The Gradient Discretisation Method

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Plan

1. **Presentation of the gradient discretisation method: linear stationary diffusion**
   - From FEM to GDM
   - Measures of gradient scheme accuracy, error estimate
2. **Gradient schemes for non-linear models**
   - Semi-linear equation
   - Quasi-linear equations (and time-stepping)
3. **Do gradient discretisations exist?**
   - A few examples
   - Proving coercivity, limit-conformity, compactness: polytopal toolboxes
   - Proving consistency: local linearly exact GD
4. **About time-dependent problems**
5. **New results obtained through gradient schemes**

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Model problem

$$\begin{cases} -\mathrm{div}(\Lambda \nabla \overline{u}) = f & \text{in } \Omega, \\ \overline{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

- Ω open bounded in $\mathbb{R}^d$,
- $\Lambda : \Omega \to M_d(\mathbb{R})$ bounded uniformly coercive,
- $f \in L^2(\Omega)$.

Weak formulation

Find $\overline{u} \in H_0^1(\Omega)$ s.t., for all $\overline{v} \in H_0^1(\Omega)$, $\displaystyle\int_\Omega \Lambda \nabla \overline{u} \cdot \nabla \overline{v} = \int_\Omega f \overline{v}$.

- Existence and uniqueness of the weak solution *(Riesz representation theorem, or Lax-Milgram)*.

Conforming Finite Element Method (e.g., $\mathbb{P}_1$)

Find $\overline{u} \in H_0^1(\Omega)$ s.t., for all $\overline{v} \in H_0^1(\Omega)$, $\quad \int_\Omega \Lambda \nabla \overline{u} \cdot \nabla \overline{v} = \int_\Omega f \overline{v}.$

Computational issue: $H_0^1(\Omega)$ is an infinite-dimensional space.
$\rightsquigarrow$ Cannot be understood/manipulated by computer.

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Find $\overline{u} \in H^1_0(\Omega)$ s.t., for all $\overline{v} \in H^1_0(\Omega)$, $\quad \int_\Omega \Lambda \nabla \overline{u} \cdot \nabla \overline{v} = \int_\Omega f \overline{v}$.

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Easy solution: replace $H^1_0(\Omega)$ be a finite-dimensional subspace $E = V_h$.

Find $u_h \in V_h$ s.t., for all $v_h \in V_h$, $\quad \int_\Omega \Lambda \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h$.

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- If $u_h = \sum_i U_i\phi_i$ on a basis $(\phi_i)_i$ of V_h, this leads to the square system

$$AU = B$$

where $A_{ij} = \displaystyle\int_\Omega \Lambda\nabla\phi_i\cdot\phi_j$ and $B_i = \displaystyle\int_\Omega f\phi_i$.

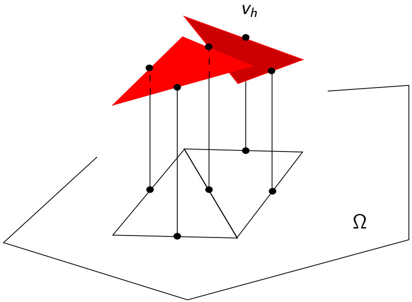
Non-conforming FEM (e.g., non-conforming $\mathbb{P}_1$/Crouzet-Raviart)

Non-conforming approximation: V_h is a space of functions, but not a subspace of $H_0^1(\Omega)$.

$\rightsquigarrow$ Need to define ∇v_h if $v_h \in V_h$?

Example: non-conforming $\mathbb{P}_1$:

$$V_h = \{v_h : \Omega \to \mathbb{R} \; : \; v_h \text{ piecewise linear, continuous at edge midpoints, zero at boundary edge midpoints}\}$$



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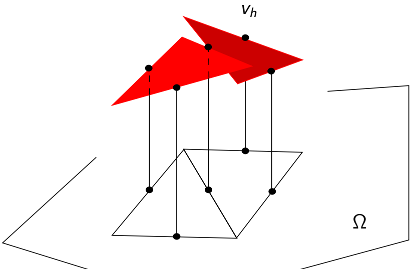
Example: non-conforming $\mathbb{P}_1$:

- ∇u_h replaced with broken gradient $\nabla_h u_h$, computed in each cell.

Find $u_h \in V_h$ s.t., for all $v_h \in V_h$, $\quad \int_\Omega \Lambda \nabla_h u_h \cdot \nabla_h v_h = \int_\Omega f v_h.$



Mass-lumping

Consider diffusion-reaction (from, e.g., Euler time-stepping):

$$-\mathrm{div}(\Lambda\nabla\overline{u}) + \overline{u} = f.$$

Issues:

- Requires computation of $M_{ij} = \int_\Omega \phi_i\phi_j$: non-diagonal mass-matrix (costly for explicit time-stepping).
- Also problematic for non-linear models $-\mathrm{div}(\Lambda\nabla\overline{u}) + \beta(\overline{u}) = f$ (e.g. Richards' equation).

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Standard mass-lumping: replace M with diagonal matrix by summing over columns. Why is that justified?

Better vision: reconstruct piecewise constant functions $\Pi_h u_h$ and $\Pi_h v_h$ from u_h and v_h.

Find $u_h \in V_h$ s.t., for all $v_h \in V_h$,

$$\int_\Omega \Lambda\nabla_h u_h \cdot \nabla_h v_h + \int_\Omega \Pi_h u_h \Pi_h v_h = \int_\Omega f \Pi_h v_h.$$

The gradient discretisation method (GDM) in a nutshell

In the weak formulation of the PDE, replace continuous space and operators (gradient, function) by discrete space and reconstructed operators.

- Set of discrete space and reconstructed operators: *gradient discretisation* (GD).
- A large number of possible gradient discretisations.

Gradient discretisation and gradient scheme

Definition (GD)

A gradient discretisation is $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ with

- $X_{\mathcal{D},0}$ finite dimensional space (of degrees of freedom), taking into account the Dirichlet BC,
- $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)$ reconstruction of function,
- $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)^d$ reconstructed gradient,

such that $v \to \|\nabla_{\mathcal{D}} v\|_{L^2(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$.

