > 0$, where $\nu_{r,\delta}$ the uniform law on $\{y \in \mathbb{R}^d, r < \|y\| < r + \delta\}$. Then, for all compact set $K$ of $\mathbb{R}^d \times \mathcal{Y}$, there exists $t_0, \alpha > 0$ such that for all $(x, y), (\tilde{x}, \tilde{y}) \in K$ and all $t \geq t_0$,

$$
\|P_t((x, y), \cdot) - P_t((\tilde{x}, \tilde{y}), \cdot)\|_{\text{TV}} \leq 2(1 - \alpha).
$$

**Proof.** The proof is exactly similar to the proof of Lemma 12. Indeed it suffices to consider a coupling of two BPS $(X_t, Y_t)_{t \geq 0}$ and $(\tilde{X}_t, \tilde{Y}_t)_{t \geq 0}$ defined similarly to the processes defined in the proof of Lemma 12 but $G^{(1)}, G^{(2)}$ are chosen according to Lemma 13 in place of Lemma 11. \hfill \Box

Finally, let us precise Lemma 10, in preparation of the low-temperature study of Section 4.2.

**Lemma 15.** Assume $A$. Then, for all compact set $K \subset \mathbb{R}^d \times \mathcal{Y}$, there exist $t_0, \varepsilon, C, R > 0$, which depend on $K$, $\mu_x$, and $\lambda_x$, but not on $U$, such that for all $(x, y), (\tilde{x}, \tilde{y}) \in K$ and all $t \geq t_0$,

$$
\|P_t((x, y), \cdot) - P_t((\tilde{x}, \tilde{y}), \cdot)\|_{\text{TV}} \leq 2 \left[ 1 - \varepsilon \exp \left( -C \|\nabla U\|_{\infty, B(0,R)} \right) \right].
$$

**Proof.** In the Gaussian case, the proof follows from the statement of Lemma 12. In the general case, we only give a sketch of proof, since this is a direct adaptation of [35, Theorem 5.1]. First, in the spirit of the proof of Lemma 12 or of [35, Lemma 5.2], we study a BPS with no potential, i.e. with $U = 0$, and we show that we may couple them so that, with some probability $\alpha > 0$, they merge in a given time $t_0$, without leaving a given compact set. Then we add independent bounces, and say that the coupling is still a success if no bounce happens before time $t_0$, which gives the desired dependency with respect to $U$. \hfill \Box

### 3.4. Proof of Theorem 5

The proof follows from Lemma 7 and Lemma 10, and an application of [32, Theorem 6.1]. However, [32, Theorem 6.1] is non quantitative and for the proofs of Section 4.2 need explicit bounds for the convergence of $(P_t)_{t \geq 0}$ to $\pi$. To this end, we give a quantitative version of Theorem 5 in Appendix B based on [22, Theorem 1.2].
3.5. Proofs of Theorem 1. In each case, we apply Theorem 5. Set \( H(t) = t^2 \) for \( t \in \mathbb{R} \). Consider \( r > 0 \) such that \( \delta = P(Y_1 > r) > 0 \) where \( Y = (Y_1, \ldots, Y_d) \in \mathcal{Y} \) is distributed according to \( \mu_\nu \). Note that A8-(ii) is automatically satisfied in all the cases.

Under A3, set \( \bar{U}(x) = U(x) \) and \( \ell(x) = 1 \) for all \( x \in \mathbb{R}^d \). All the conditions of A8 are satisfied and so is (12) by Remark 6 since \( \lim_{\|x\| \to +\infty} \|\nabla U(x)\| = +\infty \).

Under A4, set \( \bar{U}(x) = U^\varsigma(x) \) and \( \ell(x) = 1/(1+\|\nabla U(x)\|) \) for all \( x \in \mathbb{R}^d \). All the conditions of A8 are satisfied and (12) holds by Remark 6 since \( \lim_{\|x\| \to +\infty} \ell(x) = 0 \).

3.6. Proof of Theorem 2. We apply Theorem 5 again. Set \( H(t) = t^2 \) for \( t \in \mathbb{R} \). Consider \( r > 0 \) such that \( \delta = P(Y_1 > r) > 0 \) where \( Y = (Y_1, \ldots, Y_d) \in \mathcal{Y} \) is distributed according to \( \mu_\nu \). Note that A8-(ii) is automatically satisfied. Set \( \bar{U}(x) = U(x) \) and \( \ell(x) = 1 \) for any \( x \in \mathbb{R}^d \). Then, the conditions of A8 hold with \( c_4 \) arbitrarily small. Therefore, (12) is satisfied if \( \lambda_3 \) is small enough.

3.7. Proof of Theorem 4. We apply Theorem 5. Set \( H(t) = \eta t^2 \) for \( \eta \) small enough such that A8-(ii) is satisfied. Set \( \bar{U}(x) = U^\varsigma(x) \) for any \( x \in \mathbb{R}^d \). Note that

\[
\left\{ \sup_{x \in \mathcal{A}_\nu} \|y\|^2 \right\} \|\nabla^2 U(x)\| \leq 3\eta^{-1} \bar{U}(x) \|\nabla^2 \bar{U}(x)\| \leq CU^\varsigma(x) \left( \|\nabla^2 U(x)\| U^{\varsigma-1}(x) + \|\nabla U(x)\|^2 U^{\varsigma-2}(x) \right)
\]

for some \( C > 0 \), hence is bounded. Then, the proof follows the same lines as the proof of Theorem 1 under A4, and is omitted.

4. Miscellaneous

4.1. A precise and explicit bound for a toy model. Following carefully the proofs of Theorem 5, it is possible to get explicit bounds on the values of \( C, \rho \) > 0 such that (5) holds. Nevertheless, the obtained bounds are exponential in the dimension \( d \). In particular, in Section 3.3, when we try to couple two processes, we do not make any use of the potential \( U \). In fact, at this step, \( U \) only plays the role of an hindrance in the minorization condition given by Lemma 10 based on Lemma 12-Lemma 15. We try to couple the processes using only the refreshment jumps, and hope that, during this attempt, no bounce occurs. We now illustrate on a toy model, how an analysis which is model specific can circumvent this flaw. It shows that the explicit bounds we obtain in Lemma 12 may be far from optimality for some problems.

Consider the smooth manifold \( D = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\eta \mathbb{Z})^{d-1} \) for \( d \geq 2 \) and \( \eta > 0 \), and let \( \text{proj}_D : \mathbb{R}^d \to D \) be the corresponding projection. We set in this section \( \pi \) to be the uniform distribution on \( D, Y = \mathbb{R}^d \) and \( \mu_\nu \) to be the zero-mean \( d \)-dimensional Gaussian distribution on \( \mathbb{R}^d \) with covariance matrix \( \sigma^2 \) Id, \( \sigma^2 > 0 \). In this setting, \( U \) is simply the function which is identically equal to 0 on \( D \). A BPS sampler \( (X_t, Y_t)_{t \geq 0} \) is defined as in Section 2.1 to target \( \pi \odot \mu_\nu \). The construction is in all the respect the same, just by replacing the state space \( \mathbb{R}^d \times Y \) by \( D \times Y \). To show the convergence of the corresponding semi-group \( (P_t^D)_{t \geq 0} \), we show a uniform Doeblin condition [31, Chapter 16] holds using a direct coupling argument.

