Limite diffusive pour des equations cinetiques stochastiques

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Kinetic models

Many physical systems are described by a kinetic equation:

$$\partial_t f + a(v) \cdot \nabla_x f = Q(f),$$

- $v \in V$ represents the various degrees of freedom of a particle, $a(v)$ is its velocity (often $a(v) = v$).
- $f(x, v)$ is the distribution function of the particles with degrees of freedom v at position $x \in \mathbb{T}^N$.
- V is endowed with a probability measure μ and the averaged velocity is zero : $\bar{a} = \int_V a(v) d\mu = 0$.
- Q accounts for the interaction between particles or between a particle and the medium.
- In general, it has a family of equilibrium F such that: $Q(f) = 0$ iff $f = \bar{f} F = \left(\int_V f d\mu\right) F$ with $F > 0$, $\bar{F} = 1$.
- Often, a small parameter ε is present in the equation and, after rescaling, the following equation is obtained:

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} Q(f^\varepsilon),$$

Radiative transfer and Rosseland approximation

- $$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}) L f^\varepsilon,$$

  with $L(f) = \bar{f} F - f$ describes the interaction between a surrounding continuous medium and a flux of photons radiating through it in the absence of hydrodynamical motion.
- The unknown $f^\varepsilon(t, x, v)$ then stands for a distribution function of photons having position x and velocity v at time t.
- The function σ is the opacity of the matter.
- When the surrounding medium becomes very large compared to the mean free paths ε of photons, f^ε is known to behave like ρ the solution of the Rosseland equation

  $$\partial_t \rho - \mathrm{div}_x(\sigma(\rho)^{-1} K \nabla_x \rho) = 0, \qquad (t, x) \in [0, T] \times \mathbb{T}^N.$$

  with $K := \int_V a(v) \otimes a(v)\, dv$. This is called the Rosseland approximation. (Bardos, Golse, Perthame, Sentis)

Deterministic equation, diffusive limit, $F = 1$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}) L(f^\varepsilon), \quad L(f) = \bar{f} - f.$$

Hilbert expansion (formal): $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$

$\rightsquigarrow$ order -2 : $L f_0 = \bar{f}_0 - f_0 = 0$ and $f_0 = \bar{f}_0 = \rho$.

(We assume $0 < \sigma_* \leq \sigma(\rho) \leq \sigma^*, \ \rho \in \mathbb{R}$).

$\rightsquigarrow$ order -1 : $a(v) \cdot \nabla_x \rho = \sigma(\rho) L(f_1)$.

The equation

$$L(g) = \bar{g} - g = \int_V g d\mu - g = h$$

can be solved iff $\int_V h d\mu = 0$ and in this case, we can take

$$g = -h.$$

Recall that $\int_V a(v) d\mu = 0 \rightarrow f_1 = -\sigma(\rho)^{-1} a(v) \cdot \nabla_x \rho$

Deterministic equation, diffusive limit, $F = 1$

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}^\varepsilon) L f^\varepsilon, \quad L(f) = \bar{f} - f.$$

Hilbert expansion (formal): $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$

$\rightsquigarrow$ order -2 : $L(f_0) = 0$ and $f_0 = \int_V f_0 d\mu = \rho$.

$\rightsquigarrow$ order -1 : $a(v) \cdot \nabla_x \rho = L(f_1) \longrightarrow f_1 = -\sigma(\rho)^{-1} a(v) \cdot \nabla_x \rho$.

$\rightsquigarrow$ order 0 : $\partial_t \rho + a(v) \cdot \nabla_x f_1 = \sigma(\rho) L(f_2)$

$$\longrightarrow \quad \partial_t \rho - \int_V a(v) \cdot \nabla_x (\sigma(\rho)^{-1} a(v) \cdot \nabla_x \rho) d\mu = 0$$

$$\longrightarrow \quad \partial_t \rho - \mathrm{div}\left(\sigma(\rho)^{-1} K \nabla_x \rho\right) = 0,$$

with

$$K := \int_V a(v) \otimes a(v) d\mu(v).$$

Deterministic equation, diffusive limit

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}^\varepsilon) L f^\varepsilon, \; L(f) = \bar{f} F - f.$$

When $\varepsilon \to 0$, the density $\rho^\varepsilon := \int_V f^\varepsilon d\mu$ converges to the solution ρ of the diffusion equation

$$\partial_t \rho - \mathrm{div}(\sigma(\rho)^{-1} K \nabla \rho) = 0$$

with initial data $\rho_0 = \int_V f_0 d\mu$. We assume $\int_V a(v) F(v) d\mu(v) = 0$ and:

$$\forall \varepsilon > 0, \forall (\xi, \alpha) \in S^{N-1} \times \mathbb{R}, \;\; \mu\left(\{v \in V, |a(v) \cdot \xi + \alpha| < \varepsilon\}\right) \leq \varepsilon^\theta,$$

for some $\theta > 0$.

The stochastic case

We first consider a similar model with time white noise, it depends on space but is smooth in x:

$$df^\varepsilon \ + \ \frac{1}{\varepsilon} a(v)\cdot\nabla_x f^\varepsilon \ dt \ = \ \frac{1}{\varepsilon^2}\sigma(\bar{f}^\varepsilon) L f^\varepsilon dt \ + f^\varepsilon \circ Q dW_t,$$

$$x \in \mathbb{T}^N,\ v \in V,\ Lf = \bar{f} F - f.$$

- The noise represents randoms creations/absorptions of photons.
- We expect to obtain a stochastic quasilinear parabolic equation at the limit.
- We adapt the Hilbert expansion method.
- We first have to prove existence of f^ε, we need non degeneracy of a:

$$\forall \varepsilon > 0,\ \forall(\xi,\alpha) \in S^{N-1} \times \mathbb{R},\ \ \mu\left(\{v \in V, |a(v)\cdot\xi + \alpha| < \varepsilon\}\right) \leq \varepsilon^\theta,$$

for some $\theta > 0$.