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Definition (GS)

If $\mathcal{D}$ is a GD, the corresponding gradient scheme for $-\mathrm{div}(\Lambda\nabla\overline{u}) = f$ with homogeneous Dirichlet BC is

$$\text{Find } u_{\mathcal{D}} \in X_{\mathcal{D},0} \text{ s.t., for all } v_{\mathcal{D}} \in X_{\mathcal{D},0},$$

$$\int_\Omega \Lambda\nabla_{\mathcal{D}} u_{\mathcal{D}} \cdot \nabla_{\mathcal{D}} v_{\mathcal{D}} = \int_\Omega f \Pi_{\mathcal{D}} v_{\mathcal{D}}.$$

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3 measures of accuracy

Measure of coercivity (discrete Poincaré's constant)

$$C_{\mathcal{D}} = \max_{v_{\mathcal{D}} \in X_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2}}{\|\nabla_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2}}.$$

Measure of consistency ("interpolation error" in FEM vocabulary)

$$S_{\mathcal{D}}(\varphi) = \min_{v_{\mathcal{D}} \in X_{\mathcal{D},0}} \left(\|\Pi_{\mathcal{D}} v_{\mathcal{D}} - \varphi\|_{L^2} + \|\nabla_{\mathcal{D}} v_{\mathcal{D}} - \nabla\varphi\|_{L^2} \right).$$

Measure of limit-conformity

$$W_{\mathcal{D}}(\psi) = \max_{v_{\mathcal{D}} \in X_{\mathcal{D},0} \setminus \{0\}} \frac{1}{\|\nabla_{\mathcal{D}} v_{\mathcal{D}}\|_{L^2}} \left| \int_{\Omega} \nabla_{\mathcal{D}} v_{\mathcal{D}} \cdot \psi + \Pi_{\mathcal{D}} v_{\mathcal{D}} \mathrm{div} \psi \right|.$$

Error estimate

$$\|\Pi_{\mathcal{D}} u_{\mathcal{D}} - \overline{u}\|_{L^2} + \|\nabla_{\mathcal{D}} u_{\mathcal{D}} - \nabla \overline{u}\|_{L^2} \\ \leq C(1 + C_{\mathcal{D}}) \left[S_{\mathcal{D}}(\overline{u}) + W_{\mathcal{D}}(\Lambda \nabla \overline{u}) \right].$$

Error estimate

$$\|\Pi_{\mathcal{D}} u_{\mathcal{D}} - \overline{u}\|_{L^2} + \|\nabla_{\mathcal{D}} u_{\mathcal{D}} - \nabla\overline{u}\|_{L^2} \\ \le C(1 + C_{\mathcal{D}})\left[S_{\mathcal{D}}(\overline{u}) + W_{\mathcal{D}}(\Lambda\nabla\overline{u})\right].$$

Convergence: if a sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretisations is

(P1) *Coercive*: $(C_{\mathcal{D}_m})_{m\in\mathbb{N}}$ bounded,

(P2) *Consistent*: $S_{\mathcal{D}_m}(\varphi) \to 0$ for all $\varphi \in H_0^1(\Omega)$,

(P3) *Limit-conforming*: $W_{\mathcal{D}_m}(\psi) \to 0$ for all $\psi \in H_{\mathrm{div}}(\Omega)$,

then $\Pi_{\mathcal{D}_m} u_m \to \overline{u}$ and $\nabla_{\mathcal{D}_m} u_m \to \nabla\overline{u}$ in L^2.

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No more work to do to write a GS

Semi-linear model:

$$-\mathrm{div}(\Lambda(\overline{u})\nabla\overline{u}) = f \quad \text{with homogeneous Dirichlet BC.}$$

No more work to do to write a GS

Semi-linear model: Weak form:

Find $\overline{u} \in H_0^1(\Omega)$ such that, for all $\overline{v} \in H_0^1(\Omega)$,

$$\int_\Omega \Lambda(\overline{u})\nabla\overline{u}\cdot\nabla\overline{v} = \int_\Omega f\overline{v}.$$

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$$\text{Find } \overline{u} \in H^1_0(\Omega) \text{ such that, for all } \overline{v} \in H^1_0(\Omega)\,,$$
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Gradient scheme: if $\mathcal{D}$ is a GD,

$$\text{Find } u_\mathcal{D} \in X_{\mathcal{D},0} \text{ s.t., for all } v_\mathcal{D} \in X_{\mathcal{D},0},$$
$$\int_\Omega \Lambda(\Pi_\mathcal{D} u_\mathcal{D})\nabla_\mathcal{D} u_\mathcal{D}\cdot\nabla_\mathcal{D} v_\mathcal{D} = \int_\Omega f\,\Pi_\mathcal{D} v_\mathcal{D}.$$

Convergence result

- To deal with low-order non-linearities, $(\mathcal{D}_m)_{m\in\mathbb{N}}$ must be

  (P4) **Compact**: (*discrete Rellich theorem*) for all $v_m \in X_{\mathcal{D}_m,0}$ such that $(\|\nabla_{\mathcal{D}_m} v_m\|_{L^2})_{m\in\mathbb{N}}$ is bounded, the sequence $(\Pi_{\mathcal{D}_m} v_m)_{m\in\mathbb{N}}$ is relatively compact in L^2.

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Theorem (Convergence of the GDM for semi-linear equations)

If $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is coercive, consistent, limit-conforming and compact, then, up to a subsequence,

$$\Pi_{\mathcal{D}_m} u_m \to \overline{u} \quad \textit{and} \quad \nabla_{\mathcal{D}_m} u_m \to \nabla\overline{u} \textit{ strongly in } L^2(\Omega).$$

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Model

With $\beta(s)s \geq 0$,

$$-\mathrm{div}(\Lambda\nabla\overline{u}) + \beta(\overline{u}) = f.$$

- Weak form:

$$\overline{u} \in H_0^1(\Omega) \text{ s.t. } \forall v \in H_0^1(\Omega)\,,\ \int_\Omega \Lambda\nabla\overline{u}\cdot\nabla\overline{v} + \int_\Omega \beta(\overline{u})\overline{v} = \int_\Omega f\overline{v}.$$

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Application of GDM:

- Option 1:

$$u \in X_{\mathcal{D},0} \text{ s.t. } \forall v \in X_{\mathcal{D},0}\,,$$
$$\int_\Omega \Lambda \nabla_{\mathcal{D}} u \cdot \nabla_{\mathcal{D}} v + \int_\Omega \beta(\Pi_{\mathcal{D}} u)\Pi_{\mathcal{D}} v = \int_\Omega f\,\Pi_{\mathcal{D}} v.$$

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- Option 2:

$$u \in X_{\mathcal{D},0} \text{ s.t. } \forall v \in X_{\mathcal{D},0}\,,$$
$$\int_\Omega \Lambda\nabla_{\mathcal{D}}u\cdot\nabla_{\mathcal{D}}v + \int_\Omega \Pi_{\mathcal{D}}\beta(u)\Pi_{\mathcal{D}}v = \int_\Omega f\,\Pi_{\mathcal{D}}v$$

with $\beta(u_{\mathcal{D}}) \in X_{\mathcal{D},0}$ constructed DOF by DOF.

Stability vs. Computability

Stability: with Option 1, since $\beta(s)s \geq 0$, with $v_{\mathcal{D}} = u_{\mathcal{D}}$,

$$\int_\Omega \beta(\Pi_{\mathcal{D}} u_{\mathcal{D}}) \Pi_{\mathcal{D}} u_{\mathcal{D}} \geq 0.$$

- Problem: $\beta(\Pi_{\mathcal{D}} u)$ non-linear function of $\Pi_{\mathcal{D}} u$, no exact quadrature.