Note that by a deterministic transformation of this process from \( D \) to \([0, 1] \times [0, \eta]^{d-1} \), we end up with the reflected PDMP process targeting the uniform distribution on \([0, 1] \times [0, \eta]^{d-1} \) described in [2]. This can be seen as a toy model for convex potentials. If \( \eta \) is small, which
is the analogous of multi-scales problems, then the proof of Theorem 5 would yield a mixing time of order $\eta^d$. Indeed, in Section 3.3, the coupling is considered a failure as soon as one of the processes bounce (or, here, is reflected at the boundary). Hence, a successful coupling would need that, at the first refreshment time, the new Gaussian velocity is directed mainly of the processes bounce (or, here, is reflected at the boundary). Hence, a successful coupling would need that, at the first refreshment time, the new Gaussian velocity is directed mainly according to the first dimension, which is unlikely. As we will see, this is a too pessimistic bound.

**Proposition 1.** For all $x, \tilde{x} \in D$, $y, \tilde{y} \in \mathbb{R}^d$ and $t > 0$,

$$
\|\delta_{(x,y)}P_t^D - \delta_{(\tilde{x},\tilde{y})}P_t^D\|_{TV} \leq 2 \left[ \mathbb{P}(N_t \leq 1) + \mathbb{E} \left[ \mathbb{I}_{[2,\infty]}(N_t) \left( 1 - 2\Phi \left( \frac{(1 + \eta^2(d - 1)^{1/2})}{2(S_{N_t} - S_1)} \right) \right) \right] \right].
$$

where $\Phi$ is the cumulative distribution function of the standard Gaussian distribution on $\mathbb{R}$, $(N_t)_{t \geq 0}$ is a Poisson process with rate $\lambda$, and jump times $(S_t)_{t \geq 0}$, with $S_0 = 0$.

**Proof.** Let $(N_t)_{t \geq 0}$ be a Poisson process with rate $\lambda$, and jump times $(S_t)_{t \geq 0}$, with $S_0 = 0$. Set first for $t \in [0, S_1)$, $X_t = \text{proj}^D(x + ty)$, $Y_t = y$, $X_{S_t} = \text{proj}^D(x + S_t y)$, $\tilde{X}_t = \text{proj}^D(\tilde{x} + ty)$, $\tilde{Y}_t = \tilde{y}$, $\tilde{X}_{S_t} = \text{proj}^D(\tilde{x} + \tilde{S}_t \tilde{y})$. By [29, Section 2], given $(S_t)_{t \geq 0}$, there exist two Brownian motions $(W_t)_{t \geq 0}$ and $(\tilde{W}_t)_{t \geq 0}$ on $D$ such that for any $t > 0$,

$$
P \left( X_{S_t} + W_t = \tilde{X}_{S_t} + \tilde{W}_t \mid (S_k)_{k \geq 0} \right) = P \left( T_c \leq t \mid (S_k)_{k \geq 0} \right) = 1 - 2\Phi \left( - \frac{\|X_{S_t} - \tilde{X}_{S_t}\|}{(2t^{1/2})} \right),
$$

and

$$
T_c = \inf \{ s \geq 0 : X_{S_t} + W_s = \tilde{X}_{S_t} + \tilde{W}_s \}.
$$

We can define then for any $i \in \mathbb{N}^*$, $G_i = (W_{(S_{i+1} - S_i)^2} - W_{(S_i - S_{i-1})^2})/(S_{i+1} - S_i), G_i = (\tilde{W}_{(S_{i+1} - S_i)^2} - \tilde{W}_{(S_i - S_{i-1})^2})/(S_{i+1} - S_i).

Note that by the Markov property of $(W_t)_{t \geq 0}$ and $(\tilde{W}_t)_{t \geq 0}$, $(G_i)_{i \in \mathbb{N}^*}$ and $(\tilde{G}_i)_{i \in \mathbb{N}^*}$ are sequences of i.i.d. $d$-dimensional standard Gaussian random variables.

Define $Y_{S_t} = G_t$, $\tilde{Y}_{S_t} = \tilde{G}_t$ and now assume that $(X_t, Y_t), (\tilde{X}_t, \tilde{Y}_t)$ are defined for $t \in [0, S_k], k \geq 1$. Set for $t \in [S_k, S_{k+1}]$, $X_t = \text{proj}^D(X_{S_k} + (t - S_k)G_{k+1})$, $\tilde{X}_t = \text{proj}^D(\tilde{X}_{S_k} + (t - S_k)\tilde{G}_{k+1})$, for $t \in [S_k, S_{k+1}]$, $Y_t = Y_{S_k}$, $\tilde{Y}_t = \tilde{Y}_{S_k}$ and $Y_{S_{k+1}} = G_{k+1}$, $\tilde{Y}_{S_{k+1}} = \tilde{G}_{k+1}$. It follows then by construction that for any $t \geq 0$, $(X_t, Y_t)_{t \geq 0}$ is distributed according to $P_{t}^D((x, y), \cdot)$ and $(\tilde{X}_t, \tilde{Y}_t)_{t \geq 0}$ is distributed according to $P_{t}^D((\tilde{x}, \tilde{y}), \cdot)$. Then it remains to bound $P \left( (X_t, Y_t) = (\tilde{X}_t, \tilde{Y}_t) \right)$ by definition of the total variation norm.

Note that if $(S_{i+1} - S_i)^2 \geq (t - S_i)^2 \geq (S_i - S_1)^2 \geq T_c \geq (S_{i-1} - S_i)^2$, $i \geq 2$, we have by (59)-(60) and construction $(X_t, Y_t) = (\tilde{X}_t, \tilde{Y}_t)$. Therefore, we get $\{ (S_{N_t} - S_1)^2 \geq T_c \} \cap \{ N_t > 1 \} \subset \{ (X_t, Y_t) = (\tilde{X}_t, \tilde{Y}_t) \}$ and we obtain

$$
P \left( (X_t, Y_t) = (\tilde{X}_t, \tilde{Y}_t) \right) \leq P \left( \{ S_{N_t} \leq S_1 + T_c \} \cap \{ N_t \leq 1 \} \right) \leq P \left( N_t \leq 1 \right) + P \left( \{ N_t \geq 2 \} \cap \{ (S_{N_t} - S_1)^2 \geq T_c \} \right).
$$

The proof is then concluded by conditioning with respect to $(S_k)_{k \in \mathbb{N}}$ using (58) and for any $x \in D$, $\|x\| \leq (1 + \eta^2(d - 1)^{1/2}).$ 

□
Corollary 16. There exist $C \geq 0$ and $\varepsilon \in (0,1]$ independent of $d$ such that setting $t_\varepsilon = Cd^{1/2}$, for all $x, \tilde{x} \in D$ and $y, \tilde{y} \in \mathbb{R}^d$, 
$$
\|\delta_{(x,y)}P^D_{t_\varepsilon} - \delta_{(\tilde{x},\tilde{y})}P^D_{t_\varepsilon}\|_{TV} \leq (1 - \varepsilon) .
$$