Hilbert expansion, stochastic case

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v)\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon^2}\sigma(\bar{f}^\varepsilon) L f^\varepsilon + f^\varepsilon \circ Q dW_t, \quad L(f) = \bar{f} - f.$$

Hilbert expansion (formal): $f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$

$\rightsquigarrow$ order -2 : $Lf_0 = 0$ and $f_0 = \int_V f_0 d\mu = \rho$.

$\rightsquigarrow$ order -1 : $a(v)\cdot\nabla_x \rho = \sigma(\rho) L f_1 \longrightarrow f_1 = -\sigma(\rho)^{-1} a(v)\cdot\nabla_x\rho$.

$\rightsquigarrow$ order 0 : $\partial_t \rho + a(v)\cdot\nabla_x f_1 = \sigma(\rho) L f_2 + \rho\circ Q dW_t$

$$\longrightarrow \partial_t\rho - \mathrm{div}\left(\sigma(\rho)^{-1}\Big(\int_V a(v)\otimes a(v) d\mu\Big)\nabla_x\rho\right) = \rho\circ Q dW_t.$$

and

$$\mathrm{div}\left(\sigma(\rho)^{-1}\left(a(v)\otimes a(v) - \int_V a(v)\otimes a(v) d\mu\right)\nabla_x\rho\right) = \sigma(\rho) L f_2.$$

Hilbert expansion, rigorous proof

- We take the solution of the SPDE:

$$\partial_t \rho - \operatorname{div}\left(\sigma(\rho)^{-1}\left(\int_V a(v)\otimes a(v)d\mu\right)\nabla_x\rho\right) = \rho\circ QdW_t.$$

  It is smooth is space provided the noise and initial data are also smooth. (D., De Moor, Hofmanova).

- Define: $f_1 = -\sigma(\rho)^{-1}a(v)\cdot\nabla_x\rho$ and

$$f_2 = -\operatorname{div}\left(\sigma(\rho)^{-1}\left(a(v)\otimes a(v) - \int_V a(v)\otimes a(v)d\mu\right)\nabla_x\rho\right).$$

- Set

$$r^\varepsilon = f^\varepsilon - \rho - \varepsilon f_1 - \varepsilon^2 f_2$$

  then, with $df_1 = f_{1,d}dt + \Psi_1^\flat dW$,

$$\begin{aligned} dr^\varepsilon + \frac{1}{\varepsilon}a(v)\cdot\nabla_x r^\varepsilon dt &= \frac{1}{\varepsilon^2}\left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]dt \\ &- \varepsilon a(v)\cdot\nabla_x f_2 dt + (f^\varepsilon - \rho - \varepsilon f_1)\, QdW_t \\ &+ G(f^\varepsilon - \rho)\, dt - \varepsilon f_{1,d}dt - \varepsilon\Psi_1^\flat dW_t - \varepsilon^2 df_2. \end{aligned}$$

Hilbert expansion, rigorous proof

$$
\begin{aligned}
dr^\varepsilon + \frac{1}{\varepsilon} a(v)\cdot\nabla_x r^\varepsilon dt \ &= \frac{1}{\varepsilon^2}\left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right] dt \\
&- \varepsilon a(v)\cdot\nabla_x f_2 dt + (f^\varepsilon - \rho - \varepsilon f_1)\, Q dW_t \\
&+ G(f^\varepsilon - \rho)\, dt - \varepsilon f_{1,d} dt \\
&- \varepsilon \Psi_1^\flat dW_t - \ \varepsilon^2 df_2.
\end{aligned}
$$

The terms: $\frac{1}{\varepsilon} a(v)\cdot\nabla_x r^\varepsilon$ and $\frac{1}{\varepsilon^2}\left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]$ behave well in L^1:

$$
\frac{1}{\varepsilon}\int_{\mathbb{T}^N\times V} \left(a(v)\cdot\nabla_x r^\varepsilon\right) sign(r^\varepsilon) d\mu dx = 0,
$$

$$
\frac{1}{\varepsilon^2}\int_{\mathbb{T}^N\times V} \left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right] sign(r^\varepsilon) d\mu dx \leq 0.
$$

Problem: we cannot use Itô formula for $\|r^\varepsilon\|_{L^1}$.

Hilbert expansion, rigorous proof

$$
\begin{aligned}
dr^\varepsilon + \frac{1}{\varepsilon} a(v)\cdot\nabla_x r^\varepsilon dt \; &= \frac{1}{\varepsilon^2}\left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right] dt \\
&- \varepsilon a(v)\cdot\nabla_x f_2 dt + (f^\varepsilon - \rho - \varepsilon f_1)\, Q dW_t \\
&+ G(f^\varepsilon - \rho)\, dt - \varepsilon f_{1,d} dt \\
&- \varepsilon \Psi_1^\flat dW_t - \; \varepsilon^2 df_2.
\end{aligned}
$$

The terms: $\frac{1}{\varepsilon} a(v)\cdot\nabla_x r^\varepsilon$ and $\frac{1}{\varepsilon^2}\left[\sigma(\bar{f}^\varepsilon)L(f^\varepsilon) - \sigma(\rho)L(f^\varepsilon - r^\varepsilon)\right]$ behave well in L^1. Problem: we cannot use Itô formula for $\|r^\varepsilon\|_{L^1}$.

- We use Itô formula for a δ smoothed version of the L^1 norm.
- This introduces singular terms in the Itô correction: the second derivative of this smoothed L^1 norm is of order $\frac{1}{\delta}$ multiplied by ε^2.
- The use of a modified L^1 norm introduces a term of order $\frac{\delta}{\varepsilon^2}$.

→ We need to kill the noise term of order ε.

