Uniqueness in a class of Hamilton-Jacobi equations with constraints

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Received *****; accepted after revision +++++

Presented by

Abstract

In this note, we discuss a class of time-dependent Hamilton-Jacobi equations depending on a function of time, this function being chosen in order to keep the maximum of the solution to the constant value 0. The main result of the note is that the full problem has a unique classical solution. The motivation is a selection-mutation model which, in the limit of small diffusion, exhibits concentration on the zero level set of the solution of the Hamilton-Jacobi equation. The uniqueness result that we prove implies strong convergence and error estimates for the selection-mutation model. *To cite this article: S. Mirrahimi, J.-M. Roquejoffre, C. R. Acad. Sci. Paris, Ser. I 340 (2015).*

Résumé

Unicité pour une classe d'équations de Hamilton-Jacobi avec contraintes. Dans cette note, on discute une classe d'équations de Hamilton-Jacobi dépendant du temps, et d'une fonction inconnue du temps choisie pour que le maximum de la solution de l'équation de Hamilton-Jacobi prenne tout le temps la valeur 0. Le résultat principal de cette note est que le problème complet admet une unique solution classique. La motivation est un modèle de sélection-mutation qui, dans la limite d'une diffusivité nulle, prsente une concentration sur la ligne de niveau 0 de la solution de l'équation de Hamilton-Jacobi. Le rsultat d'unicité que nous démontrons implique une convergence forte avec estimations d'erreur pour le modèle de sélection-mutation. *Pour citer cet article : S. Mirrahimi, J.-M. Roquejoffre, C. R. Acad. Sci. Paris, Ser. I 340 (2015).*

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Version française abrégée

On présente dans cette note un résultat d'unicité pour le problème de Hamilton-Jacobi suivant, d'inconnues $(u(t,x), I(t))$:

$$\begin{cases} \partial_t u = |\nabla u|^2 + R(x,I), \quad (t>0, x\in\mathbb{R}^d), \quad \max_x u(t,x) = 0, \\ I(0) = I_0 > 0, \quad u(0,x) = u_0(x). \end{cases} \tag{1}$$

La donnée $R(x,I)$ vérifie des hypothses de stricte concavité par rapport à x et de monotonie par rapport à I explicites plus bas. La donnée initiale u_0 vérifie aussi des hypothèses spéciales de concavité. Ainsi, la fonction $I(t)$ doit être choisie pour que la solution $u(t,x)$ de l'équation de Hamilton-Jacobi ait, à chaque instant, un maximum égal à 0.

Nous avons alors le

Theorem 0.1 *On choisit $R \in C^2$, et on suppose l'existence de $I_M > 0$ tel que $\max_{x\in\mathbb{R}^d} R(x,I_M) = 0 = R(0,I_M)$. De plus, R est supposée strictement concave et, pour $|x|$ grand, comprise entre deux paraboles. La donne initiale u_0 est également à dcroissance quadratique et strictement concave.*

Le problème (1) a une unique solution (u,I), où u est une solution classique de l'équation de Hamilton-Jacobi. De plus $u \in L^\infty_{\text{loc}}\big(R^+; W^{3,\infty}_{\text{loc}}(\mathbb{R}^d)\big) \cap W^{1,\infty}_{\text{loc}}\big(R^+; L^\infty_{\text{loc}}(\mathbb{R}^d)\big) \times W^{1,\infty}(\mathbb{R})$.

L'existence pour (1) a été démontrée en plusieurs endroits, voir par exemple [8] ou [1]. L'unicité est donc notre résultat principal, c'était un problème ouvert. L'unicité était en effet connue seulement pour un cas très particulier (voir [8]).

Le modèle (1) intervient dans la limite $\varepsilon \to 0$ des solutions de

$$\partial_t n_\varepsilon - \varepsilon\Delta n_\varepsilon = \frac{n_\varepsilon}{\varepsilon} R\big(x, I_\varepsilon(t)\big) \ (t>0,\ x\in\mathbb{R}^d), \quad I_\varepsilon(t) = \int_{\mathbb{R}^d} \psi(x) n_\varepsilon(t,x)dx, \tag{2}$$

où $n_\varepsilon(t,x)$ est la densité d'une population caractérisée par un trait biologique x d-dimensionnel. La compétition pour une ressource unique est représentée par $I_\varepsilon(t)$, $\psi > 0$ régulière donnée. Le terme $R(x,I)$ est le taux de reproduction. Les hypothses de concavité sont des hypothèses techniques, mais pertinentes au plan biologique. Par une transformation de Hopf-Cole $n_\varepsilon = \exp(u_\varepsilon/\varepsilon)$ on se ramène à l'équation sur u_ε suivante :

$$\partial_t u_\varepsilon = \varepsilon\Delta u_\varepsilon + |\nabla u_\varepsilon|^2 + R(x, I_\varepsilon) \tag{3}$$

qui, dans la limite $\varepsilon \to 0$, donne l'équation sur u. On s'attend alors à ce que n_ε se concentre aux points où u est proche de 0. Et, dans cette limite, I_ε apparaît comme une sorte de multiplicateur de Lagrange.

