Observability, control and stabilization of water waves

Thomas Alazard

CNRS & École normale supérieure

Séminaire Laboratoire Jacques-Louis Lions

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EXACT CONTROLLABILITY, STABILIZATION AND PERTURBATIONS FOR DISTRIBUTED SYSTEMS *

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Controllability: consider

$$\partial_t^2 u - \Delta u = \chi(x) f(t,x) \quad ; \quad u(0,x) = 0, \quad \partial_t u(0,x) = 0.$$

Given $T > 0$ and two functions $u_T = u_T(x)$, $v_T = v_T(x)$, find f s.t.

$$u(T,x) = u_T(x), \quad \partial_t u(T,x) = v_T(x).$$

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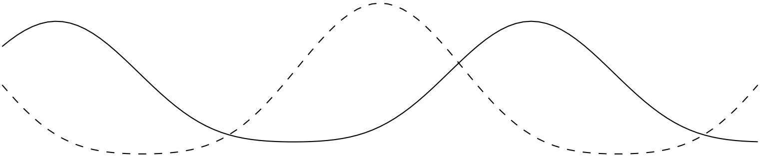
Stabilization: study of the damped wave PDE

$$\partial_t^2 u - \Delta u + \chi(x)\partial_t u = 0 \quad ; \quad u(0,x) = u_0, \quad \partial_t u(0,x) = v_0.$$

Prove that the energy converges to zero when t goes to $+\infty$.

Question: control and stabilization of water waves



Question: which water waves can be generated in a **rectangular tank**?

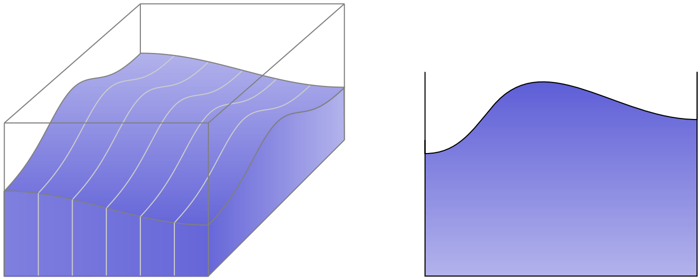


Figure: 3d and 2d waves in a rectangular tank

Main results

With **Baldi** and **Han-Kwan**: **controllability** of 2d gravity-capillary waves.

Observability and **Stabilization** of 2d or 3d water waves, with or without surface tension.

There are many results for equations describing water waves :

- Benjamin-Ono, KdV, Saint-Venant;

see works by Cerpa, Crépeau, Coron, Dubois, Glass, Guerrero, Laurent, Linares, Ortega, Petit, Rosier, Rouchon, Russell, Zhang....

Here we consider the dynamics of an incompressible, irrotational liquid flow

- moving under the force of gravitation and surface tension,
- in a time-dependent domain Ω with a free boundary.

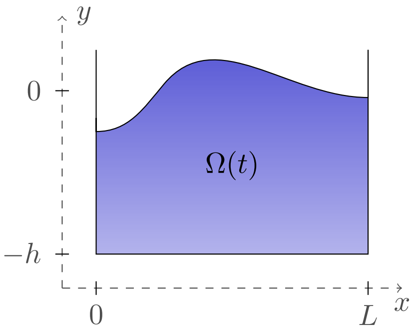
Problem determined by two unknowns :

- the free surface elevation η,
- the fluid velocity v.

The fluid domain Ω has a **free surface**. At time $t \geq 0$,

$$\Omega(t) = \{\, (x, y) \in [0, L] \times \mathbb{R} \;:\; -h < y < \eta(t, x) \,\},$$

where η is an unknown.



$$\Omega(t) = \{ (x,y) \in [0,L] \times \mathbb{R} : -h < y < \eta(t,x) \}.$$

$$\partial_t v + v \cdot \nabla v + \nabla(P + gy) = 0 \quad \text{in } \Omega$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega$$

$$v \cdot n = 0 \quad \text{on the bottom and walls}$$

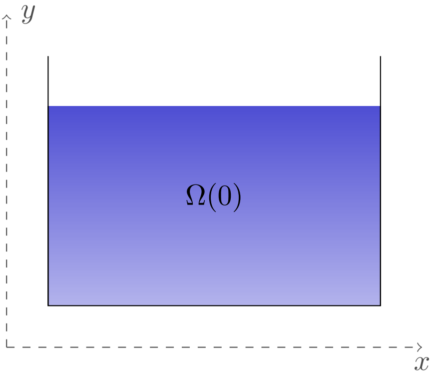
$$\partial_t \eta = \sqrt{1 + \eta_x^2}\, v \cdot n \quad \text{on the free surface}$$

$$P - P_{ext} = \kappa \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right) \quad \text{on the free surface}$$