→ We need a third corrector f_3 such that

$$\varepsilon^2 df_3 - \sigma(\rho)L(f_3)dt = \Psi_1^\flat dW_t$$

The convergence result

Theorem Let f^ε denote the solution of the kinetic problem

$$df^\varepsilon \ + \ \frac{1}{\varepsilon}a(v)\cdot\nabla_x f^\varepsilon \ dt \ = \ \frac{1}{\varepsilon^2}\sigma(\bar{f}^\varepsilon)(\bar{f}^\varepsilon F - f^\varepsilon))dt \ + f^\varepsilon \circ QdW_t,$$
$$x\in\mathbb{T}^N, \ v\in V.$$

and ρ the solution of the non-linear stochastic partial differential equation

$$\partial_t\rho - \mathrm{div}\left(\sigma(\rho)^{-1}K\nabla_x\rho\right) = \rho\circ QdW_t,$$

where K denotes the matrix $\left(\int_V a(v)\otimes a(v)d\mu\right)$. Then, the solution f^ε converges as ε tends to 0 to the fluid limit ρ and we have the estimate:

$$\sup_{t\in[0,T]} \mathbb{E}\|f_t^\varepsilon - \rho_t\|_{L^1_{x,v}} \le C\varepsilon.$$

Another model with "real noise"

We now start with a noise with non vanishing correlation length:

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}^\varepsilon) L f^\varepsilon + \frac{1}{\varepsilon} f^\varepsilon m(\frac{t}{\varepsilon^2}, x),$$

where $m(t, \cdot)$ is an ergodic centered markov process with values in a space of functions of x.

- We assume (V, μ) is a measured space, μ is a probability measure, $a \in L^\infty(V; \mathbb{R}^N)$, $N \geq 1$ and $x \in \mathbb{T}^N$.
- The equation is set in $\mathbb{R}_t^+ \times \mathbb{T}_x^N \times V_v$, with initial data $f^\varepsilon(0) = f_0$.
- As before, $L = \bar{f} F - f$ and the velocities are centered: $\int_V a(v) d\mu(v) = \int_V a(v) F(v) d\mu(v) = 0$ and non degenerate:

  $$\forall \varepsilon > 0, \forall (\xi, \alpha) \in S^{N-1} \times \mathbb{R}, \ \ \mu\left(\{v \in V, |a(v) \cdot \xi + \alpha| < \varepsilon\}\right) \leq \varepsilon^\theta.$$
- Existence and uniqueness of f^ε is classical under these assumptions.

Diffusion approximation :

We consider a differential equation in $\mathbb{R}^d$ with random coefficients:

$$\frac{dx_t^\varepsilon}{dt} = F(x_t^\varepsilon, m_t^\varepsilon) + \frac{1}{\varepsilon} G(x_t^\varepsilon, m_t^\varepsilon).$$

The driving process m_t^ε scales like $m_t^\varepsilon = m(\varepsilon^{-2} t)$ where m_t is a $\mathbb{R}^d$ valued homogeneous stationary and mixing Markov process. If $G \equiv 0$, then $x_t^\varepsilon \to \overline{x}_t$ where

$$\frac{d\overline{x}}{dt} = \overline{F}(\overline{x}_t), \quad \overline{F}(x) := \int_{\mathbb{R}} F(x, n) d\nu(n),$$

and ν is the invariant measure of m_t. We are interested in the case:

$$G \not\equiv 0, \quad \int_{\mathbb{R}} G(\cdot, n) d\nu(n) \equiv 0 \ ?$$

We concentrate on the case: $G(x, m) = G(x)m$.

The perturbed test function method.

Problem : We assume that the driving process m_t^ε scales like $m_t^\varepsilon = m(\varepsilon^{-2}t)$ where m_t is homogeneous and stationary Markov process. We assume that it is mixing with invariant measure ν. Let

$$\frac{d}{dt}x_t^\varepsilon = F(x_t^\varepsilon) + \frac{1}{\varepsilon}G(x_t^\varepsilon)m_t^\varepsilon, .$$

We expect that at the limit $\varepsilon \to 0$, x^ε converges in law to the solution of:

$$dx_t = F(x_t) + G(x_t) \circ dW_t.$$

To prove this we use the generator of $(x^\varepsilon, m^\varepsilon)$. We denote by M the generator of m, then $(x_t^\varepsilon, m_t^\varepsilon)$ has the following generator:

$$\mathcal{L}^\varepsilon \Phi(x,n) = \left(F(x) + \frac{1}{\varepsilon}G(x)n, D_x\Phi(x,n) \right) + \frac{1}{\varepsilon^2}M\Phi(x,n), \quad \Phi \in C_b^2(\mathbb{R}^{2d}).$$

Let $v^\varepsilon(t,x,n) = \mathbb{E}(\varphi(x_t^\varepsilon(x), m_t^\varepsilon(n)))$, then $\frac{d}{dt}v^\varepsilon = \mathcal{L}^\varepsilon v^\varepsilon$

The perturbed test function method.

Evolution of $\mathbb{E}(\varphi(x_t^\varepsilon))$:

$$\mathcal{L}^\varepsilon \varphi(x^\varepsilon) = \left(F(x^\varepsilon) + \frac{1}{\varepsilon} G(x^\varepsilon, n), D_x \varphi(x) \right)$$

$\rightsquigarrow$ No information as $\varepsilon \to 0$.
$\rightsquigarrow$ We try to find correctors $\varphi_1, \varphi_2 \in C_b^2(\mathbb{R}^d \times \mathbb{R}^d)$ such that the perturbed test function

$$\varphi^\varepsilon := \varphi + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2,$$

satisfies

$$\mathcal{L}^\varepsilon \varphi^\varepsilon(x, n) = \mathcal{L}\varphi(x) + \mathcal{O}(\varepsilon)$$

(Papanicolaou, Stroock, Varadhan 77. See the book by Fouque, Garnier, Papanicolaou and Solna)

The perturbed test function method.