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Computability: with Option 2

$$\int_\Omega \Pi_{\mathcal{D}} \beta(u_{\mathcal{D}}) \Pi_{\mathcal{D}} v_{\mathcal{D}}$$

is fully computable since $\Pi_{\mathcal{D}}$ usually reconstructs piecewise polynomal functions.

- Problem: $\Pi_{\mathcal{D}} \beta(u_{\mathcal{D}}) \Pi_{\mathcal{D}} u_{\mathcal{D}} \geq 0$? Stability?

Fifth and last property

To have $\Pi_{\mathcal{D}}\beta(v_{\mathcal{D}}) = \beta(\Pi_{\mathcal{D}} v_{\mathcal{D}})$:

(P5) *Piecewise constant reconstruction*: there is a basis $(e_i)_{i\in I}$ of $X_{\mathcal{D},0}$ and a partition $(\omega_i)_i$ of Ω such that, if $v = \sum_i v_i e_i \in X_{\mathcal{D},0}$,

$$\forall i \in I\,, \quad (\Pi_{\mathcal{D}} v_{\mathcal{D}})_{|\omega_i} = v_i.$$

5 properties to rule them all, and in the light of conciseness bind them

- (P1) *Coercivity*
- (P2) *Consistency*
- (P3) *Limit-conformity*
- (P4) *Compactness*
- (P5) *Piecewise constant reconstruction*

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Conforming Galerkin (incl. $\mathbb{P}_k$ FE)

V finite dimensional subspace of $H^1_0(\Omega)$, $(\phi_i)_{i\in I}$ basis of V.

- $X_{\mathcal{D},0} = \{v_\mathcal{D} = (v_i)_{i\in I}\}$,
- $\Pi_\mathcal{D} v_\mathcal{D} = \sum_{i\in I} v_i\phi_i$,
- $\nabla_\mathcal{D} v_\mathcal{D} = \nabla(\Pi_\mathcal{D} v_\mathcal{D}) = \sum_{i\in I} v_i\nabla\phi_i$.

▶ Remark: $C_\mathcal{D} \le C_P$ (Poincaré constant in H^1_0) and $W_\mathcal{D} \equiv 0$ (conforming method).

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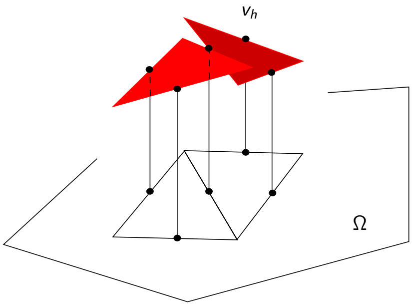
▶ Remark: $C_\mathcal{D} \leq C_P$ (Poincaré constant in H_0^1) and $W_\mathcal{D} \equiv 0$ (conforming method).

▶ Mass-lumped method: change $\Pi_\mathcal{D}$.

Non-conforming $\mathbb{P}_1$

$\mathcal{T}_h$ triangulation, $(\phi_\sigma)_{\sigma\in\mathcal{E}_{\rm int}}$ non-conforming $\mathbb{P}_1$ basis (one element per interior edge).

- $X_{\mathcal{D},0} = \{v_\mathcal{D} = (v_\sigma)_{\sigma\in\mathcal{E}_{\rm int}}\}$,
- $\Pi_\mathcal{D} v_\mathcal{D} = \sum_{\sigma\in\mathcal{E}_{\rm int}} v_\sigma\phi_\sigma$,
- $\nabla_\mathcal{D} v_\mathcal{D} = \nabla_h(\Pi_\mathcal{D} v_\mathcal{D})$ broken gradient.



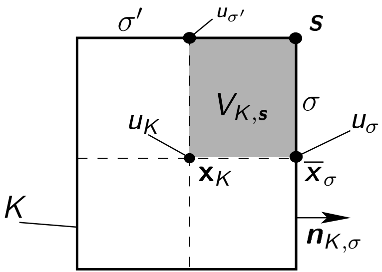
- Mass-lumped non-conforming $\mathbb{P}_1$: change $\Pi_\mathcal{D}$.

Finite volume/finite difference scheme ($\Lambda = \mathrm{Id}$)

$\mathfrak{T}$ rectangular mesh.

- $X_{\mathcal{D},0} = \{v = ((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}_{\rm int}}\}$, cell and interior edge DOFs.
- $\Pi_{\mathcal{D}} v = v_K$ in $K \in \mathcal{M}$ (piecewise constant),
- On $V_{K,\boldsymbol{s}}$ as in the figure,

$$(\nabla_{\mathcal{D}} v)_{|V_{K,\boldsymbol{s}}} = \frac{v_\sigma - v_K}{d(\overline{\mathbf{x}}_\sigma, \overline{\mathbf{x}}_K)} \boldsymbol{n}_{K,\sigma} + \frac{v_{\sigma'} - v_K}{d(\overline{\mathbf{x}}_{\sigma'}, \overline{\mathbf{x}}_K)} \boldsymbol{n}_{K,\sigma'}.$$



Also...

- Mixed $\mathbb{RT}_k$ finite elements,
- Multi-point flux approximation-O (MPFA-O) finite volumes on cartesian or simplicial meshes,
- Discrete duality finite volumes,
- Hybrid mimetic mixed (HMM) schemes, including mixed-hybrid mimetic finite difference (MFD) schemes,
- Nodal mimetic finite difference methods,
- Vertex approximate gradient (VAG) schemes...

Take-home message: what can GDM do for you?

- (S1) Develop/take your favourite method, say $\mathcal{FM}$, for linear diffusion stationary equation (E),
- (S2) Identify a gradient discretisation $\mathcal{D}_{\mathcal{FM}}$ such that the correspond gradient scheme for (E) is precisely $\mathcal{FM}$,
- (S3) Prove that $\mathcal{D}_{\mathcal{FM}}$ satisfies the *coercivity*, *consistency*, *limit-conformity*, *compactness* and has, perhaps, *piecewise constant reconstruction*,

$\rightsquigarrow$ through $\mathcal{D}_{\mathcal{FM}}$, $\mathcal{FM}$ can be applied to any model studied in the framework of gradient scheme, and yields a converging scheme (without additional work).

Exemple of models: stationary and transient Leray–Lions (p-Laplace), doubly degenerate parabolic, Stokes, Navier-Stokes, variational inequalities, diphasic flows in fractured networks...

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Polytopal toolbox

Set of scheme-independent results, for polytopal meshes, to prove (P1), (P3), (P4) (coercivity, limit-conformity, compactness).