Proof. By Proposition 1 and using the same notations, for all $x, \tilde{x} \in D$, $y, \tilde{y} \in \mathbb{R}^d$ and $t > 0$, we have since for any $s \geq 0$, $1/2 - \Phi(-s) \leq 1 \wedge (s/(2\pi)^{1/2})$,
$$
2^{-1}\|\delta_{(x,y)}P^D_{t} - \delta_{(\tilde{x},\tilde{y})}P^D_{t}\|_{TV} \leq \mathbb{P}(S_2 \geq t/4) + \mathbb{P}(S_2 \leq t/4, S_{N_t} - S_2 \leq t/2) \\
+ \mathbb{E}\left[\mathbb{I}_{[0,t/4]}(S_2)\mathbb{I}_{[t/2,\infty)}(S_{N_t} - S_2) \left\{ 1 - 2\Phi\left(\frac{(1 + \eta^2(d - 1))^{1/2}}{2(S_{N_t} - S_1)}\right)\right\}\right] \\
\leq \mathbb{P}(S_2 \geq t/4) + \mathbb{P}(S_{N_t} \leq 3t/4) + \frac{2(1 + \eta^2(d - 1))^{1/2}}{t\pi^{1/2}} .
$$
Since $\{S_{N_t} \leq 3t/4\} \subset \{N_t - N_{3t/4} = 0\}$, and $N_t - N_{3t/4}$ follows a Poisson distribution with parameter $t\lambda/4$, we get for all $x, \tilde{x} \in D$, $y, \tilde{y} \in \mathbb{R}^d$ and $t > 0$
$$
2^{-1}\|\delta_{(x,y)}P^D_{t} - \delta_{(\tilde{x},\tilde{y})}P^D_{t}\|_{TV} \leq \mathbb{P}(S_2 \geq t/4) + e^{-\lambda t/4} + \frac{2(1 + \eta^2(d - 1))^{1/2}}{t\pi^{1/2}} .
$$
The proof then follows from a straightforward computation. □

A direct consequence of Corollary 16 is that, with the same notations, for all $\nu \in \mathcal{P}(D \times \mathbb{R}^d)$ and $t \geq 0$,
$$
\|\nu P^D_t - \pi \otimes \mu_\nu\|_{TV} \leq (1 - \varepsilon)^{|t/t_\varepsilon|} .
$$
As a conclusion, for the considered toy model, we get that the rate of convergence scales only as $d^{1/2}$. Note that this result is optimal since the process has unit constant speed and the diameter of $D$ is $d^{1/2}$.

4.2. The metastable regime and annealing. The simulated annealing algorithm is a variation of the MCMC algorithm which, rather than computing expectation with respect to the distribution $\pi = \exp(-U)$, aims to find a global minimum of $U$. We will study in this section a simulated annealing algorithm based on the BPS, extending the results of [35, Theorem 1.5]. For the sake of simplicity, the study is restricted to the following case:

A9. (i) The potential $U \in C^2(\mathbb{R}^d)$ satisfies
$$
\int_{\mathbb{R}^d} \exp(-U(x)/2)dx < \infty , \\
\lim_{\|x\| \to \infty} U(x) = +\infty , \\
\liminf_{\|x\| \to \infty} \|\nabla U(x)\| > 0 , \\
\sup_{x \in \mathbb{R}^d} \|\nabla^2 U(x)\| < \infty .
$$
Moreover, without loss of generality, $U(0) = \min_{x \in \mathbb{R}^d} U = 0$.

(ii) $Y = B(0, M)$ for $M > 0$ and the distribution $\mu_\nu$ on $Y$ is rotation invariant.

In the rest of this section, A9 is enforced. However, note the arguments also work under A8 (in particular when $Y = \mathbb{R}^d$, $\mu_\nu$ has a Gaussian moment and $U$ is a perturbation of an $\chi$-homogeneous potential with $\chi > 1$, as in Proposition 3), which is not implied by A9.

For a measurable function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, referred to in the following as cooling schedule, we consider in this section the simulated annealing BPS process $(X_\beta^t, Y_\beta^t)$ defined as follows. Consider some initial point $(x, y) \in \mathbb{R}^d \times Y$, and the family of i.i.d. random variables $(E_i^1, E_i^2, G_i)i \in \mathbb{N}_0$ introduced in Section 2.1. Let $\lambda_\tau > 0$, $(X_0^\beta, Y_0^\beta) = (x,y)$ and $S_0^\beta = 0$. We
Lemma 18. For any \((\beta, s)\) should be close to a global minimum of \(U\).

Theorem 17. Note that under A9, \(Y\) is bounded and therefore by [14, Proposition 10], \(\sup_{n\in\mathbb{N}} S_\beta^n = +\infty\).

Therefore almost surely \((X_t^\beta, Y_t^\beta)_{t\geq0}\) is a \((\mathbb{R}^d \times Y)\)-valued càdlàg process. By [9, Theorem 25.5], the BPS process \((X_t^\beta, Y_t^\beta)_{t\geq0}\) defines a non-homogeneous strong Markov semi-group \((P_t)_{t\geq0}\) given for all \(s, t \in \mathbb{R}^+\), \((x, y) \in \mathbb{R}^d \times Y\) and \(A \in \mathcal{B}(\mathbb{R}^d \times Y)\) by

\[
P_t^\beta(x, y, A) = \mathbb{P}\left((X_s^\beta, Y_s^\beta) \in A\right),
\]

where \((X_u^\beta, Y_u^\beta)_{u\in\mathbb{R}^+}\) is the annealed BPS process started from \((x, y)\) and cooling schedule \(s \mapsto \beta(t+s)\). \((P_{s,t})_{s,t\geq0}\) is associated with the family of generator \((A_{\beta(t)})_{t\geq0}\) where for any \(\beta > 0\), \(A_\beta\) is defined for any \(f \in C^1(\mathbb{R}^d \times Y)\) by

\[
A_\beta f(x, y) = \langle y, \nabla f(x, y) \rangle + \beta(\langle y, \nabla U(x) \rangle \rangle + \{ f(x, R(x, y)) - f(x, y) \}
\]

\[+ \lambda \int_Y f(x, w) d\mu_v(w) - f(x, y) \right\}.
\]

As it is usual in simulated annealing if \(t \mapsto \beta(t)\) goes to infinity sufficiently slowly for the process \((X_t^\beta, Y_t^\beta)\) to approach its instantaneous equilibrium \(\exp(-\beta(t)U) \otimes \mu_v\), then \(X_t^\beta\) should be close to a global minimum of \(U\) with high probability.

A 10. The function \(t \mapsto \beta(t)\) is increasing, satisfies \(\lim_{t \to +\infty} \beta(t) = +\infty\), \(\beta(0) \geq 1\) and there exist \(s_0, D_1, D_2 > 0\) with \(D_1 \geq D_2\) such that for all \(t\) large enough, \(\beta(t) \geq D_2 \ln t\) and \(\beta(t+s_0) - \beta(t) \leq D_1/t\).

Theorem 17. Assume A9. There exists \(\theta > 0\) such that if A10 holds with \(D_1 \leq \theta^{-1}\), then for any \((x, y) \in \mathbb{R}^d \times Y\) and any levels \(\eta > \eta' > 0\), there exists \(A > 0\) such that, for all \(t > 0\),

\[
\mathbb{P}\left(U(X_t^\beta) > \eta + \min_{\mathbb{R}^d} U \right) \leq A \exp(U(x)/2)/t^p,
\]

where \(p = (1 - \theta D_1) \wedge (D_2 \eta') > 0\) and \((X_t^\beta, Y_t^\beta)\) is the annealed BPS process starting from \((x, y)\).