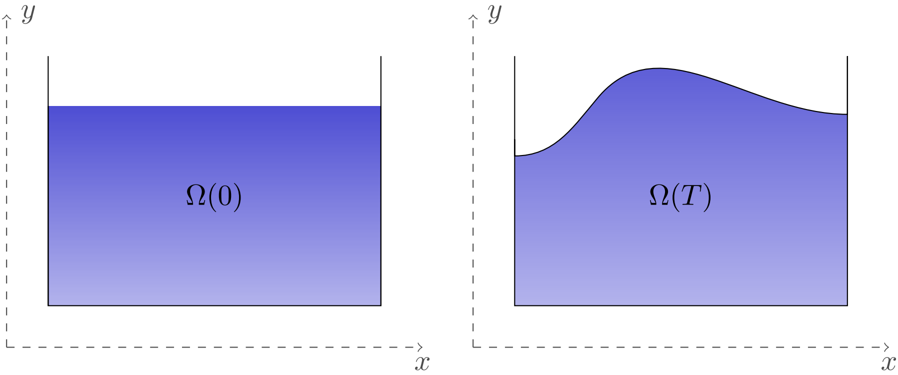
g gravity, P pressure, P_{ext} external pressure, κ surface tension.

$\kappa = 1$ (except at the end of the talk).

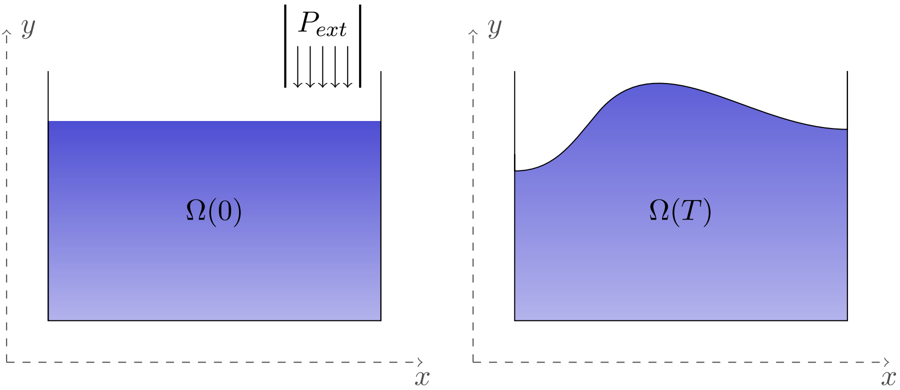
Moreover $\operatorname{curl} v = 0$ so that $v = \nabla \phi$.

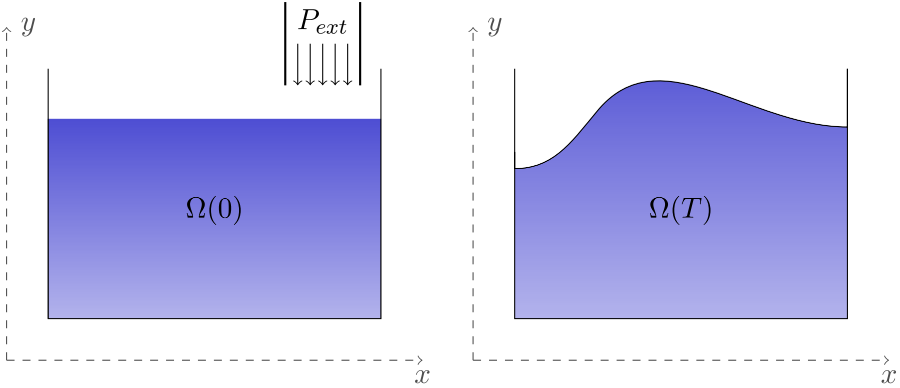


y
Ω(0)
x



y
x
$\Omega(0)$
y
x
$\Omega(T)$


P_{ext}
y
x
$\Omega(0)$
y
x
$\Omega(T)$



<u>Goal</u> : Given

- a time $T > 0$,
- a final state η_{final}, v_{final} in some space X,
- a domain $\omega = (a, b)$,

find $P_{ext}(t, x)$ supported in $[0, T] \times \omega$ such that the solution to (WW) with initial data $(\eta, v) = (0, 0)$ satisfies $(\eta, v)|_{t=T} = (\eta_{final}, v_{final})$.