Polytopal toolbox: discrete objects

- $\mathfrak{T} = (\mathcal{M}, \mathcal{E}, \mathcal{V}, \mathcal{P})$ polytopal mesh: cells, faces, vertices, cell "centers".
- Face and cell DOF gathered in $X_{\mathfrak{T},0}$:

$$X_{\mathfrak{T},0} = \{v = ((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}) \text{ with } v_\sigma = 0 \text{ if } \sigma \subset \partial\Omega\}.$$

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- $\Pi_{\mathfrak{T}} : X_{\mathfrak{T},0} \to L^\infty(\Omega)$ defined by

$$(\Pi_{\mathfrak{T}} v)_{|K} = v_K \text{ for all } K \in \mathcal{M},$$

- $\overline{\nabla}_{\mathfrak{T}} : X_{\mathfrak{T},0} \to (L^\infty(\Omega))^d$ defined by

$$(\overline{\nabla}_{\mathfrak{T}} v)_{|K} = \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}_K} |\sigma| v_\sigma \boldsymbol{n}_{K,\sigma} \text{ for all } K \in \mathcal{M}.$$

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- "H^1_0-norm" on $X_{\mathfrak{T},0}$:

$$\|v\|_{\mathfrak{T}} = \left(\sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} |\sigma| d_{K,\sigma} \left| \frac{v_\sigma - v_K}{d_{K,\sigma}} \right|^2 \right)^{1/2}.$$

Polytopal toolbox: discrete functional analysis

Under standard regularity assumptions on $\mathfrak{T}$:

- Poincaré inequality:

$$\|\Pi_{\mathfrak{T}} v\|_{L^2(\Omega)} \leq C \|v\|_{\mathfrak{T}}.$$

Polytopal toolbox: discrete functional analysis

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- Poincaré inequality:

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- Discrete (approximate) Stokes formula: for all $\varphi \in H_{\mathrm{div}}(\Omega)^d$,

$$\left|\int_\Omega \left(\overline{\nabla}_{\mathfrak{T}} v \cdot \varphi + \Pi_{\mathfrak{T}} v \,\mathrm{div}(\varphi)\right)\right| \leq C\|\nabla\varphi\|_{L^2(\Omega)^d}\|v\|_{\mathfrak{T}} h_{\mathcal{M}}.$$

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- Discrete Rellich theorem: if $(\|v_m\|_{\mathfrak{T}_m})_{m\in\mathbb{N}}$ is bounded then $(\Pi_{\mathfrak{T}_m} v_m)_{m\in\mathbb{N}}$ is relatively compact in $L^2(\Omega)$.

Polytopal toolbox: control of a GD

Definition (Control of a GD by a polytopal toolbox)

A control of $\mathcal{D}$ by $\mathfrak{T}$ is a linear mapping $\mathbf{\Phi} : X_{\mathcal{D},0} \to X_{\mathfrak{T},0}$.

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- Regularity factors:

$$\|\mathbf{\Phi}\|_{\mathcal{D},\mathfrak{T}} = \max_{v \in X_{\mathcal{D},0}\backslash\{0\}} \frac{\|\mathbf{\Phi}(v)\|_{\mathfrak{T}}}{\|\nabla_{\mathcal{D}} v\|_{L^2}},$$

$$\omega^{\Pi}(\mathcal{D},\mathfrak{T},\mathbf{\Phi}) = \max_{v \in X_{\mathcal{D},0}\backslash\{0\}} \frac{\|\Pi_{\mathcal{D}} v - \Pi_{\mathfrak{T}}\mathbf{\Phi}(v)\|_{L^2(\Omega)}}{\|\nabla_{\mathcal{D}} v\|_{L^2}},$$

$$\omega^{\nabla}(\mathcal{D},\mathfrak{T},\mathbf{\Phi}) = \max_{v \in X_{\mathcal{D},0}\backslash\{0\}} \frac{\left(\displaystyle\sum_{K\in\mathcal{M}} |K|^{-1} \left| \int_K \left[\nabla_{\mathcal{D}} v - \overline{\nabla}_{\mathfrak{T}}\mathbf{\Phi}(v)\right] \right|^2\right)^{\frac{1}{2}}}{\|\nabla_{\mathcal{D}} v\|_{L^2}}.$$

Polytopal toolbox: control of a GD

Estimates through control:

$$C_{\mathcal{D}} \leq \omega^{\Pi}(\mathcal{D}, \mathfrak{T}, \mathbf{\Phi}) + C\|\mathbf{\Phi}\|_{\mathcal{D},\mathfrak{T}}.$$

$$W_{\mathcal{D}}(\boldsymbol{\varphi}) \leq \|\boldsymbol{\varphi}\|_{H^1(\Omega)^d}\Big[Ch_{\mathcal{M}}(1 + \|\mathbf{\Phi}\|_{\mathcal{D},\mathfrak{T}}) + \omega^{\Pi}(\mathcal{D}, \mathfrak{T}, \mathbf{\Phi}) + \omega^{\nabla}(\mathcal{D}, \mathfrak{T}, \mathbf{\Phi})\Big].$$

Polytopal toolbox: control of a GD

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$$C_{\mathcal{D}} \le \omega^{\Pi}(\mathcal{D},\mathfrak{T},\mathbf{\Phi}) + C\|\mathbf{\Phi}\|_{\mathcal{D},\mathfrak{T}}.$$

$$W_{\mathcal{D}}(\varphi) \le \|\varphi\|_{H^1(\Omega)^d}\Big[Ch_{\mathcal{M}}(1+\|\mathbf{\Phi}\|_{\mathcal{D},\mathfrak{T}}) + \omega^{\Pi}(\mathcal{D},\mathfrak{T},\mathbf{\Phi}) + \omega^{\nabla}(\mathcal{D},\mathfrak{T},\mathbf{\Phi})\Big].$$

Consequence for a sequence $(\mathcal{D}_m)_m$ of GD: if $\mathbf{\Phi}_m$ control of $\mathcal{D}_m$ by $\mathfrak{T}_m$ such that

$$h_{\mathcal{M}_m} \to 0\,, \qquad \sup_{m\in\mathbb{N}} \|\mathbf{\Phi}_m\|_{\mathcal{D}_m,\mathfrak{T}_m} < +\infty\,,$$

$$\lim_{m\to\infty} \omega^{\Pi}(\mathcal{D}_m,\mathfrak{T}_m,\mathbf{\Phi}_m) = 0\,, \qquad \lim_{m\to\infty} \omega^{\nabla}(\mathcal{D}_m,\mathfrak{T}_m,\mathbf{\Phi}_m) = 0,$$

then $(\mathcal{D}_m)_m$ is coercive, limit-conforming and compact.