$$\mathcal{L}^\varepsilon \varphi^\varepsilon(x,n) = \mathcal{L}\varphi(x) + \mathcal{O}(\varepsilon), \quad \varphi^\varepsilon := \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2.$$

Write:

$$\begin{aligned}&\mathbb{E}(\varphi^\varepsilon(x_t^\varepsilon, m_t^\varepsilon)) \\ &= \mathbb{E}(\varphi^\varepsilon(x_s^\varepsilon, m_s^\varepsilon)) + \mathbb{E}\left(\int_s^t \mathcal{L}^\varepsilon \varphi^\varepsilon(x_\sigma^\varepsilon, m_\sigma^\varepsilon) d\sigma\right)\end{aligned}$$

$$\varepsilon \to 0 \quad \rightsquigarrow \quad \mathbb{E}(\varphi(x_t)) = \mathbb{E}(\varphi(x_s)) + \mathbb{E}\left(\int_s^t \mathcal{L}\varphi(x_\sigma) d\sigma\right)$$

$\rightsquigarrow$ $\mathcal{L}$ is the generator of the limit process.

Equations for the correctors

$$\begin{aligned}\mathcal{L}^{\varepsilon}\varphi^{\varepsilon}(x,n) &= (F(x) + \frac{1}{\varepsilon}G(x,n), D_x\varphi^{\varepsilon}(x,n)) + \frac{1}{\varepsilon^2}M\varphi^{\varepsilon}(x,n) \\ &= \mathcal{L}\varphi(x) + \mathcal{O}(\varepsilon), \quad \varphi \in C_b^2(\mathbb{R}^2),\end{aligned}$$

$$\varphi^{\varepsilon} := \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2.$$

We derive

$$M\varphi(x) = 0, \tag{1}$$

$$(G(x)n, D_x\varphi(x)) + M\varphi_1(x,n) = 0, \tag{2}$$

$$(F(x), D_x\varphi) + (G(x)n, D_x\varphi_1(x)) + M\varphi_2(x,n) = \mathcal{L}\varphi(x). \tag{3}$$

The first equation is satisfied since φ does not depend on n. To solve the second equation, we need to solve the Poisson equation associated to M.

The Poisson equation

We assume that for a large class of functions ψ such that $\int \psi(n) d\nu(n) = 0$ (ν is the invariant law of m_t), the equation

$$M\theta = \psi, \quad \psi \in C_b(\mathbb{R})$$

has a solution in $\theta \in C_b(E)$, unique under the condition $\int \theta(n) d\nu(n) = 0$. It is given by:

$$\theta(n) = M^{-1}\psi(n) := -\int_0^\infty e^{Mt}\psi(n)dt = \int_0^\infty \mathbb{E}\psi(m_t|m(0) = n)dt.$$

e^{Mt} is the transition semi-group associated to m_t.

Equations for the correctors

$$(G(x)n, D_x\varphi(x)) + M\varphi_1(x,n) = 0, \quad (4)$$

$$(F(x), D_x\varphi) + (G(x)n, D_x\varphi_1(x)) + M\varphi_2(x,n) = \mathcal{L}\varphi(x). \quad (5)$$

We have assumed $\int_E G(x) n d\nu(n) = 0 \rightsquigarrow$ we obtain

$$\varphi_1 = -M^{-1}(G(x)n, D_x\varphi)$$

$$\rightsquigarrow \mathcal{L}\varphi(x) = (F(x), D_x\varphi) - \int_E (G(x)n, D_x(M^{-1}G(x)n, D_x\varphi(x)))d\nu(n).$$

This is the generator associated to the SDE:

$$dx = f(X)dt + G(X)oC^{1/2}d\beta$$

where β is a d dimensional brownian motion and C is computed from $\int_E n \otimes M^{-1} n \, d\nu(n)$.

Back to the stochastic kinetic equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \sigma(\bar{f}^\varepsilon) L f^\varepsilon + \frac{1}{\varepsilon} f^\varepsilon m^\varepsilon(t),$$

- $m^\varepsilon(t)$ is a centered, mixing markov process with values in a space E of functions of x.
- (V, μ) is a measured space and μ is a probability measure.
- $a \in L^\infty(V; \mathbb{R}^d)$
- $d \geq 1$ and $x \in \mathbb{T}^d$ the d dimensional torus.
- L is a dissipative operator.
- We assume $\int_V a(v) d\mu(v) = 0$ and

$$\forall \varepsilon > 0, \forall (\xi, \alpha) \in S^{N-1} \times \mathbb{R}, \quad \mu\left(\{v \in V, |a(v) \cdot \xi + \alpha| < \varepsilon\}\right) \leq \varepsilon^\theta,$$

for some $\theta > 0$.

Back to the stochastic kinetic equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v)\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon^2}\sigma(\bar{f}^\varepsilon) L f^\varepsilon + \frac{1}{\varepsilon} f^\varepsilon m^\varepsilon(t),$$

We denote by M the generator of m, then the generator of $(f^\varepsilon, m^\varepsilon)$ is:

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi^\varepsilon) + \frac{1}{\varepsilon^2}(\sigma(\bar{f})Lf, D\varphi^\varepsilon) + \frac{1}{\varepsilon}(nf, D\varphi^\varepsilon) + \frac{1}{\varepsilon^2} M\varphi^\varepsilon \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi^\varepsilon + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi^\varepsilon\end{aligned}$$

where $Af = a(v)\cdot\nabla_x f$, D is the gradient with respect to f and

$$\mathcal{L}_1\varphi = -(Af, D\varphi) + (nf, D\varphi)$$

and

$$\mathcal{L}_2\varphi = (\sigma(\bar{f})Lf, D\varphi) + M\varphi.$$

Perturbed test function method:

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi^\varepsilon) + \frac{1}{\varepsilon^2}(\sigma(\bar{f})Lf, D\varphi^\varepsilon) + \frac{1}{\varepsilon}(nf, D\varphi^\varepsilon) + \frac{1}{\varepsilon^2}M\varphi^\varepsilon \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi^\varepsilon + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi^\varepsilon\end{aligned}$$

$$\mathcal{L}_1\varphi^\varepsilon = -(Af, D\varphi^\varepsilon) + (nf, D\varphi^\varepsilon), \quad \mathcal{L}_2\varphi^\varepsilon = (\sigma(\bar{f})Lf, D\varphi^\varepsilon) + M\varphi^\varepsilon.$$

We expect a limit model which is a SPDE with unknown $\rho = \int_V f d\mu(v)$

$\rightsquigarrow$ We use test functions of the form $\varphi(f) = \varphi(\rho)$. And consider $\mathcal{L}^\varepsilon\varphi^\varepsilon$, $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$.