La convergence de (3) vers (1) à une sous-suite près est connue depuis [8]. Le Théorème 0.1 donne la convergence de toute la famille, ainsi que des estimations d'erreur. Soit $x_\varepsilon(t)$ le point où $u_\varepsilon(t,.)$ atteint son maximum. On suppose l'existence de I^0 tel que $0 < I^0 \leq I_\varepsilon(0) := \int_{\mathbb{R}^d} \psi(x) n^0_\varepsilon(x)dx < I_M$, et on suppose

$$n^0_\varepsilon = e^{u^0_\varepsilon/\varepsilon} = \frac{r}{\varepsilon^{d/2}}\, e^{u_0/\varepsilon}, \qquad \text{with } \ u_0 \in C^2(\mathbb{R}^d) \quad \text{and} \quad \max_{x\in\mathbb{R}^d} u_0(x) = 0.$$

Le résultat est alors le

Theorem 0.2 *Soit n_ε la solution de (2) et u_ε définie par (3). Nous avons les développements asymptotiques suivants*

$$I_\varepsilon = I + \varepsilon I_1 + o(1), \quad x_\varepsilon = \overline{x} + \varepsilon\,\overline{x}_1 + o(1), \quad u_\varepsilon = u + \varepsilon \log\big(\frac{r}{\varepsilon^{\frac{d}{2}}}\big) + \varepsilon\, u_1 + o(1).$$

Les termes I_1, $\overline{x}_1$ et u_1 vont être présentés dans [7]. Ce résultat implique le corollaire

Corollary 0.3 *Nous avons l'approximation suivante pour n_ε :*

$$n_\varepsilon(t,x) = \frac{r}{\varepsilon^{\frac{d}{2}}} \left(\exp(u_1 + \frac{u}{\varepsilon}) + o(1) \right).$$

En particular, lorsque $\varepsilon \to 0$, toute la suite $(n_\varepsilon)_\varepsilon$ converge :

$$n_\varepsilon(t,x) \longrightarrow \bar{\rho}(t)\, \delta\big(x - \bar{x}(t)\big) \qquad \text{au sens des mesures,}$$

avec $\rho(t) = \dfrac{I(t)}{\psi(\overline{x}(t))}$.

En d'autres termes, la population se concentre sur un trait dominant qui évolue avec le temps. On note que la convergence de n_ε à une sous-suite prés était déjà établie dans [5].
Tous ces résultats seront détaillés dans [6] et [7].

1. Introduction

The purpose of this note is to discuss uniqueness in the following problem, with unknowns $(I(t), u(t,x))$:

$$\begin{cases} \partial_t u = |\nabla u|^2 + R(x,I) \quad (t>0, x \in \mathbb{R}^d), \quad \max\limits_x u(t,x) = 0, \\ I(0) = I_0 > 0, \quad u(0,x) = u_0(x), \end{cases} \tag{4}$$

where $I_0 > 0$ and u_0 is a concave, quadratic function:

$$-\underline{L}_0 - \underline{L}_1 |x|^2 \le u_0(x) \le \overline{L}_0 - \overline{L}_1 |x|^2, \quad -2\underline{L}_1 \le D^2 u_0 \le -2\overline{L}_1, \quad D^3 u_0 \in L^\infty(\mathbb{R}^d).$$

The constraint on the maximum of $u(t,.)$ makes the problem nonstandard. Our main result is
Theorem 1.1 *Choose $R \in C^2$, and suppose that there is $I_M > 0$ such that $\max\limits_{x \in \mathbb{R}^d} R(x, I_M) = 0 = R(0, I_M)$. Also assume the following concavity and regularity properties for R:*

$$-\underline{K}_1 |x|^2 \le R(x,I) \le \overline{K}_0 - \overline{K}_1 |x|^2, \qquad \text{for } 0 \le I \le I_M,$$
$$-2\underline{K}_1 \le D^2 R(x,I) \le -2\overline{K}_1 < 0 \text{ and } D^3 R(\cdot, I) \in L^\infty(\mathbb{R}^d) \text{ for } 0 \le I \le I_M,$$
$$-\underline{K}_2 \le \partial_I R \le -\overline{K}_2,$$
$$|\partial^2_{Ix_i} R(x,I)| + |\partial^3 R_{Ix_ix_j}(x,I)| \le K_3, \qquad \text{for } 0 \le I \le I_M, \text{ and } i, j = 1,2,\cdots,d.$$