First, we establish a Foster-Lyapunov drift condition for \(A_\beta\) uniformly on \(\beta \geq 1\).

Lemma 18. Assume A9. There exist \(A_1, A_2, A_3 > 0\), \(\beta_* \geq 1\) and \(V_1, V_2 \in C^1(\mathbb{R}^d \times Y)\), with \(V_1 \exp(-U/2)\) bounded above and below by positive constants for \(i = 1, 2\), such that for all \(\beta \geq \beta_*\),

\[
A_3 V_1 \leq A_1 (A_2 - V_1),
\]
and for all $\beta \geq 1$,

$$A_\beta V_2 \leq A_3 V_2.$$  

**Proof.** We check that $A_8$ holds for $\beta$ large enough, with $U = U/2$ and the potential $x \mapsto U_\beta(x)$. Indeed, set $\ell(x) = 1$ for all $x \in \mathbb{R}^d$ and $H(t) = t^2$ for $t \in \mathbb{R}$. Then all the conditions of $A_8$ are clearly satisfied, with $c_1$, $c_2$ and $c_4$ which does not depend on $\beta$, and $c_3 = \beta$. Let $\beta_*$ be large enough so that (12) holds for $\beta \geq \beta_*$ and $\kappa = 1$ defined in (32).

Let $V_1$ be the function defined by (21). According to Lemma 7, there exist $A_1, A_2 > 0$ such that

$$A_{\beta_*} V_1 \leq A_1 (A_2 - V_1).$$

Now, for $\beta \geq \beta_*$, keeping the notations of Section 3.2,

$$(A_\beta - A_{\beta_*}) V_1(x, y) = e^{-U(x)/2} \langle y, \nabla U(x) \rangle_+ (\varphi(-\theta) - \varphi(\theta)) \leq 0.$$  

Second, set for any $(x, y) \in \mathbb{R}^d \times Y$, $V_2(x, y) = \exp(U(x)/2) \varphi_2((y, \nabla U(x)))$, where $\varphi_2 \in C^1(\mathbb{R})$ is an increasing function such that $\varphi(s) = 1$ for $s \leq -1$ and $\varphi(s) = 3$ for $s \geq 1$. Then, for all $\beta > 1$,

$$e^{-U(x)/2} A_\beta V_2(x, y) \leq \langle y, \nabla U(x) \rangle \varphi_2((y, \nabla U(x))) + M^2 \|\nabla^2 U\|_\infty \|\varphi_2\|_\infty + 2 \lambda_r + \beta \langle y, \nabla U(x) \rangle_+ (\varphi(-\langle y, \nabla U(x) \rangle) - \varphi((y, \nabla U(x))))$$

$$\leq 3 + M^2 \|\nabla^2 U\|_\infty \|\varphi_2\|_\infty + 2 \lambda_r ,$$

and we conclude by noting that $\exp(U(x)/2) \leq V_2(x, y)$ for any $(x, y) \in \mathbb{R}^d \times Y$. \hfill \Box

**Corollary 19.** Assume that the assumptions of Theorem 17 hold. Then there exists $A_4 > 0$ such that for all $t, s \geq 0$ and $(x, y) \in \mathbb{R} \times Y$,

$$P_{t,t+s} V_1(x, y) \leq A_4 e^{A_3 s} V_1(x, y),$$

and for all $t \geq 0$ such that $\beta(t) \geq \beta_*$,

$$P_{t,t+s} V_1(x, y) \leq e^{-A_1 s} V_1(x, y) + (1 - e^{-A_1 s}) A_2 ,$$

where $V_1 \in C^1(\mathbb{R}^d \times Y)$, $A_1, A_2, A_3$ are given by Lemma 18.

**Proof.** The proof follows the same line as the proof of Corollary 9, using Lemma 18 and $V_1/V_2$ is bounded above and below by positive constants. \hfill \Box

**Lemma 20.** Assume that the assumptions of Theorem 17 hold. Then, for all compact set $K$ of $\mathbb{R}^d \times Y$, there exist $s_1, \chi, A_5 > 0$ which depend on $K$, $\mu_y$, $\lambda_r$ and $U$ but not on $t \mapsto \beta(t)$, such that for all $(x, y), (\bar{x}, \bar{y}) \in K$, all $t \geq 0$ and all $s \geq s_1$,

$$\left\| P_{t,t+s}((x, y), \cdot) - P_{t,t+s}(\bar{x}, \bar{y}), \cdot) \right\|_{TV} \leq 2 \left[ 1 - \chi \exp \left( -A_5 \int_0^{t+s} \beta(u) du \right) \right].$$

**Proof.** The arguments are exactly those of the proof of Lemma 15, hence of [35, Theorem 5.1], so that we only give a sketch of proof. First, considering the case $\beta = 0$, we have already shown in Section 3.3 that, starting from two different points in a given compact $K$, it is possible to merge two processes in a time $s_1 > 0$ while staying in a compact $K'$, with some probability $\chi > 0$. Call $E$ this event. Then, considering the case $\beta > 0$, we follow the same coupling up to the first bounce time. The processes have merged if this first bounce happens after time $s_1$, which occurs with probability

$$\mathbb{P} \left( \int_t^{t+s_1} \beta(u) \left\langle Y_u^\beta, \nabla U(X_u^\beta) \right\rangle \geq \mathbb{E} \left[ E \right] \right) \geq \exp \left( -M \|\nabla U\|_\infty, K' \int_t^{t+s_1} \beta(u) du \right),$$

where $M = \sup_{(u, z) \in K'} \|z\|$. \hfill \Box
Lemma 21. Assume that the conditions of Theorem 17 hold. Then for all $t_0$, let $t \geq t_0$, define
\begin{equation}
\mathbf{n}(t) = \left( (t-t_0)/s_1 \right).
\end{equation}
Consider the following decomposition,
\begin{equation}
P_{0,t} = P_{0,t-n(t)s_1}Q_0Q_1\cdots Q_{n(t)-1}Q_{n(t)},
\end{equation}
where $Q_0$ is the identity kernel and for $k \in \{1, \ldots, n(t)\}$, we set
\begin{equation}
Q_k = P_{-((t-n(t)+1)s_1, t-(n(t)-k)s_1)}.
\end{equation}
For any measurable function $\varphi : \mathbb{R}^d \times Y \to \mathbb{R}$ and $\zeta \geq 0$, we set
\begin{equation}
\|\varphi\|_{\zeta, V_1} = \sup_{(x,y) \in \mathbb{R}^d \times Y} \left\{ \frac{|\varphi(x,y)|}{1 + \zeta V_1(x)} \right\},
\end{equation}
and consider the weighted $V_1$-norm on $P_{V_1}(\mathbb{R}^d \times Y) = \{ \mu \in \mathcal{P}(\mathbb{R}^d \times Y) : \mu(V_1) < \infty \}$, defined for $\mu_1, \mu_2 \in P_{V_1}(\mathbb{R}^d \times Y)$ by
\begin{equation}
\rho_\zeta(\mu_1, \mu_2) = \sup \{ \mu_1(\varphi) - \mu_2(\varphi) : \|\varphi\|_{\zeta, V_1} \leq 1 \}.
\end{equation}
Note that $\rho_\zeta(\mu_1, \mu_2)$ increases with $\zeta$ and that $p_0 = \| \cdot \|_{TV}$. In addition, for any $\mu_1, \mu_2 \in P_{V_1}(\mathbb{R}^d \times Y)$,
\begin{equation}
\rho_\zeta(\mu_1, \mu_2) \leq \|\mu_1 - \mu_2\|_{V_1} \leq (1 + \zeta)^{-1} \rho_\zeta(\mu_1, \mu_2).
\end{equation}

Corollary 19. Proof. It is a direct application to $Q_k$ for all $k$ of Theorem 24 based on Lemma 20 and Corollary 19.

