Anomalous solutions to the Monge Ampère equation in two dimensions and pre-strained elasticity

Reza Pakzad

University of Pittsburgh

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Université Pierre et Marie Curie, Laboratoire Jacques Louis Lions

Motivation

- **The Monge-Ampère constrained plate model:**
  (Friesecke-James-Müller 2006; Lewicka-Ochoa-P. 2014)
- $\omega \subset \mathbb{R}^2$, $f : \omega \to \mathbb{R}$.
  Minimize $I_f(u) = \int_\omega |D^2 u|^2$ in the class of Monge-Ampère solutions:
  $$MA_f := \{u : \omega \to \mathbb{R}; \quad \det D^2 u = f \quad a.e., \quad \int_\omega |D^2 u|^2 < \infty\}$$
- **Main problems**:
  Existence of admissible solutions,
  (with or without boundary conditions).
  Uniqueness and regularity of minimizers/ critical points.
  Study of Sobolev ($W^{2,2}$), weak or very weak MA solutions.

Weak solutions to the Monge-Ampère equation:

- $\omega \subset \mathbb{R}^2$.
- Weak Solutions: Weak Hessian $Det\, D(Du) = f$,
  where for $\Psi = (\Psi^1, \Psi^2) : \omega \to \mathbb{R}^2$, the distributional Jacobian is:
  $$Det\, D(\Psi) := (\Psi^1 \Psi^2_{,2})_{,1} - (\Psi^1 \Psi^2_{,1})_{,2}.$$
  Well defined for $u \in W^{2,p}_{loc}$, $p \geq \frac{4}{3}$ or $u \in W^{2,1}_{loc} \cap W^{1,\infty}_{loc}$. Coincides with $\det(D^2 u)$ when $u \in W^{2,2}$.
- Very Weak Solutions: The very weak Hessian determinant for $u \in W^{1,2}_{loc}$:
  $$\mathcal{D}et D^2 u := -\frac{1}{2}\left[(|u_{,1}|^2)_{,22} + (|u_{,2}|^2)_{,11} - 2(u_{,1} u_{,2})_{,12}\right]$$
- (Iwaniec 2001) $\mathcal{D}et D^2$ is weakly sequentially discontinuous on $W^{1,p}$ for all $p \geq 2$ at $u_0 = 0$.

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- Very Weak Solutions: The very weak Hessian determinant for $u \in W^{1,2}_{loc}$:
  $$\mathcal{D}et D^2 u := -\frac{1}{2} \mathrm{curl}^t \mathrm{curl}(Du \otimes Du)$$
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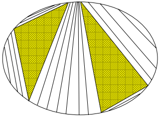
- $\omega \subset \mathbb{R}^2$, $f : \omega \to \mathbb{R}$. No convexity assumptions for solution u:

$$det(D^2 u) = f \quad \text{in} \quad \omega. \tag{1}$$

- Rigidity for $u \in C^2$:



$f \equiv 0 \implies u$ is developable.
$f > 0 \implies u$ is locally convex (modulo sign) in ω.

- (Kirchheim 2001, P. 2004, Šverák 1991, Lewicka-Mahadevan-P. 2013.) Rigidity holds for $W^{2,2}$.

Theorem (Lewicka-P. 2015)

$\omega \subset \mathbb{R}^2$ open, bounded, simply connected and of sufficient regularity. $f \in L^2(\omega)$. Then for all $u_0 \in C^0(\bar{\omega})$, and all $\alpha < 1/7$, there exists a sequence of weak solutions $u_k \in C^{1,\alpha}$ to $\mathcal{D}et D^2 u_k = f$, converging uniformly to u_0. If $u_0 \in C^1(\bar{\omega})$, then u_k are uniformly bounded in C^1.

Existence of anomalous very weak solutions

- An *h*-principle:

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- Compare with:

Theorem (Nash and Kuiper 1955)

Let $\mathbf{g}$ be a smooth Riemannian metric on ω and $y_0 : (\omega, \mathbf{g}) \to \mathbb{R}^3$ be a smooth short immersion:

$$0 < y_0^* e := (\nabla y_0)^t \nabla y_0 < \mathbf{g}.$$

Then there exists a sequence of C^1 isometric embeddings of $(\omega, \mathbf{g})$ into $\mathbb{R}^3$ uniformly converging to y_0.