$\rightsquigarrow$ Order -2 : $\mathcal{L}_2\varphi = 0 \rightarrow$ automatically satisfied.

$\rightsquigarrow$ Order -1 : $\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0$

Inversion of $\mathcal{L}_2$:

$$\mathcal{L}_2\psi = (\sigma(\bar{f})Lf, D\psi) + M\psi = \Phi$$

- This is the generator of the process $(g(t; f, n), m(t; f, n))$:

$$\frac{d}{dt}g = \sigma(\bar{g})Lg = \sigma(\bar{g})\left(\int_V g\, d\mu(v) - g\right) = \sigma(\bar{g})\left(\rho - g\right),\ g(0) = f,$$

  where m is the driving process starting form n at $t = 0$.

- Explicit solution : $\rho = \int_V g(t)d\mu(v) = \int_V fd\mu(v)$

  $\rightsquigarrow$ $g(t) = e^{-\sigma(\bar{g})t}f + (1 - e^{-\sigma(\bar{g})t})\rho.$

  $\rightsquigarrow$

$$\psi(f, n) = \mathcal{L}_2^{-1}\Phi(f, n) = -\int_0^\infty \mathbb{E}\left(\Phi(g(t; f, n); m(t; f, n))\right) dt$$

$$\text{if } \int_E \Phi(\rho, n)d\nu(n) = 0.$$

Perturbed test function method:

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(Lf, D\varphi) + \frac{1}{\varepsilon}(m, D\varphi) + \frac{1}{\varepsilon^2}M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi) + (m, D\varphi), \quad \mathcal{L}_2\varphi = (Lf, D\varphi) + M\varphi$$

$\rightsquigarrow$ Order -2 : $\mathcal{L}_2\varphi = 0$ $\to$ automatically satisfied

$\rightsquigarrow$ Order -1 : $\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0$

$$\int_E \mathcal{L}_1\varphi(\rho, n)d\nu(n) = \int_E -(A\rho, D\varphi) + (n\rho, D\varphi)d\nu(n) = 0$$

$$\to \varphi_1 = -\mathcal{L}_2^{-1}\mathcal{L}_1\varphi = -\left(A\left(\sigma(\rho)^{-1}f\right), D\varphi\right) - (fM^{-1}n, D\varphi).$$

Perturbed test function method:

$$\begin{aligned}\mathcal{L}^{\varepsilon}\varphi^{\varepsilon}(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(Lf, D\varphi) + \frac{1}{\varepsilon}(m, D\varphi) + \frac{1}{\varepsilon^2}M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi) + (m, D\varphi), \quad \mathcal{L}_2\varphi = (Lf, D\varphi) + M\varphi$$

$\rightsquigarrow$ Order -2 : $\mathcal{L}_2\varphi = 0 \rightarrow$ automatically satisfied

$\rightsquigarrow$ Order -1 :
$\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0, \quad \varphi_1 = -(A\left(\sigma(\rho)^{-1}f\right), D\varphi) - (fM^{-1}n, D\varphi).$

$\rightsquigarrow$ Order 0 : $\mathcal{L}_1\varphi_1 + \mathcal{L}_2\varphi_2 = \mathcal{L}\varphi \rightarrow \mathcal{L}\varphi(\rho) = \int_E \mathcal{L}_1\varphi_1(\rho, n)d\nu(n).$

Limit generator:

$$\begin{aligned}\mathcal{L}\varphi &= \int_E \mathcal{L}_1\varphi_1 d\nu(n)\\ &= (\mathcal{A}\rho, D\varphi) - \int_E \Big((\rho n M^{-1} n, D\varphi(\rho)) + D^2\varphi(\rho)\cdot(\rho M^{-1} n, \rho n) \Big) d\nu(n).\end{aligned}$$

where

$$\mathcal{A}\rho = \mathrm{div}\big(\big(\sigma(\rho)^{-1}\big)\, K\nabla\rho\big)$$

This is the generator of

$$\begin{aligned} d\rho &= \mathrm{div}(K\nabla\rho)dt + \rho\circ Q^{1/2}dW(t)\\ &= \mathrm{div}(K\nabla\rho)dt + \frac{1}{2}F\rho + \rho Q^{1/2}dW(t).\end{aligned}$$

Limit $\varepsilon \to 0$

- To complete the proof, we need to prove tightness of the laws of $\rho^\varepsilon = \bar{f}^\varepsilon$.
- Bound in $L^2(\mathbb{T}^N)$: Take $\varphi(f) = \|f\|_{L^2}^2$
  (weigthed norm: $\|f\|_{L^2}^2 = \int_{\mathbb{T}^N \times V} f^2(x,v)F^{-1}(v)dxdv$.)
- It is not a function of $\rho = \bar{f}$ but it is possible to compute correctors and obtain a bound on $\|f^\varepsilon\|_{L^2(\mathbb{T}^N)}$ in $L^\infty(0,T)$.
- This implies tightness in $C([0,T]; H^{-\eta}(\mathbb{T}^N))$, $\eta > 0$.
- This is not sufficient to deal with the nonlinear term.