Reduction to the case

$$\Omega(t) = \{ (x, y) \in \mathbb{R} \times \mathbb{R} \, : \, -h < y < \eta(t, x) \}.$$

Periodization

Justification by ABZ / Thibault de Poyferré.

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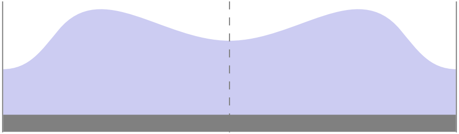

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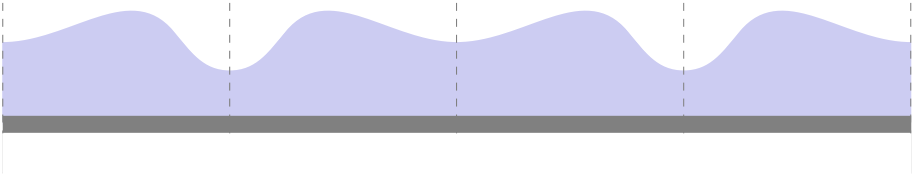

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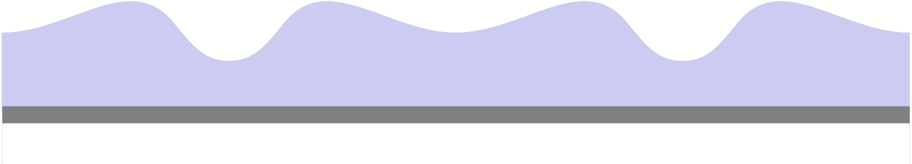

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Local controllability of 2D gravity-capillary water waves

Notations: $\psi(t,x) = \phi(t,x,\eta(t,x))$, $H^\mu(\mathbb{T})$ is the usual Sobolev space of order μ ($H^\mu_m(\mathbb{T})$ is the subspace of functions with mean value zero.)

Theorem (T.A., Pietro Baldi, Daniel Han-Kwan)

Let $T > 0$ and consider $\omega \subset \mathbb{T}$. There exist $s > 0$ (large) and $M_0 > 0$ (small) s.t. for any (η_{in}, ψ_{in}), $(\eta_{final}, \psi_{final})$ in $H_m^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$

$$\|\eta_{in}\|_{H^{s+\frac{1}{2}}} + \|\psi_{in}\|_{H^s} < M_0, \quad \|\eta_{final}\|_{H^{s+\frac{1}{2}}} + \|\psi_{final}\|_{H^s} < M_0,$$

there exists P_{ext} in $C^0([0,T]; H^s(\mathbb{T}))$ supported in $[0,T] \times \omega$, such that the Cauchy problem with data $(\eta|_{t=0}, \psi|_{t=0}) = (\eta_{in}, \psi_{in})$ has a unique solution

$$(\eta, \psi) \in C^0([0,T]; H_m^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T}))$$

satisfying

$$(\eta|_{t=T}, \psi|_{t=T}) = (\eta_{final}, \psi_{final}).$$

Linearized equation (neglect gravity):

$$u = \psi - i|D_x|^{\frac{1}{2}}\eta, \qquad |D_x|^{\frac{1}{2}} e^{ix\xi} = |\xi|^{\frac{1}{2}} e^{ix\xi},$$

satisfies the **dispersive** equation

$$\partial_t u + i\,|D_x|^{\frac{3}{2}}\,u = P_{ext}.$$

Similar **diagonalization** of the nonlinear equations, based on

- study in Eulerian coordinates (Zakharov, Craig-Sulem, Lannes)
- complete paralinearization of the equations (A-Métivier)
- Symmetrization (A-Burq-Zuily)
- normal forms (A-Delort; A-Baldi)

(**Oversymplifying**) One can rewrite the WW system as:

$$\frac{\partial u}{\partial t} + V(u)\partial_x u + i\,|D_x|^{\frac{3}{4}}\left(c(u)\,|D_x|^{\frac{3}{4}}\,u\right) = P_{ext}$$

where V, c are real-valued functions.

The linearized system at the origin has constant coefficients and can be controlled by means of Fourier analysis, Reid (1995), Lissy (2015) or multipliers Biccari (2015). But this is not enough since the problem is quasi-linear. We seek P_{ext} as the limit of solutions to approximate control problems with variable coefficients (similar scheme used by Coron for Saint-Venant).