Lemma 21. Assume that the conditions of Theorem 17 hold. Then for all $\nu_1, \nu_2 \in P_{V_1}(\mathbb{R}^d \times Y)$, $t \geq t_0$ and all $k \in \{1, \ldots, n(t)\}$,
\begin{equation}
\rho_{\epsilon_k}(\nu_1Q_k, \nu_2Q_k) \leq \kappa_k \rho_{\epsilon_k}(\nu_1, \nu_2),
\end{equation}
where
\begin{align*}
\epsilon_k &= \frac{\chi}{(1 - \gamma)A_2} \exp \left( -A_5 \int_{t-(n-k)s_1}^{t-(n-k)s_1} \beta_u du \right), \\
\kappa_k &= 1 - \frac{\chi}{2} \gamma \frac{1 - \gamma}{4} \exp \left( -A_5 \int_{t-(n-k)s_1}^{t-(n-k)s_1} \beta_u du \right), \gamma = \exp(-s_1A_1).
\end{align*}

Proof. It is a direct application to $Q_k$ for all $k$ of Theorem 24 based on Lemma 20 and Corollary 19.

For a fixed $\beta \geq 0$, let $(P_t^{(\beta)})_{t \geq 0}$ be the semi-group of the BPS sampler associated with the potential $x \mapsto \beta U(x)$ and, for $t \geq t_0$ and $k \in \{0, \ldots, n(t)\}$, let
\begin{equation}
Q_k' = P_{s_1}^{(\beta_k)},
\end{equation}
where for ease of notation simplicity we denote
\begin{equation}
\beta_k = \beta_{-(n(t)-k)s_1}.
\end{equation}
In other words, $Q_k'$ is similar to $Q_k$ except that the inverse temperature is frozen. Let $\pi_k$ be the invariant measure of $Q_k'$, namely
\begin{equation}
\pi_k = \pi_k \otimes \mu_v,
\end{equation}
where $\pi_k$ admits a density with respect to the Lebesgue measure given for any $x \in \mathbb{R}^d$ by
\begin{equation}
\pi_k(x) = Z_k^{-1} \exp(-\beta_k U(x)) dx, \quad Z_k = \int_{\mathbb{R}^d} \exp(-\beta_k U(\tilde{x})) d\tilde{x}.
\end{equation}
We know that the mass of $\pi_k$ concentrates, as $k \to \infty$, around the vicinity of the global minima of $U$. To get the same with $P_{0,t}((x,y),\cdot)$, we need to show that $\|\tilde{\pi}_n(t) - P_{0,t}((x,y),\cdot)\|_{\text{TV}}$ vanishes as $t \to \infty$. Denoting, for $t \geq t_0$ and $k \in \{0, \ldots, n(t)\}$, $\nu_k = \delta_{(x,y)} P_{0,t-n(t)+1} Q_0 Q_1 \cdots Q_{k-1} Q_k$, where $Q_k$ is defined in (64), it is then natural to study

$$u_k = \rho_k(\nu_k, \tilde{\pi}_k) .$$

From (66), for any $t \geq t_0$, $k \in \{1, \ldots, n(t)\}$

$$u_k \leq \kappa_k \rho_k(\nu_{k-1}, \tilde{\pi}_{k-1}) + \rho_k(\tilde{\pi}_{k-1} Q_k, \tilde{\pi}_k) \leq \kappa_k u_{k-1} + e_k$$

where we defined

$$e_k = \rho_k(\tilde{\pi}_{k-1} Q_k, \tilde{\pi}_k)$$

and used that $\rho_k(\nu_{k-1}, \tilde{\pi}_{k-1}) \leq \rho_{k-1}(\nu_{k-1}, \tilde{\pi}_{k-1})$ since $(\epsilon_k)_{k \geq 0}$ is non-increasing.

**Lemma 22.** Assume that the conditions of Theorem 17 hold. Then, there exists $A_0 > 0$ such that for all $t \geq t_0$, all $k \in \{1, \ldots, n(t)\}$ and $l \geq 1$, there exists $A_l > 0$ such that

$$e_k \leq A_l (\sqrt{\beta_k - \beta_{k-1}} + \beta_k - \beta_{k-1}) + A_6 e^{\frac{l}{2} (\beta_{k-1} - 1)} ,$$

where $\beta_k, n$ and $t_0$ are defined by (68), (63) and (62) respectively.

**Proof.** Let $t \geq t_0$, $k \in \{1, \ldots, n(t)\}$ and $l \geq 1$. In the proof, $C$ stands for a constant which may change from line to line but does not depend on $k$, $l$ and $\beta$. We bound

$$e_k \leq \rho_k(\tilde{\pi}_{k-1} Q_k, \tilde{\pi}_k) + \rho_k(\tilde{\pi}_{k-1} Q_k, \tilde{\pi}_{k-1})$$

and deal with each terms of the right hand side apart. Indeed, for the first one, the first marginal of $\tilde{\pi}_{k-1}$ and $\tilde{\pi}_k$ having an explicit density, and their second marginal being equal, we bound

$$\rho_k(\tilde{\pi}_{k-1}, \tilde{\pi}_k) = \int_{\mathbb{R}^d \times Y} (1 + e_k V_1(x,y))|\pi_k(x) - \pi_{k-1}(x)|d\mu_x(dy)$$

$$\leq C \int_{\mathbb{R}^d} e^{U(x)/2} \left| \frac{e^{-\beta_k U(x)}}{Z_k} - \frac{e^{-\beta_{k-1} U(x)}}{Z_{k-1}} \right| dx$$

$$\leq C e^{l/2} \int_{\mathbb{R}^d} \left| \frac{e^{-\beta_k U(x)}}{Z_k} - \frac{e^{-\beta_{k-1} U(x)}}{Z_{k-1}} \right| dx$$

$$+ C \int_{\{U > l\}} \frac{e^{-(\beta_k - \frac{1}{2}) U(x)}}{Z_k} + \frac{e^{-(\beta_k - \frac{1}{2}) U(x)}}{Z_{k-1}} - dx .$$