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Local linearly exact (LLE) gradient discretisations

Definition (LLE GD)

Let $\mathfrak{T}$ be a mesh. A GD $\mathcal{D}$ is local linearly exact if $X_{\mathcal{D},0} = \mathbb{R}^I$ and, for each cell K, there is $I_K \subset I$ s.t.

$$\forall \mathbf{x} \in K\,,\ \Pi_{\mathcal{D}} v(\mathbf{x}) = \sum_{i\in I_K} v_i \alpha_K^i(\mathbf{x}) \ \text{ and }\ \nabla_{\mathcal{D}} v(\mathbf{x}) = \sum_{i\in I_K} v_i \mathcal{G}_K^i(\mathbf{x})$$

where, for some $(\mathbf{x}_i)_{i\in I_K}$ close to K,

$$\forall \mathbf{x} \in K\,,\ \sum_{i\in I_K} \alpha_K^i(\mathbf{x}) = 1 \text{ and } \forall q_1 \in \mathbb{P}_1\,,\ \sum_{i\in I_K} q_1(\mathbf{x}_i)\mathcal{G}_K^i(\mathbf{x}) = \nabla q_1.$$

- A parameter $\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D})$ measures, for all $K \in \mathcal{M}$,
  - **(i)** how far $(\mathbf{x}_i)_{i\in I_K}$ are from K,
  - **(ii)** scaled $L^2(K)$ norms of α_K^i and $\mathcal{G}_K^i$.

Local linearly exact (LLE) gradient discretisations

Theorem (LLE GD are consistent (i.e. satisfy (P2)))

If $(\mathcal{D}_m)_m$ are LLE GD associated with meshes $(\mathfrak{T}_m)_m$ such that $h_{\mathcal{M}_m} \to 0$ and $(\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m))_m$ is bounded, then $(\mathcal{D}_m)_m$ is consistent.

Entire proof of (P1)–(P4) for non-conforming $\mathbb{P}_1$

Proof of the property ($\mathcal{P}$) for non-conforming $\mathbb{P}_1$ gradient discretisations. We drop the index m from time to time for sake of legibility, and all constants below do not depend on m or the considered cells/edges. Let us define a control of $\mathcal{D}$ by $\mathcal{T}$ in the sense of Definition 2.29, where $\mathcal{T}$ is the simplicial mesh associated to $\mathcal{D}$, with $\boldsymbol{x}_K = \overline{\boldsymbol{x}}_K = \frac{1}{d+1}\sum_{\sigma\in\mathcal{E}_K}\overline{\boldsymbol{x}}_\sigma$ the centres of gravity of the cells K. We define the linear (injective) mappings $\boldsymbol{\Phi} : X_{\mathcal{D}_m,0} \longrightarrow X_{\mathcal{T}_m,0}$ by $\boldsymbol{\Phi}(u)_K = \frac{1}{d+1}\sum_{\sigma\in\mathcal{E}_K} u_\sigma = \Pi_{\mathcal{D}}u(\boldsymbol{x}_K)$ and $\boldsymbol{\Phi}(u)_\sigma = u_\sigma = \Pi_{\mathcal{D}}u(\overline{\boldsymbol{x}}_\sigma)$.

Since $\boldsymbol{\Phi}(u)_K = \Pi_{\mathcal{D}}u(\boldsymbol{x}_K)$ and $\mathcal{G}_K u = \nabla(\Pi_{\mathcal{D}}u)$ in K, we get

$$\boldsymbol{\Phi}(u)_\sigma - \boldsymbol{\Phi}(u)_K = \mathcal{G}_K u \cdot (\overline{\boldsymbol{x}}_\sigma - \boldsymbol{x}_K). \tag{3.2}$$

Therefore, since $\frac{|\overline{\boldsymbol{x}}_\sigma - \boldsymbol{x}_K|}{d_{K,\sigma}} \le \frac{h_K}{d_{K,\sigma}} \le \theta_{\mathcal{T}}$,

$$\sum_{\sigma\in\mathcal{E}_K} |\sigma| d_{K,\sigma}\left|\frac{\boldsymbol{\Phi}(u)_\sigma - \boldsymbol{\Phi}(u)_K}{d_{K,\sigma}}\right|^p \le \theta_{\mathcal{T}}^p d|K|\,|\mathcal{G}_K u|^p.$$

This implies (2.34). We now observe that the affine function α_σ reaches its extremal values at the vertices of K. It is easy to see that $\alpha_\sigma(\mathsf{v}_\sigma) = 1-d$, where v_σ is the vertex opposite to the face σ, and that $\alpha_\sigma(\mathsf{v}_{\sigma'}) = 1$ for all $\sigma' \ne \sigma$. Therefore, for $\boldsymbol{x}\in K$,

$$|\Pi_{\mathcal{D}}u(\boldsymbol{x}) - \boldsymbol{\Phi}(u)_K| = \left|\sum_{\sigma\in\mathcal{E}_K}(\boldsymbol{\Phi}(u)_\sigma - \boldsymbol{\Phi}(u)_K)\alpha_\sigma(\boldsymbol{x})\right| \le (d+1)\max(1,d-1)\max_{\sigma\in\mathcal{E}_K}|\mathcal{G}_K u\cdot(\overline{\boldsymbol{x}}_\sigma - \boldsymbol{x}_K)|.$$

This inequality implies $\omega^{\Pi}(\mathcal{D},\mathcal{T},\boldsymbol{\Phi}) \le (d+1)\max(1,d-1)h_{\mathcal{M}}$ and therefore (2.35) holds. Finally, recalling that $\Pi_{\mathcal{D}}u$ is affine in each simplex K and that $\nabla_{\mathcal{T}}$ is exact on interpolants of affine functions (cf. Lemma 2.28), we see that $\nabla_{\mathcal{D}}u = \nabla_{\mathcal{T}}\boldsymbol{\Phi}(u)$ in Ω. Hence $\omega^{\nabla}(\mathcal{D},\mathcal{T},\boldsymbol{\Phi}) = 0$ and (2.36) holds. Proposition 2.31 therefore shows that $(\mathcal{D}_m)_{m\in\mathbf{N}}$ is coercive in the sense of Definition 2.2, limit-conforming in the sense of Definition 2.4, and compact in the sense of Definition 2.5.