Limit $\varepsilon \to 0$

- We also have a bound $\frac{1}{\varepsilon}\|Lf^\varepsilon\|_{L^2}$ in $L^2(0,T)$.
- $\varepsilon\partial_t f^\varepsilon + a(v)\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon}\sigma(\bar{f}^\varepsilon)Lf^\varepsilon + m^\varepsilon f^\varepsilon$ is bounded in $L^2(0,T;L^2(\mathbb{T}^N))$.
- Averaging Lemma: *Assume*

$$\forall\varepsilon>0,\ \forall(\xi,\alpha)\in S^{N-1}\times\mathbb{R},\ \ \mu\left(\{v\in V, |a(v)\cdot\xi+\alpha|<\varepsilon\}\right)\le\varepsilon^\theta,$$

  for some $\theta>0$.
  If f^ε and $\varepsilon\partial_t f^\varepsilon + a(v)\cdot\nabla_x f^\varepsilon$ are bounded in $L^2(0,T;L^2(\mathbb{T}^N))$ then $\rho^\varepsilon=\bar{f}^\varepsilon$ is bounded in $L^2(0,T;H^s)$, for $s<\theta/2$.

- We get tightness in $L^2(0,T;L^2(\mathbb{T}^N))$

Limit $\varepsilon \to 0$

Theorem Let $f_0^\varepsilon \in L^2_{x,v}$ and

$$\rho_0 := \int_V f_0 \, d\mu.$$

Under the above assumptions on the velocities a and on the driving process m^ε, we have: for all $\eta > 0$, the density $\rho^\varepsilon := \int_V f^\varepsilon d\mu$ converges in law in $C([0,T]; H^{-\eta})$ and in $L^2(0,T; H^s)$ to the solution ρ of the equation

$$\begin{aligned} d\rho &= \mathrm{div}(\sigma(\rho)^{-1} K \nabla \rho) dt + \rho \circ Q^{1/2} dW(t), \text{ in } \mathbb{R}_t^+ \times \mathbb{T}^d, \\ &= \mathrm{div}(\sigma(\rho)^{-1} K \nabla \rho) dt + \frac{1}{2} F\rho + \rho Q^{1/2} dW(t), \text{ in } \mathbb{R}_t^+ \times \mathbb{T}^d, \end{aligned}$$

with initial data ρ_0, where W is a cylindrical Wiener process on $L^2(\mathbb{T}^d)$, Q is a nuclear operator on $L^2(\mathbb{T}^d)$ determined by the correlation of m.

Coefficient Q in the limit model

It is associated to a kernel k:

$$Qf(x) = \int_{\mathbb{T}^d} k(x,y)f(y)dy, \quad f \in L^2(\mathbb{T}^d),$$

where

$$k(x,y) := \mathbb{E}\int_{\mathbb{R}} m(0)(y)m(t)(x)dt, \quad x,y \in \mathbb{T}^d.$$

The Itô correction :

$$F(x) = k(x,x).$$

The noise as a force (linear case):

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} a(v) \cdot \nabla_x f^\varepsilon + \frac{1}{\varepsilon^2} m^\varepsilon \cdot \nabla_v f = \frac{1}{\varepsilon^2} L f^\varepsilon.$$

- $m^\varepsilon(t)$ is a centered mixing markov process with values in a space of functions E.
- $v \in V = \mathbb{T}^d$.
- $a(v) = v$.
- $Lf = \rho F - f$ where F is an equilibrium function satisfying:

$$\int_V vF(v) d\mu(v) = 0.$$

The noise as a force (linear case):

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon + \frac{1}{\varepsilon^2} m^\varepsilon \cdot \nabla_v f = \frac{1}{\varepsilon^2} L f^\varepsilon .$$

We denote by M the generator of m, then the generator of $f^\varepsilon, m^\varepsilon$ is given by:

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi^\varepsilon) + \frac{1}{\varepsilon^2}(Lf, D\varphi^\varepsilon) - \frac{1}{\varepsilon^2}(mBf, D\varphi^\varepsilon) + \frac{1}{\varepsilon^2} M\varphi^\varepsilon \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi^\varepsilon + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi^\varepsilon\end{aligned}$$

where $Af = v \cdot \nabla_x f$, $Bf = \nabla_v f$, D is the gradient with respect to f and now

$$\mathcal{L}_1 \varphi = -(Af, D\varphi)$$

and

$$\mathcal{L}_2 \varphi = (Lf, D\varphi) - (mBf, D\varphi) + M\varphi$$

Perturbed test function method

$$\begin{aligned}\mathcal{L}^\varepsilon \varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi^\varepsilon) + \frac{1}{\varepsilon^2}(Lf, D\varphi^\varepsilon) - \frac{1}{\varepsilon}(mBf, D\varphi^\varepsilon) + \frac{1}{\varepsilon^2} M\varphi^\varepsilon \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi^\varepsilon + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi^\varepsilon\end{aligned}$$

$$\mathcal{L}_1\varphi^\varepsilon = -(Af, D\varphi^\varepsilon), \quad \mathcal{L}_2\varphi^\varepsilon = (Lf, D\varphi^\varepsilon) - (mBf, D\varphi^\varepsilon) + M\varphi^\varepsilon$$

We expect a limit model which is a SPDE with unknown $\rho = \int_V f d\mu(v)$

$\rightsquigarrow$ We use test functions of the form $\varphi(f) = \varphi(\rho)$.

$\rightsquigarrow$ Order -2 : $\mathcal{L}_2\varphi = 0 \rightarrow$ automatically satisfied

$\rightsquigarrow$ Order -1 : $\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0$

Inversion of $\mathcal{L}_2$:

$$\mathcal{L}_2\varphi = (Lf, D\varphi) + (mBf, D\varphi) + M\varphi$$

This is the generator of the process $(g(t; f, n), m(t; f, n))$:

$$\frac{d}{dt}g = Lg - mBg = \rho F - g - m \cdot \nabla_v g, \quad g(0) = f,$$

where m is the driving process starting form n at $t = 0$.