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Fix $\underline{u} = \underline{u}(t, x)$ and consider

$$P = \partial_t + V\partial_x + i\,|D_x|^{\frac{3}{4}}\left(c\,|D_x|^{\frac{3}{4}}\,\cdot\,\right)$$

where $V = V(\underline{u})$ and $c = c(\underline{u})$ are real-valued and $c - 1$ is small enough.

Using a change of variables (preserving the L^2-norm in x)

$$(1 + \partial_x \kappa(t, x))^{\frac{1}{2}}\, h(t, x + \kappa(t, x))$$

we replace P by

$$Q = \partial_t + W\partial_x + i\,|D_x|^{\frac{3}{2}} + R \qquad R \text{ is of order zero.}$$

where one can further assume that $\int_{\mathbb{T}} W(t, x)\, dx = 0$.

- Nontrivial since the equation is nonlocal and cancellation of the term of order $1/2$.

To study $\partial_t + W\partial_x + i\,|D_x|^{\frac{3}{2}} + R'$, we seek an operator A such that

$$i\big[A, |D_x|^{\frac{3}{2}}\,\big] + W\partial_x A \quad \text{is a zero order operator}$$

We find an operator of the form

$$A = \operatorname{Op}\left(q(t,x,\xi)e^{i\beta(t,x)|\xi|^{\frac{1}{2}}}\right)$$

with

$$\beta = \beta_0(t) + \frac{2}{3}\partial_x^{-1}W.$$

Then

$$\big(\partial_t + W\partial_x + i\,|D_x|^{\frac{3}{2}}\,\big)A = A\big(\partial_t + i\,|D_x|^{\frac{3}{2}} + R''\big)$$

with R'' of order 0.

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with R'' of order 0.

Notice that $A \in \mathrm{Op}\,S^0_{\rho,\rho}$ with $\rho = 1/2$ (quasi-linear). For Benjamin-Ono one has a similar conjugation but with $A \in \mathrm{Op}\,S^0_{1,0}$ (semi-linear).

Ingham type inequality

Plancherel type inequality for pseudo-periodic function of time (the variable x is seen as a parameter).

For some given real-valued function $\beta \in C^3(\mathbb{R})$, set

$$\mu_n(t) = \operatorname{sign}(n) \left[|n|^{\frac{3}{2}} t + \beta(t) |n|^{\frac{1}{2}} \right].$$

For any $T \in (0,1]$ there are $C(T)$ and $\delta(T)$ such that, if

$$\left\| (\partial_t \beta, \partial_t^2 \beta, \partial_t^3 \beta) \right\|_{L^\infty} \leq \delta(T)$$

then, for any $(w_n) \in \ell^2(\mathbb{Z})$,

$$C(T) \sum_{n \in \mathbb{Z}} |w_n|^2 \leq \int_0^T \left| \sum_{n \in \mathbb{Z}} w_n e^{i\mu_n(t)} \right|^2 dt.$$

For $\beta = 0$: Ingham, Kahane, Ball-Slemrod, Haraux.

$\beta(t)|n|^{\frac{1}{2}}$ is sub-principal but not perturbative.

Corollary (Observability)

Consider $\omega = (a,b) \subset \mathbb{T}$ *and* $T > 0$. *Assume* v *solves*

$$\partial_t v + V\partial_x v + i\,|D_x|^{\frac{3}{4}}\left(c\,|D_x|^{\frac{3}{4}}\,v\right) = 0, \quad v(0) = v_0$$

with

$$\|V\|_{C^0([0,T];H^s)} + \|c-1\|_{C^0([0,T];H^s)} \leq \varepsilon_0.$$

Then

$$\int_0^T \int_{\omega} |v(t,x)|^2 \, dxdt \geq K \int_{\mathbb{T}} |v_0(x)|^2 \, dx.$$

To conclude : Hilbert Uniqueness Method (HUM), this is a duality argument (J.L. Lions) based on Riesz representation theorem. It deduces controllability of a linear operator L from the observability of its adjoint operator L^*.

Stabilization

After wave generation, wave absorption is the most important mechanism in a wave tank.

Many problems in water-wave theory require to study the behavior of waves propagating in an unbounded domain, like those encountered in the open sea. On the other hand, the numerical or experimental analysis of the water-wave equations requires to work in a bounded domain.