Since non-conforming $\mathbb{P}_1$ gradient discretisations are LLE gradient discretisations, the consistency of $(\mathcal{D}_m)_{m\in\mathbf{N}}$ follows from Proposition 2.14 by noticing that $\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m)$ is controlled by $\theta_{\mathcal{T}_m}$. □

Entire proof of (P1)–(P4) for MPFA-O

Proof of the property ($\mathcal{P}$) *for MPFA-O gradient discretisations.* We drop the indices m for sake of legibility. We consider the polytopal mesh $\mathcal{T} = (\mathcal{M}, \mathcal{E}', \mathcal{P}, \mathcal{V}')$ where the sets $(\mathcal{M}, \mathcal{P})$ are those of the original polytopal mesh, $\mathcal{E}' = \{\sigma_{\mathsf{v}}\,;\, \sigma \in \mathcal{E},\ \mathsf{v} \in \mathcal{V}\}$, and $\mathcal{V}'$ is the set of all vertices of the elements of $\mathcal{E}'$. We define a control of $\mathcal{D}$ by $\mathcal{T}$ (in the sense of Definition 2.29) as the isomorphism $\mathbf{\Phi} : X_{\mathcal{D},0} \longrightarrow X_{\mathcal{T},0}$ given by $\mathbf{\Phi}(u)_K = u_K$ and $\mathbf{\Phi}(u)_{\sigma_{\mathsf{v}}} = u_{(\sigma,\mathsf{v})}$. We observe that

$$\int_K |\nabla_{\mathcal{D}} u(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x} \geq C_3 \sum_{\sigma \in \mathcal{E}_K} \sum_{\mathsf{v} \in \mathcal{V}_\sigma} |\sigma_{\mathsf{v}}| d_{K,\sigma} \left| \frac{u_{(\sigma,\mathsf{v})} - u_K}{d_{K,\sigma}} \right|^p,$$

with $C_3 = 1$ for parallelepipedic meshes, and $C_3 > 0$ depends on an upper bound of the regularity of the mesh for simplicial meshes. Therefore $\|\nabla_{\mathcal{D}} u\|^p_{L^p(\Omega)^d} \geq C_3 \|\mathbf{\Phi}(u)\|^p_{\mathcal{T},0,p}$ and (2.34) is proved. Since $\Pi_{\mathcal{D}} u = \Pi_{\mathcal{T}} \mathbf{\Phi}(u)$, we get $\omega^{\Pi}(\mathcal{D}, \mathcal{T}, \mathbf{\Phi}) = 0$, which proves (2.35). Finally, we have

$$\int_K \nabla_{\mathcal{D}} u(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \sum_{\sigma \in \mathcal{E}_K} \sum_{\mathsf{v} \in \mathcal{V}_\sigma} |\sigma_{\mathsf{v}}| (u_{\sigma,\mathsf{v}} - u_K) \mathbf{n}_{K,\sigma} = \sum_{\sigma' \in \mathcal{E}'_K} |\sigma'| (\mathbf{\Phi}(u)_{\sigma'} - \mathbf{\Phi}(u)_K) \mathbf{n}_{K,\sigma'} = |K|\ \nabla_{\mathcal{T}} \mathbf{\Phi}(u)_{|K}.$$

This shows that $\omega^{\nabla}(\mathcal{D}, \mathcal{T}, \mathbf{\Phi}) = 0$, which establishes (2.36). Proposition 2.31 therefore shows that $(\mathcal{D}_m)_{m \in \mathbf{N}}$ is coercive in the sense of Definition 2.2, limit-conforming in the sense of Definition 2.4, and compact in the sense of Definition 2.5.

It is proved in [40,41] that the definitions of the approximation points S give the LLE property in both the Cartesian and simplicial cases. Hence, the consistency of $(\mathcal{D}_m)_{m \in \mathbf{N}}$ follows from Proposition 2.14. □

Entire proof of (P1)–(P4) for HMM

Proof of the property ($\mathcal{P}$) *for HMM gradient discretisations.* Let $(\mathcal{D}_m)_{m\in\mathbf{N}}$ be HMM gradient discretisations built on polytopal meshes $(\mathcal{T}_m)_{m\in\mathbf{N}}$, and let us define a control of $\mathcal{D}_m$ by $\mathcal{T}_m$ in the sense of Definition 2.29. We drop the index m from time to time. Since $X_{\mathcal{D},0} = X_{\mathcal{T},0}$, we can take $\mathbf{\Phi} = \mathrm{Id}$. Estimate (2.34) is given by (3.14). Relation (2.35) follows immediately since $\omega^{\Pi}(\mathcal{D},\mathcal{T},\mathbf{\Phi}) = 0$, owing to $\Pi_{\mathcal{D}}u = \Pi_{\mathcal{T}}u = \Pi_{\mathcal{T}}\mathbf{\Phi}(u)$. Recalling that $|D_{K,\sigma}| = \frac{|\sigma|d_{K,\sigma}}{d}$ we have

$$\int_K \nabla_{\mathcal{D}}u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = |K|\nabla_K u + \frac{1}{\sqrt{d}}\sum_{\sigma\in\mathcal{E}_K}|\sigma|[\mathcal{L}_K R_K(Q_K(u))]_\sigma \mathbf{n}_{K,\sigma}. \tag{3.15}$$

The definition of R_K and the property $\sum_{\sigma\in\mathcal{E}_K}|\sigma|\mathbf{n}_{K,\sigma}(\overline{\boldsymbol{x}}_\sigma - \boldsymbol{x}_K)^T = |K|\mathrm{Id}$ (a consequence of Stokes' formula) show that for any $\eta\in\mathrm{Im}(R_K)$ we have $\sum_{\sigma\in\mathcal{E}_K}|\sigma|\eta_\sigma\mathbf{n}_{K,\sigma} = 0$. Hence, since $\mathrm{Im}(\mathcal{L}_K) = \mathrm{Im}(R_K)$, (3.15) gives

$$\int_K \nabla_{\mathcal{D}}u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = |K|\nabla_K u = |K|\nabla_{\mathcal{T}}\mathbf{\Phi}(u)_{|K},$$

which shows that $\omega^{\nabla}(\mathcal{D},\mathcal{T},\mathbf{\Phi}) = 0$, and thus that (2.36) holds. The coercivity, limit-conformity and compactness of $(\mathcal{D}_m)_{m\in\mathbf{N}}$ therefore follow from Proposition 2.31. Since HMM gradient discretisations are LLE gradient discretisations, the consistency of $(\mathcal{D}_m)_{m\in\mathbf{N}}$ readily follows from Proposition 2.14, after noticing that the regularity assumption on $(\mathcal{D}_m)_{m\in\mathbf{N}}$ gives a bound on $(\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m))_{m\in\mathbf{N}}$. □

Entire proof of (P1)–(P4) for nodal MFD

Proof of the property ($\mathcal{P}$) for the nMFD gradient discretisation. As in previous proofs, we drop indices m from time to time. We define a control $\mathbf{\Phi}$ of $\mathcal{D}$ by $\mathcal{T}$, in the sense of Definition 2.29, by

$$\forall K \in \mathcal{M},\ \mathbf{\Phi}(u)_K = u_K = \frac{1}{|K|}\sum_{\mathsf{v}\in\mathcal{V}_K}\omega_K^{\mathsf{v}} u_{\mathsf{v}} \quad \text{and} \quad \forall \sigma \in \mathcal{E},\ \mathbf{\Phi}(u)_\sigma = \frac{1}{|\sigma|}\sum_{\mathsf{v}\in\mathcal{V}_\sigma}\omega_\sigma^{\mathsf{v}} u_{\mathsf{v}}. \tag{3.28}$$