We treat the two terms in the right-hand-side apart. The first term is the total variation distance between $\pi_k$ and $\pi_{k-1}$. Since $\beta_{k-1} \leq \beta_k$ since $\beta$ is non-decreasing, $Z_{k-1} \geq Z_k$. Using Pinsker’s inequality and this result, we get

$$\left( \int_{\mathbb{R}^d} \left| \frac{e^{-\beta_k U(x)}}{Z_k} - \frac{e^{-\beta_{k-1} U(x)}}{Z_{k-1}} \right| dx \right)^2 \leq 2 \int_{\mathbb{R}^d} \ln \left( \frac{e^{-\beta_{k-1} U(x)} Z_k}{e^{-\beta_k U(x)} Z_{k-1}} \right) \pi_{k-1}(x) dx$$

$$\leq 2(\beta_k - \beta_{k-1}) \int_{\mathbb{R}^d} U(x) \frac{e^{-\beta_{k-1} U(x)}}{Z_{k-1}} dx$$

$$\leq 2(\beta_k - \beta_{k-1})(1 + C \sqrt{\beta_{k-1} e^{-\beta_{k-1} + 1}})$$

$$\leq C(\beta_k - \beta_{k-1}) ,$$

where we used for the two last inequalities that

$$\int_{\{U > l\}} U(x) e^{-U(x)} dx \leq 2 \int_{\mathbb{R}^d} e^{-U(x)/2} dx < \infty$$
and since $U(0) = 0$, $U(x) \leq \|\nabla^2 U\|_\infty \|x\|^2$, for any $x \in \mathbb{R}^d$ by A9,

$$Z_{k-1} \geq \int_{\mathbb{R}^d} e^{-\beta_{k-1} \|\nabla^2 U\|_\infty \|x\|^2} dx \geq C \beta_{k-1}^{-d/2} > 0 .$$

Similarly, for the second term of (72) we obtain

$$\int_{\{U > l\}} \frac{e^{-(\beta_k - \frac{1}{l})(U(x))}}{Z_k} dx \leq C \beta_k^{d/2} e^{-(\beta_k - \frac{1}{l})l} \int_{\mathbb{R}^d} e^{-U(x)/2} dx .$$

Using that for any $t \geq 1$, $t^{d/2} \exp(-l(t - 1)/2) \leq (d/l)^{d/2} \exp(-(d - l)/2)$ if $d \geq l$ and $t^{d/2} \exp(-l(t - 1)/2) \leq 1$ otherwise, there exists $A_{6,1}$ which does not depend on $l$ such that

$$\int_{\{U > l\}} \frac{e^{-(\beta_k - \frac{1}{l})(U(x))}}{Z_k} dx \leq A_{6,1} e^{-(\beta_k - 1)/l^2} .$$

Combining this bound and (73) in (72), we get that there exists $A_{1,1} \geq 0$ such that

$$\rho_{\epsilon_k}(\pi_{k-1}, \pi_k) \leq A_{1,1} \sqrt{\beta_k - \beta_{k-1}} + A_{6,1} e^{-(\beta_k - 1)/l^2} .$$

The second term of (71) is treated through a synchronous coupling similar to [14, Section 6]. Indeed, $\tilde{\pi}_{k-1}$ being invariant for $Q'_k$ defined in (67) and by (65),

$$\rho_{\epsilon_k}(\tilde{\pi}_{k-1} Q_k, \tilde{\pi}_{k-1} Q'_k) = \rho_{\epsilon_k}(\pi_{k-1} Q_k, \pi_{k-1} Q'_k)$$

where $(X_t, Y_t)_{t \geq 0}$ (resp. $(X'_t, Y'_t)_{t \geq 0}$) is a BPS process with a fixed temperature $\beta_{k-1}$ (resp. a annealed BPS process with cooling schedule $s \mapsto \beta(t - (n(t) - k + 1)s_1 + s)$) and $(X_0, Y_0) = (X'_0, Y'_0)$ is distributed according to $\tilde{\pi}_{k-1}$. Following [14, Section 6], we construct such processes in such a way $(X_t, Y_t) = (X'_t, Y'_t)$ up to time $T'_0$, where $T'_0$ is the first time $(X'_t, Y'_t)_{t \geq 0}$ bounces while $(X_t, Y_t)_{t \geq 0}$ does not, defined by

$$T'_0 = \inf \left\{ \tau \geq 0, E \leq \int_0^\tau (\beta_k - \beta_{t-(n(t)-n)s_1+s}) \langle Y'_s, \nabla U(X'_s) \rangle + ds \right\} ,$$

where $E$ is a standard exponential random variable independent of $Z_k$.

Consider the compact sets $K = \{(x, y) \in \mathbb{R}^d \times Y : U(x) \leq l\}$ and $\tilde{K} = \{(x, y) \in \mathbb{R}^d \times Y : \text{dist}((x, y), K) \leq M s_1\}$, where $\text{dist}((x, y), K)$ is the distance from $K$ and $M = \sup_{z \in Y} \|z\|$.

That way, if a BPS with refreshment law $\mu_{\epsilon_k}$ over $Y$ have an initial condition in $K$, then on the time interval $[0, s_1]$ it necessarily stays in $K$.

Consider $\varphi : \mathbb{R}^d \times Y \to \mathbb{R}$ with $\|\varphi\|_{\infty, V_1} \leq 1$ and the following decomposition

$$\mathbb{E}[\varphi(X_s, Y_s) - \varphi(X'_s, Y'_s)] = \mathbb{E}[1_K(X_0, Y_0)\{\varphi(X_s, Y_s) - \varphi(X'_s, Y'_s)\}]$$

$$+ \mathbb{E}[1_{\mathbb{R}^d \times Y \setminus K}(X_0, Y_0)\{\varphi(X_s, Y_s) - \varphi(X'_s, Y'_s)\}] .$$

We bound the two terms separately. First, using that if $(X_0, Y_0) \in K$, then for any $t \in [0, s_1], (X'_t, Y'_t) \in K$, we have

$$\mathbb{E}[1_K(X_0, Y_0)\{\varphi(X_s, Y_s) - \varphi(X'_s, Y'_s)\}]$$

$$= 2(1 + \epsilon_1)\|V_1\|_{\infty, \tilde{K}} \mathbb{P}\left( (X_0, Y_0) \in K, (X_s, Y_s) \neq (X'_s, Y'_s) \right)$$

$$= 2(1 + \epsilon_1)\|V_1\|_{\infty, \tilde{K}} \mathbb{P}\left( (X_0, Y_0) \in K, T'_b < s_1 \right)$$

$$\leq 2(1 + \epsilon_1)\|V_1\|_{\infty, \tilde{K}} \mathbb{P}\left( E < M \|\nabla U\|_{\infty, \tilde{K}} \int_0^{s_1} (\beta_k - \beta_{t-(n(t)-n)s_1+s}) ds \right)$$

$$\leq 2(1 + \epsilon_1)\|V_1\|_{\infty, \tilde{K}} M \|\nabla U\|_{\infty, \tilde{K}} s_1 (\beta_k - \beta_{k-1}) ,$$

where $E$ is a standard exponential random variable independent of $Z_k$.
where \( M = \sup_{z \in Y} \| z \| \). Note that \( \| \nabla U \|_{\infty, K} \) depends on \( K \), hence on \( l \). 