Two main approaches:
- truncation of the domain by introducing an **artificial boundary**;
- damping of outgoing waves in an **absorbing zone** surrounding the computational boundary. (stabilization)

Stabilization

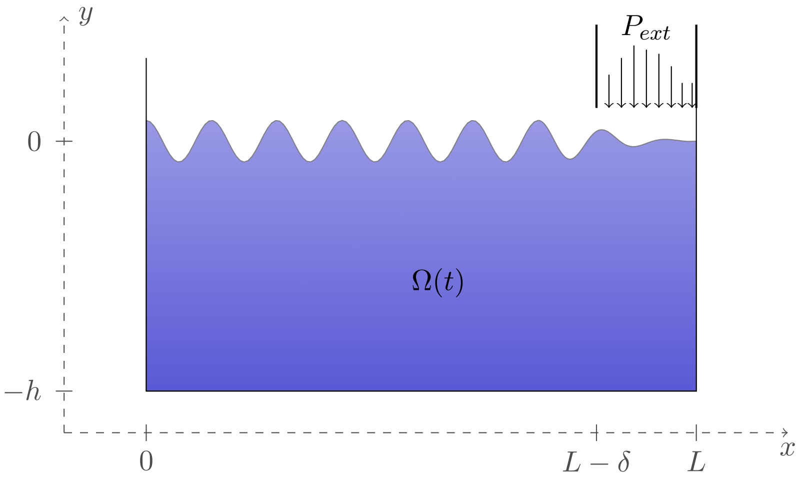

Figure: Waves generated near $x = 0$, propagating to the right, and absorbed in the neighborhood of $x = L$ by means of an external counteracting pressure produced by blowing above the free surface.

Stabilization

Denote by $\mathcal{H}(t)$ the energy of the fluid at time t:

$$\mathcal{H}(t) = \underbrace{\frac{g}{2}\int_0^L \eta(t,x)^2\,\mathrm{d}x + \kappa\int_0^L \sqrt{1+\eta_x^2}\,\mathrm{d}x}_{\text{potential energy}} + \underbrace{\frac{1}{2}\iint_{\Omega(t)} |\nabla_{x,y}\phi|^2\,\mathrm{d}y\,\mathrm{d}x}_{\text{kinetic energy}}.$$

Goal : find P_{ext} such that (i) one has $\operatorname{supp} P_{ext}(t,\cdot) \subset [L-\delta, L]$ and

(ii) $\mathcal{H}$ is decreasing and (iii) $\displaystyle\int_0^T \mathcal{H}(t)\,\mathrm{d}t \leq C\mathcal{H}(0)$.

Then

$$\underbrace{\mathcal{H}(T) \leq \frac{1}{T}\int_0^T \mathcal{H}(t)\,\mathrm{d}t \leq \frac{C}{T}\mathcal{H}(0)}_{\text{damping for } T>C} \quad \Rightarrow \quad \underbrace{\mathcal{H}(nT) \leq \left(\frac{C}{T}\right)^n \mathcal{H}(0)}_{\text{exponential decay}}.$$

Stabilization

Hamiltonian damping. Zakharov observed that the equations have the hamiltonian form

$$\frac{\partial \eta}{\partial t} = \frac{\delta \mathcal{H}}{\delta \psi}, \qquad \frac{\partial \psi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \eta} - P_{ext}.$$

Then

$$\frac{d\mathcal{H}}{dt} = \int \left[\frac{\delta \mathcal{H}}{\delta \eta} \frac{\partial \eta}{\partial t} + \frac{\delta \mathcal{H}}{\delta \psi} \frac{\partial \psi}{\partial t} \right] dx = - \int \frac{\partial \eta}{\partial t} P_{ext} \, \mathrm{d}x.$$

If $P_{ext} = \chi \partial_t \eta$, then $\dfrac{d\mathcal{H}}{dt} \leq 0.$

This choice is widespread: Cao–Beck–Schultz, Clément, Grilli, Bonnefoy, Ducrozet.

See also Baker–Meiron–Orszag, Clamond–Fructus–Grue–Kristiansen.

Another possible choice: the energy also decreases when

$$\partial_x P_{ext} = \chi(x) \int_{-h}^{\eta(t,x)} \phi_x(t,x,y) \, \mathrm{d}y.$$

Theorem

Assume that

$$P_{ext}(t,x) = \chi(x)\partial_t\eta - \int_0^L \chi(x)\partial_t\eta(t,x)\,\mathrm{d}x \quad \textit{(with } \chi(x) \textit{ well chosen).}$$

$i)$ *There exist two positive constants* δ, C*, depending on* h, L*, s.t. if*

$$\|\eta\|_{L^\infty} + \|\eta_x\|_{L^\infty} \leq \delta,$$

and if the solution exists on the time interval $[0, 2C]$*, then*

$$\mathcal{H}(T) \leq \frac{1}{2}\mathcal{H}(0).$$

$ii)$ *There exists a constant* c_* *such that, if*

$$\|\eta_0\|_{H^{7/2}} + \|\psi_0\|_{H^3} \leq \varepsilon,$$

then the solution exists and is $O(\varepsilon)$ *on a time interval of size* c_*/ε*.*

Damping (decreasing energy) is easy but stabilization (convergence to 0) is more difficult (since the equation is **quasi-linear** and **nonlocal**).