Problem (4) has a unique solution (u,I), where u solves the Hamilton-Jacobi equation in the classical sense. Moreover $u \in L^\infty_{\mathrm{loc}}\big(R^+; W^{3,\infty}_{\mathrm{loc}}(\mathbb{R}^d)\big) \cap W^{1,\infty}_{\mathrm{loc}}\big(R^+; L^\infty_{\mathrm{loc}}(\mathbb{R}^d)\big) \times W^{1,\infty}(\mathbb{R})$.
Existence to (4) has been proved in various contexts (see [8,1,5]). Thus, our contribution is uniqueness, which has up to now been an open problem. The uniqueness has indeed been known only for a very particular case (see [8]).
The rest of the note is organized as follows. In Section 2, we explain the motivation and, in particular, the meaning of the various assumptions. In Section 3, we revisit existence for (4), which will entail an unconventional ODE formulation for uniqueness. Section 4, which is the main part of the note, provides a fairly complete sketch of the uniqueness proof. In Section 5, we give an application.

2. Background and motivation

Model (4) arises in the limit $\varepsilon \to 0$ of the solutions to the problem

$$\partial_t n_\varepsilon - \varepsilon \Delta n_\varepsilon = \frac{n_\varepsilon}{\varepsilon} R\big(x, I_\varepsilon(t)\big)\ (t>0,\ x \in \mathbb{R}^d), \quad I_\varepsilon(t) = \int_{\mathbb{R}^d} \psi(x) n_\varepsilon(t,x) dx, \tag{5}$$

where $n_\varepsilon(t,x)$ is the density of a population characterized by a d-dimensional biological trait x. The population competes for a single resource, this is represented by $I_\varepsilon(t)$, where ψ is a given positive smooth function. The term $R(x,I)$ is the reproduction rate; it is, as can be expected, very negative for large x and decreases as the competition increases. Such models can be derived from individual based stochastic processes in the limit of large populations (see [2]). The concavity assumption on R is a technical one, although biologically relevant. The Hopf-Cole transformation $n_\varepsilon = \exp(u_\varepsilon/\varepsilon)$ yields the equation

$$\partial_t u_\varepsilon = \varepsilon \Delta u_\varepsilon + |\nabla u_\varepsilon|^2 + R(x, I_\varepsilon) \tag{6}$$

which, in the limit $\varepsilon \to 0$, yields the equation for u. Now, I_ε being uniformly positive and bounded in ε, the Hopf-Cole transformation leads to the constraint on u. Moreover, one expects that n_ε concentrates at the points where u is close to 0 and the function I_ε appears, in the limit, as a sort of Lagrange multiplier.

This approach, based on the Hopf-Cole transformation, to study (5) has been introduced in [4] and then developed in different contexts (see for instance [8,1,3,5]). Long time asymptotics of such models have also been studied in [9] and the references therein.

3. Existence

Existence to a solution to (4) is obtained by letting $\varepsilon \to 0$ in (6). The main step is the

Theorem 3.1 *(uniform estimates for u_ε, [5]) There exists $I_m > 0$ such that $0 < I_m \le I_\varepsilon(t) \le I_M + C\varepsilon^2$. Moreover we have the following estimates on u_ε*

$$\begin{cases} -\underline{L}_0 - \underline{L}_1 |x|^2 - \varepsilon 2d \underline{L}_1 t \le u_\varepsilon(t,x) \le \overline{L}_0 - \overline{L}_1 |x|^2 + \big(\overline{K}_0 + 2d\varepsilon \overline{L}_1\big)\, t, \\ \underline{L}_1 - 2t\underline{K}_1 \le D^2 u_\varepsilon(t,x) \le -2\overline{L}_1, \quad \|D^3 u_\varepsilon(t,\cdot)\|_{L^\infty} \le C(T), \qquad \textit{for } t \in [0,T]. \end{cases} \tag{7}$$

The bounds for u_ε can be obtained for any uniformly bounded function I_ε, not only for that of (5). This remark will be an important ingredient of the uniqueness proof.