Next using Lemma 18 and the Markov property, we get
\[
\mathbb{E}[\mathbb{I}_{\mathbb{R}^d \times Y \setminus K}(X_0, Y_0)(\varphi(X_{s_1}, Y_{s_1}) - \varphi(X_{s_1}', Y_{s_1}'))] \\
\leq (1 + \epsilon_1) \mathbb{E}[\mathbb{I}_{\mathbb{R}^d \times Y \setminus K}(X_0, Y_0)(V_1(X_{s_1}) + V_1(X_{s_1}'))] \\
\leq 2(1 + \epsilon_1) \left( \mathbb{E}[\mathbb{I}_{\mathbb{R}^d \times Y \setminus K}(X_0, Y_0) V_1(X_0)] + A_2 \right) \\
\leq 2(1 + \epsilon_1) \left( C \int_{U \geq l} e^{U(x)/2} \tilde{\pi}_{k-1}(dx) + A_2 \right) \\
\leq 2(1 + \epsilon_1) \left( C e^{-(\beta_k - \beta_{k-1})/2} + A_2 \right).
\]

where we used for the penultimate inequality that \((X_0, Y_0)\) is distributed according to \( \tilde{\pi}_{k-1} \). Combining this result and (78) in (77) and (76), we get there exist \( A_{6,2} \geq 0 \) independent of \( l \) and \( A_{l,2} \geq 0 \) satisfying
\[
\rho_{\epsilon_2}(\tilde{\pi}_{k-1} Q_k, \tilde{\pi}_{k-1}) \leq A_{l,2}(\beta_k - \beta_{k-1}) + A_{6,2} e^{-(\beta_k - \beta_{k-1})/2}.
\]

The proof is concluded combining this result and (75) in (71). \( \square \)

**Lemma 23.** Assume \( A \ref{assumption} \). There exists \( \theta > 0 \) such that if \( A \ref{assumption} \) holds with \( D_1 \leq \theta^{-1} \), then there exists \( A_{7} \geq 0 \) satisfying for all \( t \geq t_0, k \leq n(t) \) and \((x, y) \in \mathbb{R}^d \times Y, u_k \leq A_l V_1(x, y)/k^n \) where \( u_k \) is given in (69) and \( q_1 = (1/2)(1 - \theta D_1) \).

**Proof.** Let \( l \geq 1, t \geq t_0 \) and \( k \in \{1, \ldots, n(t)\} \). In the proof, \( C \) stands for a constant which may change from line to line but does not depend on \( k, l \) and \( \beta \). Denoting \( d_0 = 0 \) and \( d_k = \kappa_k d_{k-1} + e_k \), (70) reads
\[
u_k - d_k \leq \kappa_k (u_{k-1} - d_{k-1})
\]
and yields

\[
u_k \leq u_0 \prod_{j=1}^{k} \kappa_j + \sum_{i=1}^{k} \left( e_i \prod_{j=i}^{k-1} \kappa_j \right)
\]
with the convention that \( \prod_{j=1}^{k-1} \kappa_j = 1 \). From Lemma 22 applied with \( l = 1/D_2 \), and bounding
\[
\beta_k - \beta_{k-1} \leq \frac{D_1 [s_1/s_0]}{t - (n(t) - k + 1)s_1}
\]
for \( k \) large enough, we get
\[
e_k \leq C/\sqrt{k}
\]
Let \( \theta = 2 A_5 s_1 \), so that, by definition of \( \kappa_k \) given in Lemma 21 the condition \( \beta \), and using \( 1 - s \leq e^{-s} \), we have
\[
\kappa_k \leq 1 - \left( \frac{1}{2} \wedge 1 - \gamma/4 \right) \exp(-\theta \beta_k/2) \leq \exp(-C n^{-\theta D_1/2})
\]
Hence, for \( i \in \{1, \ldots, k\} \),
\[
\prod_{j=1}^{k} \kappa_j \leq \exp \left( -C \sum_{j=i}^{n} j^{-\theta D_1/2} \right) \leq \exp \left( -(C/q)\{(n + 1)^q - i^q\} \right),
\]
with \( q = 1 - \theta D_1/2 \in (1/2, 1) \) by assumption. Thus, combining this result and (80) in (79), we get
\[
u_k \leq u_0 e^{-(C/q)(n^q - 1)} + C(k^{-1/2} + I(k))
\]
with

\[ I(k) = e^{-(C/q)k} \int_1^k \frac{1}{s} e^{(C/q)s\eta} \, ds \]

\[ = \frac{1}{C} \left( \frac{1}{k^{q-\frac{3}{2}}} - e^{-(C/q)(k^{q-1})} \right) + \frac{e^{-(C/q)k}q}{C} (q - 1/2) \int_1^k s^{q-\frac{3}{2}} e^{(C/q)s\eta} \, ds \]

\[ \leq \frac{1}{Ck^{q-\frac{3}{2}}} + \frac{e^{-(C/q)k}q}{C} \int_1^{k_0} s^{q-\frac{3}{2}} e^{(C/q)s\eta} \, ds + \frac{I(k)}{Ck^{1-q}} \]

for all \( k_0 \geq 1 \). In particular, for \( k_0 \geq (2/C)^{1/(1-q)} \), this means \( I(k) \leq Ck^{1/2-q} \).

Finally, from the first part Corollary 19, \( u_0 \leq CV_1(x,y) \).

\[ \square \]

**Proof of Theorem 17.** Let \( t > t_0, n = n(t), \eta > \eta' > 0 \). In the proof, \( C \) stands for a constant which may change from line to line but does not depend on \( n, \eta, \eta', t \) and \( \beta \). First,

\[ \mathbb{P}(U(X^\beta_t) > \eta + \min U) \leq \int_{\{U \geq \eta\}} \tilde{\pi}_k(dx,dy) + (1/2)\|P_{0,t}((x,y),\cdot) - \tilde{\pi}_k\|_{TV} \]

Similarly to (74),

\[ \int_{\{U \geq \eta\}} \tilde{\pi}_k(dx,dy) \leq Ce^{-\beta k \eta'} \leq Ct^{-Dz\eta'} \]

We conclude, with Lemma 23 and the first part of Corollary 19, by

\[ \|\nu P_{0,t} - \tilde{\pi}_k\|_{TV} \leq u_k \leq CV_1(x,y)/t^q \].

\[ \square \]

**Acknowledgements**

Alain Durmus acknowledges support from Chaire BayeScale ”P. Laffitte”. Pierre Monmarché acknowledges support from the French ANR project ANR-12-JS01-0006 - PIECE. Arnaud Guillin and Pierre Monmarché acknowledge support from the French ANR-17-CE40-0030 - EFI - Entropy, flows, inequalities.

**References**


Appendix A. Postponed proof

Proof of Proposition 3. Note that since for all \( x \in \mathbb{R}^d, \|x\| \geq 1 \),
\[
U_1(x) = \|x\|^{\alpha} U_1(x/\|x\|),
\]
that it is sufficient to show that there exists \( C_1, C_2 > 0 \) such that for all \( x \in \mathbb{R}^d, \|x\| \geq 1 \), such that
\[
C_1 \|x\|^{\alpha-1} \leq \|\nabla U_1(x)\| \leq C_2 \|x\|^{\alpha-1},
\]
\[
\|\nabla^2 U_1(x)\| \leq C_2 \|x\|^{\alpha-2}.
\]