Explicit solution: $\rho = \int_V g(t) d\mu(v) = \int_V f d\mu(v)$

$\rightsquigarrow$

$$g(t, x, v) = e^{-t} f(x, v - M_t) + \int_0^t e^{-(t-s)} \rho(x) F(v + M_s - M_t) ds.$$

where $M_t = \int_0^t m(s, x, n) ds$.

$\rightsquigarrow$ $\mathcal{L}_2^{-1}\psi(f, n) = -\int_0^\infty \mathbb{E}\left(\psi(g(t; f, n); m(t; f, n))\right) dt$ if

$$\int_E \psi\Big(\int_{-\infty}^0 e^s \rho F(v - \int_s^0 m(\sigma, n) d\sigma) ds, n\Big) d\nu(n) = 0.$$

Perturbed test function method:

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi^\varepsilon(f,n) &= -\frac{1}{\varepsilon}(Af, D\varphi) + \frac{1}{\varepsilon^2}(Lf, D\varphi) + \frac{1}{\varepsilon}(m, D\varphi) + \frac{1}{\varepsilon^2}M\varphi \\ &= \frac{1}{\varepsilon}\mathcal{L}_1\varphi + \frac{1}{\varepsilon^2}\mathcal{L}_2\varphi\end{aligned}$$

$$\mathcal{L}_1\varphi = -(Af, D\varphi), \quad \mathcal{L}_2\varphi = (Lf, D\varphi) - (mBf, D\varphi) + M\varphi$$

⇝ Order -2 : $\mathcal{L}_2\varphi = 0$ → automatically satisfied

⇝ Order -1 : $\mathcal{L}_1\varphi + \mathcal{L}_2\varphi_1 = 0$. It is possible to invert $\mathcal{L}_2$:

$$\varphi_1 = -(Af, D\varphi) - (\mathrm{div}(fM^{-1}n), D\varphi(f)).$$

⇝ Order 0 : $\mathcal{L}_1\varphi_1 + \mathcal{L}_2\varphi_2 = \mathcal{L}\varphi$

$$\to \mathcal{L}\varphi = -\int_E \mathcal{L}_1\varphi_1\Big(\int_{-\infty}^0 e^s\rho F(v - \int_s^0 m(\sigma,n)d\sigma)ds\Big)d\nu(n)$$

.

Limit generator:

After some computations ... we obtain the limit SPDE:

$$d\rho \quad = \mathrm{div}((K+H)\nabla\rho)dt + \mathrm{div}(\rho G) + \mathrm{div}(\rho \circ Q^{1/2} dW(t)), \text{ in } \mathbb{R}_t^+ \times \mathbb{T}^d$$

The operator Q:

$$Qf(x) = \int_{\mathbb{T}^d} k(x,y) f(y) dy, \quad f \in L^2(\mathbb{T}^d),$$

where

$$k(x,y) := \mathbb{E} \int_{\mathbb{R}} m(0)(y) \otimes m(t)(x) dt, \quad x, y \in \mathbb{T}^d.$$

The extra (deterministic) diffusion:

$$H(x) := \mathbb{E} \int_0^\infty e^{-s} m(0)(x) \otimes m(t)(x) dt, \quad x \in \mathbb{T}^d.$$

Space-time diffusion approximation:

- $$\partial_t u^\varepsilon = \Delta u^\varepsilon + f(u^\varepsilon) + \frac{1}{\varepsilon^{3/2}}\sigma(u^\varepsilon)m(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}), \quad x \in \mathbb{T},$$

  with initial condition

  $$u^\varepsilon(0) = u_0, \quad x \in \mathbb{T}.$$

- We expect convergence to a Stratonovitch equation:

  $$du = \Delta u + f(u) + \sigma(u) \circ dW, \quad \in \mathbb{T},$$

  where W is a cylindrical Wiener proces, *i.e.* dW is a space-time white noise. It is well known that this equation does not have solutions, contrary to its Itô form.

- Following Hairer and Pardoux (see also Hairer-Shen), we introduce three correction terms:

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + f(u^\varepsilon) + H_1(u^\varepsilon) + \frac{1}{\varepsilon^{1/2}}H_2(u^\varepsilon) + \frac{1}{\varepsilon}H_3(u^\varepsilon) + \frac{1}{\varepsilon^{3/2}}\sigma(u^\varepsilon)m(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}),$$

where H_1, H_2, H_3 depend on σ are to be precised.

The case $\sigma(u) = u$

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + C_1 u + \frac{1}{\varepsilon^{1/2}} C_2 u + \frac{1}{\varepsilon} C_3 u + \frac{1}{\varepsilon^{3/2}} u^\varepsilon m(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}).$$

- We set $m^\varepsilon(t,x) = m(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ with generator $\frac{1}{\varepsilon^2} M$ where M is the generator of $m(t, \frac{x}{\varepsilon^2})$.
- The infinitesimal generator the couple $(u^\varepsilon, m^\varepsilon)$ is :

$$\begin{aligned}
&\mathcal{L}^\varepsilon \varphi(u,n) = \\
&(\Delta u, D\varphi(u,n)) + (C_1 u, D\varphi(u,n)) + \frac{1}{\varepsilon^{1/2}} (C_2 u, D\varphi(u,n)) \\
&+ \frac{1}{\varepsilon} (C_3 u, D\varphi(u,n)) + \frac{1}{\varepsilon^{3/2}} (mu, D\varphi(u,n)) + \frac{1}{\varepsilon^2} M\varphi,
\end{aligned}$$

- This a differential operators with respect to the variables $u \in L^2$, $n \in E$, D denotes differentiation with respect to u.
- For φ be a function of u, we introduce the perturbed test function:

$$\varphi^\varepsilon(u,n) = \varphi(u) + \sum_{i=1}^{4} \varepsilon^{i/2} \varphi_i(u,n).$$