4. Uniqueness

For a given continuous function $I(t)$ such that $0 < I(t) < I_M$, one may construct a solution of $\partial_t u = |\nabla u|^2 + R(x,I)$ with initial datum u_0. Just as in Theorem 3.1, this solution satisfies estimates (7). And so, $u(t,.)$ being strictly concave and quadratically decreasing, there exists a unique function $\bar{x}(t)$ such that $u(t,\bar{x}(t)) = \max_{x \in \mathbb{R}^d} u(t,x)$. Assume that $I(t)$ is chosen such that $u(t,\bar{x}(t)) = 0$. Then, from the equation on u we deduce that $R(\bar{x}(t), I(t)) = 0$. Notice also that, because $\partial_I R < 0$, we have $R(\bar{x}(t), 0) > 0$. Finally, differentiating $\nabla u(t,\bar{x}(t)) = 0$ and plugging in the equation for u we obtain an ODE for $\bar{x}$: $\dot{\bar{x}}(t) = \big(-D^2 u\big(t,\bar{x}(t)\big)\big)^{-1} \nabla_x R\big(\bar{x}(t), \bar{I}(t)\big)$.

The idea is thus to change the constrained problem (4) by the following slightly nonstandard differential system:

$$\begin{cases} R\left(\overline{x}(t), I(t)\right) = 0, & \text{for } t \in [0,T], \\ \dot{\overline{x}}(t) = \big(-D^2 u\big(t,\bar{x}(t)\big)\big)^{-1} \nabla_x R\big(\bar{x}(t), \bar{I}(t)\big), & \text{for } t \in [0,T], \\ \partial_t u = |\nabla u|^2 + R(x,I), & \text{in } [0,T] \times \mathbb{R}^d, \end{cases} \tag{8}$$

with initial conditions

$$I(0) = I_0, \qquad u(0,\cdot) = u_0(\cdot), \qquad \overline{x}(0) = \overline{x}_0, \qquad \text{such that } R(x_0, I_0) = 0. \tag{9}$$

Note that (8) is really a differential system because the assumptions on R imply that $I(t)$ can implicitely be expressed in terms of $\bar{x}(t)$. And it is slightly nonstandard because $\bar{x}$ solves an ODE whose nonlinearity depends on u. Finally, note that, as soon as u satisfies the concavity and regularity estimates (7), system (8) is equivalent to the constrained problem (4).

This suggests to use a simple fixed point argument to prove uniqueness to (8) (and so, to (4)). Which in turn suggests to set up the following scheme: starting from $x(t) \in C([0,T], \mathbb{R}^d)$, such that $x(0) = x_0$, where $R(x_0, 0) > 0$. Let $I(t)$ solve $R(x(t), I(t)) = 0$ on $[0,T]$ with $R(x_0, I_0) = 0$. Let $v(t,.)$ be the unique solution to

$$\partial_t v = |\nabla v|^2 + R(x, I), \quad v(0,x) = u_0(x). \tag{10}$$

Let $y(t)$ solve $\dot{y}(t) = \big(-D^2 v\big(t, x(t)\big)\big)^{-1} \nabla_x R\big(x(t), I(t)\big)$ on $[0,T]$ with initial datum $y(0) = x_0$. Setting $y := \Phi(x)$, we notice that uniqueness is proved as soon as we have proved that Φ has a unique fixed point. One additional feature about a solution $(\bar{I}, u, \bar{x})$ of (8):

Lemma 4.1 *The function $\bar{I}(t)$ is increasing.*

We claim that our problem reduces to proving the

Theorem 4.2 *There exists $C > 0$ universal and $\delta > 0$, which is small as $R(x_0, 0)$ tends to 0, such that Φ is a contraction from $C([0,\delta], \overline{B}_C(x_0))$ to itself; here $B_r(a)$ denotes the ball of centre $a \in \mathbb{R}^d$ and radius $r > 0$.*

Note indeed that, by Lemma 4.1, we have, because $\partial_I R < 0$:

$$R(\bar{x}(\delta), 0) = R(\bar{x}(\delta), 0) - R(\bar{x}(\delta), \bar{I}(\delta)) \geq c\bar{I}(\delta) \geq cI_0,$$

for some universal $c > 0$. Hence Theorem 4.2 can be iterated to yield global existence and uniqueness.