\( (82) \) is just a consequence of [24, Lemma 4.5]. As for \( (83) \), we have by \( (81) \) for all \( x \in \mathbb{R}^d, \|x\| \geq 1 \),
\[
\nabla U_1(x) = \alpha \|x\|^{\alpha-2} x U_1(x/\|x\|) + \|x\|^\alpha \left\{ \text{Id} - xx^T/\|x\|^2 \right\} \nabla U_1(x/\|x\|)
\]
\[
\nabla U_2(x) = \alpha \left\{ \|x\|^{\alpha-2} + (\alpha - 2) \|x\|^{\alpha-4} x^T \right\} U_1(x/\|x\|)
\]
\[
+ \alpha \|x\|^{\alpha-2} \left\{ \text{Id} - xx^T/\|x\|^2 \right\} \nabla U_1(x/\|x\|) x^T + x \nabla U_1(x/\|x\|)^T(x) \left\{ \text{Id} - xx^T/\|x\|^2 \right\}
\]
\[
+ \|x\|^{\alpha-2} \left\{ \nabla U_1(x/\|x\|) x^T + x \nabla U_1(x/\|x\|)^T + 2 \nabla U_1(x/\|x\|)^T x x^T \right\}
\]
\[
+ \|x\|^\alpha \left\{ \text{Id} - xx^T/\|x\|^2 \right\} \nabla^2 U_1(x/\|x\|).
\]

Since the \( U_1 \) is assumed to be twice continuously differentiable, the proof is finished. \( \square \)

Appendix B. Quantitative contraction rates for Markov chains

In this section, we give for completeness a quantitative version of [22, Theorem 1.2] which is used in Section 4.2. Let \( Q \) be a Markov operator on a smooth finite dimension manifold \( M \) (in our applications \( Q = P_{t_0} \) for some \( t_0 > 0 \), with \( M = \mathbb{R}^d \times Y \)) and \( V : M \to [1, +\infty) \) (which can be thought as the one given by (21)).

For any measurable function \( \varphi : M \to \mathbb{R} \) and \( \zeta \geq 0 \), we set
\[
\|\varphi\|_{\zeta,V} = \sup_{x \in M} \left\{ \frac{|\varphi(x)|}{1 + \zeta V(x)} \right\},
\]
and consider the weighted $V$-norm on $\mathcal{P}_{V}(M) = \{ \mu \in \mathcal{P}(M) : \mu(V) < \infty \}$, defined for $\mu_1, \mu_2 \in \mathcal{P}_{V}(M)$ by

$$\rho_{\zeta}(\mu_1, \mu_2) = \sup \{ \mu_1(\varphi) - \mu_2(\varphi) : \|\varphi\|_{\zeta, V} \leq 1 \}.$$  

**Theorem 24.** Suppose that there exist $\alpha, \gamma \in (0, 1)$, $C_1 > 0$ and $C_2 > 2C_1$ such that for all $x, y, z \in M$, $V(x) + V(y) \leq C_2$.

$$\|Q(x, \cdot) - Q(y, \cdot)\|_{TV} \leq 2(1 - \alpha), \quad QV(z) \leq \gamma V(z) + C_1(1 - \gamma).$$

Then there exists $\zeta > 0$ and $\kappa \in (0, 1)$ such that for all $\mu_1, \mu_2 \in \mathcal{P}_{V}(M)$,

$$\rho_{\zeta}(\mu_1 Q, \mu_2 Q) \leq \kappa \rho_{\zeta}(\mu_1, \mu_2),$$

where $\rho_{\zeta}$ is defined by (84). More precisely, if $C_2 = 4C_1$, then this holds with

$$\zeta = \alpha((1 - \gamma)C_1)^{-1}, \quad \kappa = (1 - \alpha/2) \vee ((3 + \gamma)/4).$$

**Proof.** [22, Lemma 2.1] shows that

$$\rho_{\zeta}(\mu_1, \mu_2) = \sup \{ \mu_1(\varphi) - \mu_2(\varphi) : |\varphi|_{\zeta} \leq 1 \},$$

where $\rho_{\zeta}$ is defined by (65) and

$$|\varphi|_{\zeta} = \sup_{x \neq y} \left\{ \frac{|\varphi(x) - \varphi(y)|}{2 + \zeta V(x) + \zeta V(y)} \right\} = \inf_{c \in \mathbb{R}} \|\varphi + c\|_{\zeta, V}.$$

Let $\varphi$ be a measurable function such that $|\varphi|_{\zeta} = \|\varphi\|_{\zeta, V} = 1$. We aim to show that $|Q\varphi|_{\zeta} \leq \kappa$ or, in other words, that

$$|Q\varphi(x) - Q\varphi(y)| \leq \kappa(2 + \zeta V(x) + \zeta V(y))$$

for all $x, y \in M$.

First, consider the case where $V(x) + V(y) \geq C_2$. For $\zeta > 0$, set $\kappa_1 = \gamma + (1 - \gamma)\frac{1 + \zeta C_1}{1 + \zeta C_2}$. Note that $\gamma < \kappa_1 < 1$, and

$$2(1 - \kappa_1) + (\gamma - \kappa_1)C_2 + 2\zeta C_1(1 - \gamma) \leq 0.$$  

Hence,

$$|Q\varphi(x) - Q\varphi(y)| \leq 2 + \zeta QV(x) + \zeta QV(y) \leq 2 + \zeta \gamma V(x) + \zeta \gamma V(y) + 2\zeta C_1$$

$$\leq \kappa_1(2 + \zeta V(x) + \zeta V(y)) + 2(1 - \kappa_1) + (\gamma - \kappa_1)C_1 + 2\zeta C_1$$

$$\leq \kappa_1(2 + \zeta V(x) + \zeta V(y)).$$

Second, consider the case where $V(x) + V(y) \leq C_1$. Let $(Z_x, Z_y)$ be an optimal coupling of $Q(x, \cdot)$ and $Q(y, \cdot)$. Then, writing $\kappa_2 = (1 - \alpha + \zeta C_1(1 - \gamma)/2) \vee \gamma$ (which is smaller than 1 for $\zeta$ small enough),

$$|Q\varphi(x) - Q\varphi(y)| \leq \mathbb{P}(Z_x \neq Z_y)\mathbb{E}[|\varphi(Z_x) - \varphi(Z_y)| | Z_x \neq Z_y]$$

$$\leq \mathbb{P}(Z_x \neq Z_y)(2 + \zeta \mathbb{E}[V(Z_x) + V(Z_y)])$$

$$\leq 2(1 - \alpha) + \zeta \gamma (V(x) + V(y)) + \zeta C_1(1 - \gamma)$$

$$\leq \kappa_2(2 + \zeta V(x) + \zeta V(y)),$$

which concludes the general proof.

For $C_2 = 4C_1$, we chose $\zeta = \frac{\alpha}{(1 - \gamma)C_1}$, so that $\kappa_2 = 1 - \alpha/2$ and

$$\kappa_1 = \gamma + (1 - \gamma) \left(1 - \frac{\alpha}{1 - \gamma + 2\alpha}\right) = 1 - \frac{\alpha(1 - \gamma)}{1 - \gamma + 2\alpha}.$$

Using that, for $a, b > 0$,

$$\frac{ab}{a + b} = \frac{a \wedge b}{1 + \frac{a \wedge b}{a + b}} > \frac{a \wedge b}{2},$$
we get
\[ \kappa_1 \leq 1 - \frac{(2\alpha) \wedge (1 - \gamma)}{4} = (1 - \alpha/2) \lor ((3 + \gamma)/4). \]

Remark that, under the same assumptions that Theorem 24 but with \( \alpha = 0 \), the same proof yields, for all \( \zeta > 0 \) and all \( \mu_1, \mu_2 \in \mathcal{P}_V(M) \),
\[ \rho_\zeta(\mu_1 Q, \mu_2 Q) \leq (1 + \zeta C_1) \rho_\zeta(\mu_1, \mu_2). \]