Remove:

- paradifferential calculus
- microlocal analysis.

Microlocal (x, ξ) –> **Global** $\int f(x)dx$

Seek exact identities using only global quantities:

- Luke's variational principle
- Multipliers method
- Equipartition of energy
- Pohozaev type identity
- Conservation laws
- Hamiltonian formulation.

Multiplier method

How to compute

$$\int_0^T \mathcal{H}(t)\,\mathrm{d}t.$$

<u>**Example:**</u> consider the 1D wave eq with Dirichlet boundary condition:

$$\partial_t^2 u - \partial_x^2 u = 0, \quad u(t,0) = u(t,1) = 0.$$

Multiply the equation by $x\partial_x u$ and integrate by parts

$$\int_0^T (\partial_x u(t,1))^2\,dt = 2\int_0^1 (\partial_t u)(x\partial_x u)\,dx\,\Big|_0^T + \iint_S \left[(\partial_t u)^2 + (\partial_x u)^2\right] dxdt$$

where $S = (0,T)\times(0,1)$.

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where $S = (0,T)\times(0,1)$.

Instead of $m(x) = x$ consider $m \in C^\infty([0,L])$ s.t. $m(0) = m(L) = 0$ (Alabau-Boussouira).

Lemma

Consider $m \in C^\infty([0,L])$ *such that* $m(0) = m(L) = 0$. *Set*

$$\zeta = \partial_x(m\eta) - \frac{1}{4}\eta + \frac{1-m_x}{2}\eta, \quad \rho = (m-x)\eta_x + \left(\frac{5}{4} + \frac{m_x}{2}\right)\eta.$$

Then, for smooth enough solutions of the gravity water-wave equations, defined for $t \in [0,T]$,

$$\frac{1}{2}\int_0^T \mathcal{H}(t)\,\mathrm{d}t + \mathcal{Q} = \iint P_{ext}\,\zeta\,\mathrm{d}x\,\mathrm{d}t - \int \zeta\psi\,\mathrm{d}x\,\Big|_0^T$$
$$+ \iint \left(\frac{1-m_x}{2}\psi + (x-m)\psi_x\right)\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iiint \rho_x\,\phi_x\,\phi_y\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,$$

where

$$\mathcal{Q} = \int_0^T\int_0^L \left(\frac{h}{2} + \frac{\rho}{2}\right)\phi_x^2(t,x,-h)\,\mathrm{d}x\,\mathrm{d}t + \frac{L}{2}\int_0^T\int_{-h}^{\eta(t,L)} \phi_y^2(t,L,y)\,\mathrm{d}y\,\mathrm{d}t.$$

1. Luke's variational principle

We want to introduce the following key function:

$$\Theta = -\eta \partial_t \psi - \frac{g}{2}\eta^2.$$

Following Luke, the gravity WW system can be derived by minimizing

$$\mathcal{L} = \int_{t_0}^{t_1} \iint_{\Omega(t)} p \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t.$$

Now observe that, modulo terms which do not contribute to a variational principle,

$$\mathcal{L} = \int_{t_0}^{t_1} \int \Theta \, \mathrm{d}x \, \mathrm{d}t - \frac{1}{2} \int_{t_0}^{t_1} \iint_{\Omega(t)} |\nabla_{x,y}\phi|^2 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t.$$

We want to compute

$$\int_0^T \int_0^L \Theta(t,x) \, \mathrm{d}x \, \mathrm{d}t \quad \text{or} \quad \int_0^T \int_0^L m_x(x)\Theta(t,x) \, \mathrm{d}x \, \mathrm{d}t.$$