Let us give an overview of the proof of Theorem 4.2. For $I \in \mathrm{C}([0,\delta]; [0, I_M])$, let $V(I)$ be the (unique) solution of (10). The main step is the following

Lemma 4.3 *Let I_1, $I_2 \in \mathrm{C}([0,\delta]; [0, I_M])$. Then*

$$\|V(I_1) - V(I_2)\|_{W^{2,\infty}([0,\delta]\times\mathbb{R}^d)} \leq C \|I_1 - I_2\|_{L^\infty([0,\delta])} \delta.$$

This lemma, once proved, opens the way to Theorem 4.2. Indeed the equation $R(x, I) = 0$ yields a smooth mapping $x \mapsto I$, and $I \mapsto V$ is a Lipschitz mapping thanks to Lemma 4.3. Moreover, the equation $\dot{y}(t) = \big(-D^2 v\big(t, x(t)\big)\big)^{-1} \nabla_x R\big(x(t), I(t)\big)$ yields a Lipschitz mapping $v \mapsto y$ by the estimates for v given by Theorem 3.1.

Lemma 4.3 is more involved. If I_1 and I_2 are as in the assumptions of the lemma, the function $r = V(I_1) - V(I_2)$ solves

$$\begin{cases} \partial_t r = (\nabla v_1 + \nabla v_2) \cdot \nabla r + R(x, I_2) - R(x, I_1), & \text{in } [0,\delta] \times \mathbb{R}^d \\ r(0,x) = 0, & \text{for all } x \in \mathbb{R}^d. \end{cases} \tag{11}$$

with $v_i = V(I_i)$. Note that the above equation has a unique classical solution which can be computed by the method of characteristics. The characteristics solve

$$\dot{\gamma}(t) = -\nabla v_1(t, \gamma) - \nabla v_2(t, \gamma), \tag{12}$$

and, due to the estimates of Theorem 3.1, they exist globally. So, one may successively express r given by integration along characteristics, and estimate its derivatives recursively.

5. Application

Convergence of (6) to (4) had already been proved in [8,1], along subsequences. The uniqueness part of Theorem 1.1 yields the convergence of the full family of solutions u_ε of (6), instead of convergence along a subsequence. Moreover it allows an expansion of I_ε, u_ε and x_ε (the maximum point of u_ε at each time) in terms of ε. Here are the results.

Assume that there is I^0 such that $0 < I^0 \le I_\varepsilon(0) := \int_{\mathbb{R}^d} \psi(x) n_\varepsilon^0(x) dx < I_M$, and that

$$n_\varepsilon^0 = e^{u_\varepsilon^0/\varepsilon} = \frac{r}{\varepsilon^{d/2}}\, e^{u^0/\varepsilon}, \qquad \text{with } \ u^0 \in C^2(\mathbb{R}^d) \quad \text{and} \quad \max_{x\in\mathbb{R}^d} u^0(x) = 0.$$

The result is the

Theorem 5.1 *Let n_ε be the solution of (5) and u_ε be defined by (6). We have the following asymptotic expansions*

$$I_\varepsilon = I + \varepsilon I_1 + o(1), \quad x_\varepsilon = \overline{x} + \varepsilon\, \overline{x}_1 + o(1), \quad u_\varepsilon = u + \varepsilon \log\Big(\frac{r}{\varepsilon^{\frac{d}{2}}}\Big) + \varepsilon\, u_1 + o(1).$$

The terms I_1, $\overline{x}_1$ and u_1 will be provided in [7]. This yields the corollary

Corollary 5.2 *We have the following approximation for n_ε:*

$$n_\varepsilon(t,x) = \frac{r}{\varepsilon^{\frac{d}{2}}}\Big(\exp\big(u_1(t,x) + \frac{u(t,x)}{\varepsilon}\big) + o(1)\Big).$$

In particular, as $\varepsilon \to 0$, the whole sequence $(n_\varepsilon)_\varepsilon$ converges:

$$n_\varepsilon(t,x) \longrightarrow \bar\rho(t)\; \delta\big(x - \bar x(t)\big), \qquad \textit{weakly in the sense of measures},$$

with $\rho(t) = \dfrac{I(t)}{\psi(\overline{x}(t))}$.

In other words, the population density concentrates on a dominant trait which evolves in time. We note that the convergence of n_ε along subsequences was already established in [5].
The above results will be detailed in [6] and [7].

Acknowledgements

S. Mirrahimi was partially funded by the french ANR projects KIBORD ANR-13-BS01-0004 and MODEVOL ANR-13-JS01-0009. J.-M. Roquejoffre has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 321186 - ReaDi - "Reaction-Diffusion Equations, Propagation and Modelling" held by Henri Berestycki. He was also partially supported by the French National Research Agency (ANR), within the project NONLOCAL ANR-14-CE25-0013. Both authors are grateful to the Labex CIMI for organizing a PDE-probability quarter in Toulouse, which provided quite a stimulating environment to this collaboration.

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