Rigidity/Flexibility of isometric Immersions

- (Darboux) Smooth isometric immersions of flat 2d domains into $\mathbb{R}^3$ are developable.
- (Hilbert) Smooth isometric immersions of the sphere into $\mathbb{R}^3$ are rigid motions.
- (Hartman-Nirenberg 1959) C^2 is enough for rigidity.
- (Pogorelov 1956) C^1 with bounded extrinsic curvature is enough.
- (P. 2004, Hornung-Velčić 2015) Rigidity of Sobolev isometries of class $W^{2,2}$.
- (Borisov 2004) Nash-Kuiper scheme yields $C^{1,\alpha}$ isometries, $\alpha < 1/13$ for analytic metrics, points to $\alpha < 1/7$.
- (Conti-De Lellis-Székelyhidi 2010) Same for $\alpha < 1/7$.
- (De Lellis-Inauen-Székelyhidi 2015) Same for $\alpha < 1/5$.
- (Borisov 50's) Rigidity of isometric immersions of sphere for $\alpha > 2/3$.
  (Conti-De Lellis-Székelyhidi 2010) Same result using a more analytical approach: a commutator estimate.
- Conjectures for critical α: $1/3, 1/2, 2/3$?

Euler equations:

- 3d incompressible Euler equations:
  $u : \mathbb{T}^3 \times [0, T] \to \mathbb{R}^3, p : \mathbb{T}^3 \times [0, T] \to \mathbb{R}$.

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 \\ \mathrm{div} u = 0 \end{cases}$$

- Flexibility below $C^{0,1/5}$ (Delellis-Szekelyhidi, Isett): existence of $L^\infty(0, T; C^\alpha(\mathbb{T}^3))$ solutions compactly supported in time (Buckmaster-Delellis-Isett-Szekelyhidi): existence of solutions with arbitrary temporal kinetic energy profile;
- Rigidity beyond $C^{0,1/3}$ (Constantin-E-Titi, Eyink): every $L^\infty(0, T; C^\alpha(\mathbb{T}^3))$ solution is energy conserving;
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$$\begin{cases} \partial_t u + \mathrm{div}(\mathrm{u} \otimes \mathrm{u}) + \nabla p = 0 \\ \mathrm{div} u = 0 \end{cases}$$

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Convex integration for the Monge-Ampère equation

- The Monge-Ampère constraint is a 2nd order infinitesimal isometry constraint.
- Analytically, if $f = -\Delta g$:

$$\mathcal{D}et D^2 u = -\frac{1}{2}\mathrm{curl}^t\mathrm{curl}(Du \otimes Du) = f$$
$$\Longleftrightarrow$$
$$\exists w : \omega \to \mathbb{R}^2 \quad \mathrm{sym}\, Dw + \frac{1}{2} Du \otimes Du = g\,\mathrm{Id}.$$

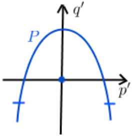
- Given u_0, we introduce w_0 so that we have $\mathrm{sym}\, Dw_0 + \frac{1}{2} Du_0 \otimes Du_0 < g\,\mathrm{Id}$.
- Nash-Kuiper iteration.
- Decompose the defect $B := g\,\mathrm{Id} - (\mathrm{sym}\, Dw_0 + \frac{1}{2} Du_0 \otimes Du_0)$ as

$$B(x) := \sum_{j=1}^{3} a_j^2(x)\eta_j \otimes \eta_j, \quad \forall j\ |\eta_j| = 1.$$

Iteration à la Nash & Kuiper 1

- Step: Introduce one dimensional oscillations to recompense a defect of the form $a^2\eta\otimes\eta$.
- $u_1(x) := u_0(x) + \frac{1}{\lambda}a(x)p(t)$
  $Du_1(x) = Du_0(x) + a(x)p'(t)\eta + O(\frac{1}{\lambda})$
- $w_1 = w_0 - \frac{1}{\lambda}ap(t)Du_0 + \frac{1}{\lambda}a^2q(t)\eta$
  $Dw_1 = Dw_0 - ap'(t)(\eta\otimes Du_0) + a^2q'(t)(\eta\otimes\eta) + O(\frac{1}{\lambda})$
- $\mathrm{sym}\,Dw_1 + \frac{1}{2}Du_1\otimes Du_1$
  $= \mathrm{sym}\,Dw_0 + \frac{1}{2}Du_0\otimes Du_0 + ap'(t)\mathrm{sym}(\eta\otimes Du_0)$
  $- ap'(t)\mathrm{sym}(\eta\otimes Du_0) + a^2(\frac{1}{2}(p')^2(t) + q'(t))\eta\otimes\eta + O(\frac{1}{\lambda})$.
- $t = \lambda x\cdot\eta$.
- Need p, q and their derivatives bounded for all $t\in\mathbb{R}$ and $\frac{1}{2}(p')^2 + q' \equiv 1$.

$\gamma = (p,q)\in C^\infty(\mathbb{R},\mathbb{R}^2)$
Need γ bounded with bounded derivatives.
γ periodic $\implies \int\gamma'(t)dt = (0,0)$.