2. Multiplier method

Proposition

Consider a smooth solution and a smooth function $m\colon [0,L] \to \mathbb{R}$ *satisfying* $m(0) = m(L) = 0$. *Then*

$$\iint m_x \Theta \, \mathrm{d}x \, \mathrm{d}t + R_1 + R_2 = -\int \partial_x(m\eta)\psi \, \mathrm{d}x \Big|_0^T + \iint P_{ext} m \eta_x \, \mathrm{d}x \, \mathrm{d}t$$

where

$$R_1 = \iint (\partial_t \eta)(m \partial_x \psi) \, \mathrm{d}x \, \mathrm{d}t$$

$$R_2 = \frac{1}{2} \iint m \eta_x \left(\phi_x^2 - \phi_y^2 + 2\phi_x \, \phi_y \eta_x \right) \big|_{y=\eta} \mathrm{d}x \, \mathrm{d}t.$$

Proof: set

$$A := \iint \left[(\partial_t \eta)(m \partial_x \psi) - (\partial_t \psi)(m \partial_x \eta) \right] \mathrm{d}x \, \mathrm{d}t$$

and compute A in two different ways.

3. Equipartition of energy

Compare

$$\iint m_x \Theta \,\mathrm{d}x\,\mathrm{d}t \quad \text{and} \quad \frac{g}{2}\iint m_x \eta^2 \,\mathrm{d}x\,\mathrm{d}t$$

starting from

$$\Theta = \underbrace{-\eta(\partial_t \psi + g\eta)}_{\partial_t\psi + g\eta = P_{ext} \text{ for linearized pb}} + \frac{g}{2}\eta^2.$$

Then compare

$$\underbrace{\frac{g}{2}\iint m_x \eta^2 \,\mathrm{d}x\,\mathrm{d}t}_{\text{averaged potential energy}} \quad \text{and} \quad \underbrace{\frac{1}{2}\iint m_x \psi G(\eta)\psi \,\mathrm{d}x\,\mathrm{d}t}_{\text{averaged kinetic energy}}$$

and write

$$\iint m_x \psi G(\eta)\psi \,\mathrm{d}x\,\mathrm{d}t = \iint \psi G(\eta)\psi \,\mathrm{d}x\,\mathrm{d}t + \iint (1-m_x)\psi G(\eta)\psi \,\mathrm{d}x\,\mathrm{d}t.$$

□

4. Pohozaev identity

Term

$$R_1 = \iint (\partial_t \eta)(m\partial_x \psi)\,\mathrm{d}x\,\mathrm{d}t.$$

We have $\partial_t \eta = \partial_n \phi|_{y=\eta} = \mathcal{N}\psi$.
Next, we split

$$\int (\mathcal{N}\psi) m \partial_x \psi \,\mathrm{d}x = \int (\mathcal{N}\psi) x \partial_x \psi \,\mathrm{d}x + \int (\mathcal{N}\psi)(m-x)\partial_x \psi \,\mathrm{d}x.$$

We have a Pohozaev identity for the Dirichlet to Neumann operator:

$$\int (\mathcal{N}\psi)(x\partial_x \psi)\,\mathrm{d}x = \Sigma + \int (\eta - x\eta_x)\left(\phi_x^2 - \phi_y^2 + 2\phi_x\,\phi_y \eta_x\right)\big|_{y=\eta}\,\mathrm{d}x$$

where $\Sigma = \Sigma(t)$ is a positive term given by

$$\Sigma(t) = \frac{h}{2}\int_0^L \phi_x^2(t,x,-h)\,\mathrm{d}x + \frac{L}{2}\int_{-h}^{\eta(t,L)} \phi_y^2(t,L,y)\,\mathrm{d}y.$$

We end up with

$$
\begin{aligned}
\frac{1}{2}\int_0^T \mathcal{H}(t)\,\mathrm{d}t + \int \Sigma(t)\,\mathrm{d}t = &- \iint P_{ext}\,\zeta\ \mathrm{d}x\,\mathrm{d}t \\
&+ \iint \left(\frac{1-m_x}{2}\psi + (x-m)\partial_x\psi\right)\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t \\
&- \int \zeta\psi\,\mathrm{d}x\,\Big|_0^T \\
&- \iint \rho\left(\phi_x^2 - \phi_y^2 + 2\phi_x\,\phi_y\eta_x\right)\,\Big|_{y=\eta}\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

where

$$
\zeta = \partial_x(m\eta) - \frac{1}{4}\eta + \frac{1-m_x}{2}\eta, \quad \rho = (m-x)\eta_x + \left(\frac{5}{4} + \frac{m_x}{2}\right)\eta.
$$

Nonlinear term:

$$\iint \rho\left(\phi_x^2-\phi_y^2+2\phi_x\,\phi_y\eta_x\right)\,\big|_{y=\eta}\,\mathrm{d}x\,\mathrm{d}t.$$

The energy controls $\iint |\nabla_{x,y}\phi|^2\,\mathrm{d}y\,\mathrm{d}x$, not $\int |\nabla_{x,y}\phi|^2\,\big|_{y=\eta}\,\mathrm{d}x$.