Let us prove (2.34). Since $\sum_{\mathsf{v}\in\mathcal{V}_\sigma}\omega_\sigma^{\mathsf{v}} = |\sigma|$ we have $\mathbf{\Phi}(u)_\sigma - \mathbf{\Phi}(u)_K = \frac{1}{|\sigma|}\sum_{\mathsf{v}\in\mathcal{V}_\sigma}\omega_\sigma^{\mathsf{v}}(u_{\mathsf{v}} - u_K)$. Therefore, using Jensen's inequality and the fact that $\frac{1}{d_{K,\sigma}} \le \frac{\theta_{\mathcal{T}}}{h_K}$ we find

$$\sum_{\sigma\in\mathcal{E}_K}|\sigma| d_{K,\sigma}\left|\frac{\mathbf{\Phi}(u)_\sigma - \mathbf{\Phi}(u)_K}{d_{K,\sigma}}\right|^p \le \sum_{\sigma\in\mathcal{E}_K} d_{K,\sigma}\sum_{\mathsf{v}\in\mathcal{V}_\sigma}\omega_\sigma^{\mathsf{v}}\left|\frac{u_{\mathsf{v}} - u_K}{d_{K,\sigma}}\right|^p \le \theta_{\mathcal{T}}^p\sum_{\sigma\in\mathcal{E}_K} d_{K,\sigma}\sum_{\mathsf{v}\in\mathcal{V}_\sigma}\omega_\sigma^{\mathsf{v}}\left|\frac{u_{\mathsf{v}} - u_K}{h_K}\right|$$

$$\le \theta_{\mathcal{T}}^p\sum_{\mathsf{v}\in\mathcal{V}_K}\left(\sum_{\sigma\in\mathcal{E}_{K,\mathsf{v}}} d_{K,\sigma}\omega_\sigma^{\mathsf{v}}\right)\left|\frac{u_{\mathsf{v}} - u_K}{h_K}\right|^p = \theta_{\mathcal{T}}^p d\sum_{\mathsf{v}\in\mathcal{V}_K}|V_{K,\mathsf{v}}|\left|\frac{u_{\mathsf{v}} - u_K}{h_K}\right|^p.$$

We conclude the proof of (2.34) thanks to (3.26). Since $\Pi_{\mathcal{D}} u = \Pi_{\mathcal{T}}\mathbf{\Phi}(u)$, we have $\omega^{\Pi}(\mathcal{D},\mathcal{T},\mathbf{\Phi}) = 0$ and (2.35) follows. For $K \in \mathcal{M}$ we have $\nabla_K u = (\nabla_{\mathcal{T}}\mathbf{\Phi}(u))_{|K}$. Therefore

$$\int_K \nabla_{\mathcal{D}} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = |K|(\nabla_{\mathcal{T}}\mathbf{\Phi}(u))_{|K} + \frac{1}{d}\sum_{\mathsf{v}\in\mathcal{V}_K}[\mathcal{L}_K R_K(Q_K(v))]_{\mathsf{v}}\sum_{\sigma\in\mathcal{E}_{K,\mathsf{v}}}\omega_\sigma^{\mathsf{v}}\mathbf{n}_{K,\sigma}. \tag{3.29}$$

Similarly as for the HMM method, for any $\eta \in \mathrm{Im}(R_K)$ we have $\sum_{\mathsf{v}\in\mathcal{V}_K}\eta_{\mathsf{v}}\sum_{\sigma\in\mathcal{E}_{K,\mathsf{v}}}\omega_\sigma^{\mathsf{v}}\mathbf{n}_{K,\sigma} = 0$. Hence, the last term in (3.29) vanishes and (2.36) holds since $\omega^{\nabla}(\mathcal{D},\mathcal{T},\mathbf{\Phi}) = 0$. Hence the hypotheses of Proposition 2.31 are verified, which shows that $(\mathcal{D}_m)_{m\in\mathbf{N}}$ is coercive, limit-conforming and compact.

By noticing that $\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m)$ remains bounded by regularity assumption on $(\mathcal{D}_m)_{m\in\mathbf{N}}$, the consistency of $(\mathcal{D}_m)_{m\in\mathbf{N}}$ is an immediate consequence of Proposition 2.14 since nMFD gradient discretisations are LLE gradient discretisations. □

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A catch-all model

$$\begin{cases} \partial_t \beta(\overline{u}) - \mathrm{div}(a(\mathbf{x}, \nabla\zeta(\overline{u}))) = f & \text{in } \Omega \times (0,T), \\ \zeta(\overline{u}) = 0 & \text{on } \partial\Omega \times (0,T), \\ \beta(\overline{u}) = \beta(u_{\rm ini}) & \text{at } t = 0. \end{cases}$$

- $a(\mathbf{x}, \xi) = |\xi|^{p-2}\xi$, $\beta(s) = \zeta(s) = s$:
  transient p-**Laplace** (also for generic Leray–Lions operator),
- $a(\mathbf{x}, \xi) = \Lambda(\mathbf{x})\xi$, $\beta(s) = s$, ζ non-decreasing:
  Stefan's model of melting material ($\overline{u}$ = enthalpy, $\zeta(\overline{u})$ = temperature),
- $a(\mathbf{x}, \xi) = \Lambda(\mathbf{x})\xi$, $\zeta(s) = s$, β non-decreasing:
  Richards' equation of underground water flow ($\overline{u}$ = pressure, $\beta(\overline{u})$ = water content).

Application of the GDM

Weak formulation: find $\overline{u}$ in the proper space s.t.

$$\int_0^T \langle \partial_t \beta(\overline{u}), v \rangle + \int_0^T \int_\Omega a(\mathbf{x}, \nabla \zeta(\overline{u})) \cdot \nabla v = \int_0^T \int_\Omega f v, \quad \forall v.$$

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Consider $\mathcal{D}$, time steps $0 = t_1 < t_2 < \cdots < t_N = T$, and interpolator $I_\mathcal{D} : L^2(\Omega) \to X_{\mathcal{D},0}$:

Find $u_\mathcal{D} = (u^n)_{n=0,\ldots,N} \in X_{\mathcal{D},0}^{N+1}$ s.t. $u^0 = I_\mathcal{D} u_{\mathrm{ini}}$ and,

for all $n = 0, \ldots, N-1$ and all $v_\mathcal{D} \in X_{\mathcal{D},0}$,

$$\int_\Omega \delta_\mathcal{D}^{n+\frac{1}{2}} \beta(u_\mathcal{D}) \Pi_\mathcal{D} v_\mathcal{D} + \int_\Omega a(\mathbf{x}, \nabla_\mathcal{D}[\zeta(u_\mathcal{D}^{n+1})]) \cdot \nabla_\mathcal{D} v_\mathcal{D}$$
$$= \frac{1}{\delta t} \int_{n\delta t}^{(n+1)\delta t} \int_\Omega f \Pi_\mathcal{D} v_\mathcal{D}$$

Convergence analysis tools

- Space–time Kolmogorov with (discrete) varying spaces,
- Discrete Aubin–Simon theorem with (discrete) varying spaces,
- Discontinuous weak Ascoli–Arzela.
- Discrete "compensated compactness" theorem (convergence of $\int f_n g_n$ from time-derivatives estimates on f_n and space-derivative estimates on g_n).