Need $(0,0)$ to be in the convex hull of P. Solve for $\gamma'(t)\in P$

Iteration à la Nash & Kuiper 2

- Stage: Given $(\hat{u}, \hat{w})$, repeat steps 3 times in three directions η_i to obtain $(\tilde{u}, \tilde{w})$ such that for
$$\hat{E} := \|\hat{B}\|_0 = \left\| g\,\mathrm{Id} - (\mathrm{sym} D\hat{w} + \frac{1}{2} D\hat{u} \otimes D\hat{u}) \right\|_0 \text{ and } \lambda > 1:$$
$$\|\tilde{u} - \hat{u}\|_0 + \|\tilde{w} - \hat{w}\|_0 \leq C \frac{\hat{E}^{1/2}}{\lambda}$$
$$\|\tilde{u} - \hat{u}\|_1 + \|\tilde{w} - \hat{w}\|_1 \leq C \hat{E}^{1/2}$$
$$\longrightarrow \qquad \|\cdot\|_1 \sim \lambda^{-m/2}$$
$$\|\tilde{u} - \hat{u}\|_2 + \|\tilde{w} - \hat{w}\|_2 \leq C(1 + \|\hat{u}\|_2 + \|\hat{w}\|_2)\lambda^3$$
$$\longrightarrow \qquad \|\cdot\|_2 \sim \lambda^{3m}$$
$$\tilde{E} := \left\| g\,\mathrm{Id} - (\mathrm{sym}\, D\tilde{w} + \frac{1}{2} D\tilde{u} \otimes D\tilde{u}) \right\|_0 \leq C \hat{E} \lambda^{-1}$$
$$\longrightarrow \qquad \tilde{E}_m \sim \lambda^{-m}$$
- Iterate the stages and interpolate to get a sequence $(\tilde{u}_m, \tilde{w}_m) \to (u, w)$ in $C^{1,\alpha}$.
- $\alpha < 1/7$ since 3 steps are needed.
  Better regularity expected if one can reduce the number of steps.
  Best to hope through this method $\alpha < 1/3$.

Isometric morphisms of a Riemann manifold (Ω, G)

Let $\Omega \subset \mathbb{R}^n$, $G \in \mathcal{C}^\infty(\Omega, \mathbb{R}^{n\times n}_{sym,+})$. Look for $U : \Omega \to \mathbb{R}^n$ so that
$(\nabla U)^T \nabla U = G$ in Ω

Theorem (Gromov 1986)

Let $U_0 : \Omega \to \mathbb{R}^n$ be smooth short immersion, i.e.:
$0 < (\nabla U_0)^T \nabla U_0 < G$ *in* Ω. *Then:*
$\forall \varepsilon > 0 \quad \exists U \in W^{1,\infty} \quad \|U - U_0\|_{\mathcal{C}^0} < \varepsilon$ *and* $(\nabla U)^T \nabla U = G$.

Theorem (Myers-Steenrod 1939, Calabi-Hartman 1970)

Let $U \in W^{1,\infty}$ satisfy $(\nabla U)^T \nabla U = G$ and $\det \nabla U > 0$ a.e. in Ω.
(For example, $U \in \mathcal{C}^1$ enough). Then $\Delta_G U = 0$ and so U is smooth.
In fact, U is unique up to rigid motions, and: $\exists U \Leftrightarrow Riem(G) \equiv 0$ in Ω.

Prestrained elasticity

$$E(U) = \int_\Omega W\big((\nabla U)G^{-1/2}(x)\big)\, dx$$

$W(F) \sim \text{dist}^2(F, SO(3))$

- $E(U) = 0 \Leftrightarrow \nabla U(x) \in SO(3)G^{1/2}(x) \;\; \forall a.e.\ x$

  $\Leftrightarrow (\nabla U)^T \nabla U = G$ and $\det \nabla U > 0$

Lemma (Lewicka-P. 2009)

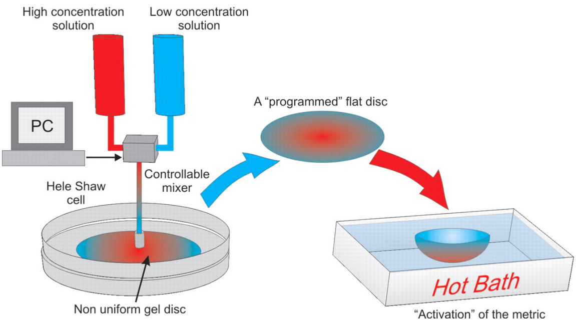
$$\inf_{U \in W^{1,2}} E(U) > 0 \Leftrightarrow Riem(G) \not\equiv 0.$$

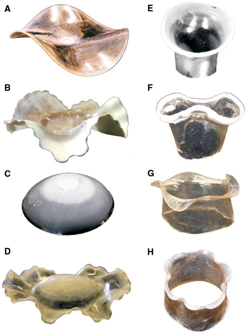
Thin prestrained plates: $\Omega = \Omega^h = \omega \times (-h/2, h/2)$, $G = G^h$, $\omega \subset \mathbb{R}^2$

- As $h \to 0$: Scaling of: $\inf E^h \sim h^\beta$? $\text{argmin} E^h \to \text{argmin}\, I_\beta$?