Guided the study of conservation laws by Benjamin–Olver, use

$$\begin{aligned}\int u(x,\eta(x))\,dx &+ \int f(x,\eta(x))\partial_x\eta\,dx\\ &= \iint(\partial_y u-\partial_x f)\,dydx+\int u(x,-h)\,dx+\int f\,dy\Big|_{x=0}^{x=L}.\end{aligned}$$

Then

$$\begin{aligned}\int \rho\left(\phi_x^2-\phi_y^2+2\phi_x\,\phi_y\eta_x\right)\,\big|_{y=\eta}\,\mathrm{d}x &= -\iint \rho_x\,\phi_x\,\phi_y\,\mathrm{d}y\,\mathrm{d}x\\ &\quad+\frac{1}{2}\int \rho\phi_x^2|_{y=-h}\,\mathrm{d}x.\end{aligned}$$

Set

$$\zeta = \partial_x(m\eta) - \frac{1}{4}\eta + \frac{1-m_x}{2}\eta, \quad \rho = (m-x)\eta_x + \left(\frac{5}{4} + \frac{m_x}{2}\right)\eta.$$

Then

$$\begin{aligned}\frac{1}{2}\int_0^T \mathcal{H}(t)\,\mathrm{d}t + \mathcal{Q} = & \iint P_{ext}\,\zeta\ \mathrm{d}x\,\mathrm{d}t - \int \zeta\psi\,\mathrm{d}x\,\Big|_0^T \\ & + \iint \left(\frac{1-m_x}{2}\psi + (x-m)\partial_x\psi\right)\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t \\ & + \iiint \rho_x\,\phi_x\,\phi_y\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where

$$\mathcal{Q} = \int_0^T\int_0^L \left(\frac{h}{2} + \frac{\rho}{2}\right)\phi_x^2(t,x,-h)\,\mathrm{d}x\,\mathrm{d}t + \frac{L}{2}\int_0^T\int_{-h}^{\eta(t,L)} \phi_y^2(t,L,y)\,\mathrm{d}y\,\mathrm{d}t.$$

Set

$$\zeta = \partial_x(m\eta) - \frac{1}{4}\eta + \frac{1-m_x}{2}\eta, \qquad \rho = (m-x)\eta_x + \left(\frac{5}{4} + \frac{m_x}{2}\right)\eta.$$

Then

$$\frac{1}{2}\int_0^T \mathcal{H}(t)\,\mathrm{d}t + \mathcal{Q} = \iint P_{ext}\,\zeta\;\mathrm{d}x\,\mathrm{d}t - \int \zeta\psi\,\mathrm{d}x\,\Big|_0^T$$
$$+ \iint \left(\frac{1-m_x}{2}\psi + (x-m)\partial_x\psi\right)\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iiint \rho_x\,\phi_x\,\phi_y\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,$$

where

$$\mathcal{Q} = \int_0^T\int_0^L \left(\frac{h}{2} + \frac{\rho}{2}\right)\phi_x^2(t,x,-h)\,\mathrm{d}x\,\mathrm{d}t + \frac{L}{2}\int_0^T\int_{-h}^{\eta(t,L)} \phi_y^2(t,L,y)\,\mathrm{d}y\,\mathrm{d}t.$$

If η and $\partial_x\eta$ are small enough then $\mathcal{Q} \geq 0$ and one can absorb $\iiint \rho_x\,\phi_x\,\phi_y\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$ in the energy.

Set

$$\zeta = \partial_x(m\eta) - \frac{1}{4}\eta + \frac{1-m_x}{2}\eta.$$

Then

$$\frac{1}{4}\int_0^T \mathcal{H}(t)\,\mathrm{d}t \le \iint P_{ext}\,\zeta\;\mathrm{d}x\,\mathrm{d}t - \int \zeta\psi\,\mathrm{d}x\,\Big|_0^T + \iint \left(\frac{1-m_x}{2}\psi + (x-m)\partial_x\psi\right)\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t$$

Similar inequality with surface tension. By choosing

$$P_{ext} = (1-m_x)\partial_t\eta - \int_0^L (1-m_x)\partial_t\eta\,\mathrm{d}x$$

we end up with

$$\int_0^T \mathcal{H}(t)\,\mathrm{d}t \le C\mathcal{H}(0).$$

Thank you!