All these results already fully developed for GD, can be used off-the-shelf.

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A few of them

- Unified analysis framework for host of methods (no need to re-invent the wheel).
- Proper generic definition and treatment of mass-lumping.
- Generic treatment of barycentric elimination of DOFs.
- Development of generic discrete functional analysis results, for the GDM and more.
- Uniform-in-time strong $L^2(\Omega)$ convergence results for degenerate parabolic equations, without regularity assumptions.

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- Generic treatment of barycentric elimination of DOFs.
- Development of generic discrete functional analysis results, for the GDM and more.
- Uniform-in-time strong $L^2(\Omega)$ convergence results for degenerate parabolic equations, without regularity assumptions.
- **Super-convergence for TPFA finite volumes.**

Improved L^2 estimate for GS

$$-\operatorname{div}(A\nabla\overline{u}) = f \text{ in } \Omega, \qquad \overline{u} = 0 \text{ on } \partial\Omega$$

$$-\operatorname{div}(A\nabla\overline{w}) = \frac{\Pi_{\mathcal{D}}u - \overline{u}}{\|\Pi_{\mathcal{D}}u - \overline{u}\|_2} \text{ in } \Omega, \qquad \overline{w} = 0 \text{ on } \partial\Omega.$$

Theorem (Improved L^2 estimate for GS)

For all $P_{\mathcal{D}}\overline{u} \in X_{\mathcal{D},0}$,

$$\|\Pi_{\mathcal{D}}u - \overline{u}\|_2 \lesssim \left[h^{-1}I_h(\overline{u}, P_{\mathcal{D}}\overline{u})\right]^2 + [S_{\mathcal{D}}(\overline{u}) + W_{\mathcal{D}}(A\nabla\overline{u})]^2 + I_h(\overline{u}, P_{\mathcal{D}}\overline{u}) + |\widetilde{W}_{\mathcal{D}}(A\nabla\overline{u}, P_{\mathcal{D}}\overline{w})| + \text{ symmetric in } \overline{w}$$

where

$$I_h(\phi, P_{\mathcal{D}}\phi) = \|\Pi_{\mathcal{D}}(P_{\mathcal{D}}\phi) - \phi\|_2 + h\|\nabla_{\mathcal{D}}(P_{\mathcal{D}}\phi) - \nabla\phi\|_2,$$

$$\widetilde{W}(\psi, P_{\mathcal{D}}\phi) = \int_\Omega \nabla_{\mathcal{D}}(P_{\mathcal{D}}\phi)\cdot\psi + \Pi_{\mathcal{D}}(P_{\mathcal{D}}\phi)\operatorname{div}\psi.$$

Application to Hybrid Mimetic Mixed schemes

- HMM = family of schemes gathering Hybrid FV, hybrid Mimetic Finite Differences, and Mixed FV.
- HMM is a GDM, based on a polytopal mesh $\mathfrak{T}$ and such that

$$X_{\mathcal{D},0} = X_{\mathfrak{T},0}\,, \quad \Pi_{\mathcal{D}} v = \Pi_{\mathfrak{T}} v \text{ (piecewise constant)}\,,$$
$$\nabla_{\mathcal{D}} v = \overline{\nabla}_{\mathfrak{T}} v + \text{ stabilisation.}$$

Theorem (Super-convergence for HMM)

Assume optimal H^2 regularity for the PDE. If the cell “centers” $\mathcal{P}$ are, on average on local patches of cells, close to the centers of mass of the cells, then, for $u_{\mathcal{D}}$ solution to HMM,

$$\|\Pi_{\mathcal{D}} u_{\mathcal{D}} - \overline{u}\|_2 = \mathcal{O}(h^2 \|f\|_{H^1}).$$

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- Starts by a modified HMM scheme that *always* super-converges.

$$\forall \mathbf{x} \in K\,,\ \Pi^*_{\mathcal{D}} v(\mathbf{x}) = \Pi_{\mathcal{D}} v(\mathbf{x}) + \overline{\nabla}_{\mathcal{D}} v(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_K).$$

Super-convergence for TPFA

- On triangular meshes with $A(\mathbf{x}) = a(\mathbf{x})\mathrm{Id}$, TPFA is an HMM, with $\mathbf{x}_K$ = circumcenter of K.

Super-convergence for TPFA

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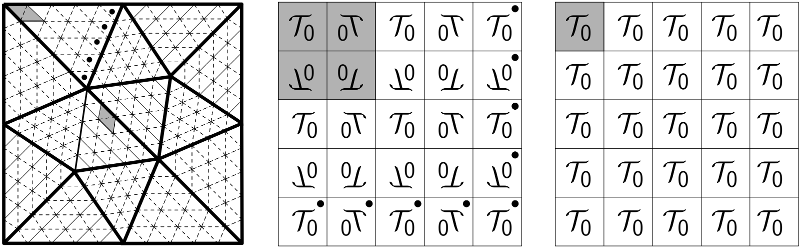
▶ For TPFA on triangular meshes as used in benchmarks, local compensation always occurs (up to a small portion).



Super-convergence for TPFA

Theorem (Super-convergence for TPFA on triangles)

In all classical triangular meshes used in benchmarking, with $\mathbf{x}_K$ *circumcenter of* K*, under optimal* H^2 *regularity,*

$$\|u_h - \overline{u}_{\mathcal{P}}\|_2 = \mathcal{O}(h^{2-\varepsilon}\|f\|_{H^1}),$$

where u_h *is the solution to the TPFA FV scheme and* $\overline{u}_{\mathcal{P}}$ *is the piecewise constant function equal to* $\overline{u}(\mathbf{x}_K)$ *on* K*.*

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Long-standing (20+ years) conjecture, only obtained by first abstracting from the specificities of the schemes.

Conclusion

- GDM = generic analysis framework for many numerical methods and many diffusion models (linear and non-linear).
- Easy proof that a given method fits into the GDM.
- Proof of convergence based on 3-5 properties.
- Discrete functional analysis results readily usable in the GDM.
- Led to novel results, thanks to a level of abstraction that liberates from the specificities of each scheme.

Main reference

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Thanks.