We introduce this expression in the generator $\mathcal{L}^\varepsilon$ and obtain since φ does not depend on n:

$$\mathcal{L}^\varepsilon \varphi(u,n) =$$

$$(\Delta u, D\varphi(u)) + (C_1 u, D\varphi(u)) + \frac{1}{\varepsilon^{1/2}}(C_2 u, D\varphi(u))$$

$$+\frac{1}{\varepsilon}(C_3 u, D\varphi(u)) + \frac{1}{\varepsilon^{3/2}}(mu, D\varphi(u))$$

$$+\varepsilon^{1/2}(\Delta u, D\varphi_1(u,n)) + \varepsilon^{1/2}(C_1 u, D\varphi_1(u,n)) + (C_2 u, D\varphi_1(u,n))$$

$$+\frac{1}{\varepsilon^{1/2}}(C_3 u, D\varphi_1(u,n)) + \frac{1}{\varepsilon}(mu, D\varphi_1(u,n)) + \frac{1}{\varepsilon^{3/2}} M\varphi_1$$

$$+\varepsilon(\Delta u, D\varphi_2(u,n)) + \varepsilon(C_1 u, D\varphi_2(u,n)) + \varepsilon^{1/2}(C_2, D\varphi_2(u,n))$$

$$+(C_3, D\varphi_2(u,n)) + \frac{1}{\varepsilon^{1/2}}(mu, D\varphi_2(u,n)) + \frac{1}{\varepsilon} M\varphi_2$$

$$+\dots$$

Order $\varepsilon^{-3/2}$

The corrector is a function of the noise which depends on the fast variable $\frac{x}{\varepsilon}$, a spatial derivative is equivalent to ε^{-1}:

$$\varepsilon^2(\Delta u, D\varphi_1) + (mu, D\varphi(u)) + M\varphi_1 = 0.$$

Take a linear function $\varphi(u) = (u, \psi)$ where $\psi(x)$ is a test function and choose $\varphi^\varepsilon(u, n) = (u, \psi^\varepsilon)$ with

$$\psi^\varepsilon(x, \frac{x}{\varepsilon}) = \psi(x) + \varepsilon^{1/2}\psi_1(x, \frac{x}{\varepsilon}) + \varepsilon\psi_2(x, \frac{x}{\varepsilon}) + \dots$$

$$\rightsquigarrow \quad \varepsilon^2(\Delta u, \psi_1) + (mu, \psi)) + M(u, \psi_1) = 0.$$

$$\rightsquigarrow \quad \varepsilon^2\Delta\psi_1 + M\psi_1 = -m\psi.$$

Introduce $y = \frac{x}{\varepsilon}$ and decompose

$$\Delta\psi_1(x, \frac{x}{\varepsilon}) = \Delta_x\psi_1(x, \frac{x}{\varepsilon}) + \frac{2}{\varepsilon}\partial_x\partial_y\psi_1(x, \frac{x}{\varepsilon}) + \frac{1}{\varepsilon^2}\Delta_y\psi_1(x, \frac{x}{\varepsilon}).$$

$$\rightsquigarrow \quad \Delta_y\psi_1 + M\psi_1 = -m\psi, \quad \psi_1 = -(\Delta_y + M)^{-1}m\psi = m_1\psi.$$

Order ε^{-1}

Using the same arguments, we obtain the equation:

$$\Delta_y \psi_2 + M\psi_2 - mm_1\psi - C_3\psi = 0.$$

This equation has a solution if and only if

$$\mathbb{E}(m(t,x)m_1(t,x)) - C_3 = 0.$$

Recall that

$$\begin{aligned} m_1(t,x) &= -(\Delta_y + M)^{-1} m(t,x) = \int_0^\infty e^{\Delta s} m(t+s,x)ds \\ &= \int_0^\infty \int k_s(z-x) m(t+s,z) dz ds. \end{aligned}$$

We obtain:

$$\begin{aligned} C_3 &= \mathbb{E}\left(\int_0^\infty \int k_s(z-x)m(t+s,z)dzds\, m(t,x)\right) \\ &= \int_0^\infty \int k_s(z-x)\mathbb{E}(m(t+s,z)m(t,x))dzds \\ &= \int_0^\infty \int k_s(z)\mathbb{E}(m(s,z)m(0,0))dzds \end{aligned}$$

- In this way, we compute the constant such that we have formally the predicted limit.
- Is it possible to prove tightness? Not possible to use Itô formula!
- The limiting equation is solved by writing :

$$u(t) = e^{\Delta t} u_0 + \int_0^t e^{\Delta(t-s)} u(s) dW(s)$$

- Use Itô isometry:

$$\mathbb{E}\left(\|u(t)\|^2_{H^\gamma(\mathbb{T})}\right) = \|e^{\Delta t} u_0\|^2_{H^s(\mathbb{T})} + \mathbb{E}\left(\int_0^t \|e^{\Delta(t-s)} u(s)\|^2_{\mathcal{L}_2(L^2;H^\gamma)} ds\right)$$

and this can be controlled for $\gamma < 1/2$. With a little more work, we can control the $W^{\gamma,p}(\mathbb{T})$ norm and for $\gamma p > 1$ we obtain Hölder continuity.

- This can be recovered by using Ito formula on

$$v(t) = e^{\Delta(T-t)}u(t),$$

it satisfies:

$$dv(t) = e^{\Delta(T-t)}\left(u(t)dW(t)\right).$$

- $$d\|v(t)\|^2_{H^\gamma(\mathbb{T})} = 2((-\Delta)^\gamma v(t), e^{\Delta(T-t)}(u(t)dW(t))) + \|e^{\Delta(T-t)}u(t)\|^2_{\mathcal{L}_2(L^2;H^\gamma)}dt$$

- Idea: Use the perturbed test function to

$$\|e^{\Delta(T-t)}u^\varepsilon(t)\|^p_{W^{\gamma,p}(\mathbb{T})}$$

.