- Hierarchy of theories I_β, where β depends on $Riem(G^h)$
(Bhattacharya, Li, Lewicka, Mahadevan, P., Raoult, Schaffner)
- When $A = Id$: dimension reduction in nonlinear elasticity, seminal analysis by
(LeDret-Raoult 1995, Friesecke-James-Müller 2006)

Manufacturing residually-strained thin films





- *Shaping of elastic sheets by prescription of Non-Euclidean metrics* (Klein, Efrati, Sharon) Science, 2007

The Monge-Ampère constrained energy

$$\text{Energy} \quad E^h(U^h) = \frac{1}{h}\int_{\Omega^h} W\big((\nabla U^h)(G^h)^{-1/2}(x)\big)\,\mathrm{d}x$$

Theorem (Lewicka-Ochoa-P. 2014)

Let: $G^h(x', x_3) = Id_3 + 2h^\gamma S(x')$ *and* $\gamma \in (1,2)$. *Then:*

- $\inf E^h \le Ch^{\gamma+2} \Leftrightarrow \exists u \in W^{2,2}(\omega), \;\; \det \nabla^2 u = -\mathit{curl}\;\mathit{curl}\; S_{2\times 2}$
- $\frac{1}{h^{\gamma+2}} E^h \xrightarrow{\Gamma} I$, *where* I *is the 2-d energy:*

  $I(u) = \int_\omega |\nabla^2 u|^2$ *for* $u \in W^{2,2}(\omega), \;\; \det \nabla^2 u = -\mathit{curl}\;\mathit{curl}\; S_{2\times 2}$

- Structure of minimizers to E^h: $U^h(x',0) = x' + h^{\gamma/2} u e_3$
  Existence in $W^{2,2} \implies E^h \le Ch^{\gamma+2}$.
- Existence in $\mathcal{C}^{1,\frac{1}{7}-} \implies \inf E^h \le Ch^{(\frac{7}{4}\gamma+\frac{1}{2})-}$ in all circumstances.
- If we had flexibility at $\mathcal{C}^{1,\frac{1}{3}-}$ which is optimal using Nash-Kuiper technique, then $\inf E^h \le Ch^{(\frac{3}{2}\gamma+1)-}$.
- The regime $h^{\gamma+2} \ll E^h \ll h^{(\frac{7}{4}\gamma+\frac{1}{2})-}$ remains unexplored.

Rigidity of the Monge-Ampère equation

- MA eqn.: fully nonlinear, 2nd order PDE, ellipticity $\Leftrightarrow$ convexity
  - (Alexandrov 1958, Bakelman 1957): existence, uniqueness of generalized (convex) solutions for $f > 0$, convex boundary data.
  - (Heinz 1961): $\mathcal{C}^{2,\alpha}$ interior estimates for $f \in \mathcal{C}^{0,\alpha}$ in 2 dimensions.
  - (Cheng-Yau 1977, Pogorelov 1978): general regularity results
  - regularity of convex generalized solutions in higher dimensions: (Caffarelli, Caffarelli-Nirenberg-Spruck, Krylov, Trudinger-Wang).

Rigidity at Hölder regularity (very weak solutions, no convexity assumpt.):

Theorem (Lewicka-P. 2015, 2016)

Let $u \in \mathcal{C}^{1,\alpha}$, $\alpha > 2/3$. If $\mathcal{D}et\nabla^2 u = 0$, then u is developable.
If $\mathcal{D}et\nabla^2 u \geq c > 0$ is Dini continuous, then u is locally convex and an Alexandrov solution in ω.

The flexibility-rigidity dichotomy

- Monge-Ampère eq: flexibility below $\mathcal{C}^{1,1/7}$; rigidity beyond $\mathcal{C}^{1,2/3}$
  - rigidity of $W^{2,2}$ solutions in the developable $f = 0$ (Kirchheim, P.) and convex $f > c > 0$ (Šverák , Lewicka-Mahadevan-P.) cases.
- Isometric immersions of $(\omega, \mathbf{g})$ in $\mathbb{R}^3$:
  - flexibility below $\mathcal{C}^{1,1/5}$ (De Lellis-Inaunen-Székelyhidi);
  - rigidity beyond $\mathcal{C}^{1,2/3}$ (Borisov);
  - rigidity of $W^{2,2}$ immersions in the developable $\kappa = 0$ (P.) and convex $\kappa > c > 0$ (Hornung-Velčić) cases.
  - Expected threshold: $\frac{1}{3}$ or $\frac{1}{2}$ or $\frac{2}{3}$.
- 3d incompressible Euler equations:
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