La limite quasineutre du système de Vlasov-Poisson

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The quasineutral limit of Vlasov-Poisson

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + E_\varepsilon \cdot \nabla_v f_\varepsilon = 0, \quad t \geq 0, \, (x, v) \in \mathbb{T}^d \times \mathbb{R}^d \\ E_\varepsilon = -\nabla_x U_\varepsilon, \\ U_\varepsilon - \varepsilon^2 \Delta_x U_\varepsilon = \displaystyle\int_{\mathbb{R}^d} f_\varepsilon \, dv - 1, \\ f_\varepsilon|_{t=0} = f_{0,\varepsilon} \geq 0, \ \displaystyle\int_{\mathbb{T}^d \times \mathbb{R}^d} f_{0,\varepsilon} \, dv \, dx = 1. \end{cases}$$

• f_ε describes the dynamics of ions, in a background of massless electrons following a linearized Maxwell-Boltzmann law :

$$n_e = e^{U_\varepsilon} \sim 1 + U_\varepsilon.$$

• The parameter $\varepsilon \in (0, 1]$ is the ratio between the **Debye length** and the observation length. In practice, $\varepsilon \ll 1$.

• **Quasineutral limit** : $\varepsilon \to 0$.

Assuming $f_{0,\varepsilon} \to f_0$ and taking $\varepsilon = 0$ yields

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \rho \cdot \nabla_v f = 0, \\ \rho = \int_{\mathbb{R}^d} f \, dv - 1, \\ f\big|_{t=0} = f_0 \geq 0, \quad \int_{\mathbb{R}^d} f_0 \, dvdx = 1. \end{cases}$$

- ... a system called **Vlasov-Dirac-Benney** by Bardos.

- **Loss of derivative ?** The force $-\nabla_x \rho$ is one derivative less regular than f.

- Is Vlasov-Dirac-Benney a good approximation of Vlasov-Poisson when $\varepsilon \to 0$?

More on Vlasov-Dirac-Benney

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \rho \cdot \nabla_v f = 0, \\ \rho = \displaystyle\int_{\mathbb{R}^d} f \, dv - 1, \\ f|_{t=0} = f_0 \geq 0, \quad \displaystyle\int_{\mathbb{R}^d} f_0 \, dvdx = 1. \end{cases}$$

(Local) Existence of solutions is known for

- **analytic** initial data (Cauchy-Kowalevski type result) ;
- in $d = 1$, Sobolev initial data that, for all x, have the shape of one bump [Bardos, Besse 2013], through a water-bag representation ;
- **Penrose stable Sobolev** initial data [DHK, Rousset, to appear]. More to come in a few minutes !

More on Vlasov-Dirac-Benney

There are **unstable equilibria** around which the **linearized** equations have **unbounded unstable spectrum** [Bardos, Nouri 2012]. This reflects the singularity of the equation.

In [DHK, T. Nguyen, preprint], we lead a detailed study of the consequences of this (designing an abstract framework to prove **ill-posedness** properties, which we also applied to the **hydrostatic Euler** equations).

In particular, we prove that the solution map $f_0 \mapsto f(t)$ of Vlasov-Dirac-Benney cannot be $C^{\alpha}_{loc}(H^s, L^2)$ for any $\alpha \in (0,1]$ and $s \geq 0$, even for $t \ll 1$ (we **build** a sequence of solutions making the **Hölder norm blow up**).

Similar to [Métivier 2005] for quasilinear symmetric **non hyperbolic** systems.

Quasineutral limit and large time behavior

• For all $\varepsilon \in (0, 1]$, the Cauchy theory for Vlasov-Poisson is very well understood (Arsenev, Ukai-Okabe, Pfaffelmoser, Schaeffer, Lions-Perthame, Batt-Rein,...), but does not provide useful **uniform** estimates.

Using conservation laws only yield a weak form of the limit with **defect measures** [Brenier, Grenier '94], [Grenier '95].

• The change of variables $(t, x, v) \mapsto (\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v)$ gives the **unscaled** Vlasov-Poisson system (that is, with $\varepsilon = 1$).

Quasineutral limit $\rightarrow$ Large Time Behavior problem

• **The stability or instability of homogeneous equilibria** play a decisive role in the derivation of Vlasov-Dirac-Benney in the quasineutral limit.

Plan of the talk

I. Invalidity of Vlasov-Dirac-Benney in the quasineutral limit

- [Grenier '99], [DHK, Hauray 2015]

II. Validity of Vlasov-Dirac-Benney in the quasineutral limit

- Uniform analytic regularity [Grenier '96]
- Zero-temperature limit [Brenier 2000], [DHK 2011]
- **General Penrose stable data** [DHK, Rousset, to appear]

Nonlinear instability

Penrose instability conditions ensure the linear spectral instability of homogeneous equilibria of Vlasov-Poisson (**two-stream instability**).
In [Guo, Strauss '95], it is proved that spectral instability implies **nonlinear instability** as well.

Theorem 1 (*[DHK, Hauray 2015]*)

Let $\mu(v)$ *be a* **smooth** *Penrose unstable equilibrium. For all* $n \geq 0$, *there is* $\theta > 0$ *such that, for all* $\delta > 0$, *there is a solution* g *of Vlasov-Poisson with*

$$\|g(0) - \mu\|_{W^{n,1}_{x,v}} \leq \delta$$

but

$$\sup_{t \in [0, t_\delta]} \|g(t) - \mu\|_{W^{-n,1}_{x,v}} \geq \theta > 0$$

with $t_\delta = O(|\log \delta|)$ *as* $\delta \to 0$.

(Adaptation of a method of a nonlinear instability method of [Grenier 2000].)

A non-derivation result in the quasineutral limit

Combining the previous **nonlinear instability theorem** and the change of variables $(t, x, v) \mapsto \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$ (**high frequency** regime), we obtain :

Theorem 2 ([DHK, Hauray 2015])

Let $\mu(v)$ be a smooth Penrose unstable equilibrium. For all $n, k \geq 0$, there exists a sequence of solutions (f_ε) such that

$$\|f_\varepsilon(0) - \mu\|_{W^{n,1}_{x,v}} \leq \varepsilon^k,$$

but

$$\liminf_{\varepsilon \to 0} \sup_{t \in [0, \varepsilon |\log \varepsilon|]} \|f_\varepsilon(t) - \mu\|_{W^{-n,1}_{x,v}} > 0.$$

We deduce that the limit equation can not admit $\mu(v)$ as a stationary solution. **In particular Vlasov-Dirac-Benney is not a good approximation near such equilibria.**
(Extended to 3D Vlasov-Maxwell in [DHK, T. Nguyen, preprint].)

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Derivation in Analytic regularity

• In [Grenier '96], it is shown that two-stream instabilities have no effect for solutions with uniform **analytic regularity**.

• Loosely speaking, the principle of his proof is to write the distribution function f_ε as the **superposition of layers of fluids**.

For some fixed probability space $(M, \mu(d\theta))$, write the decomposition

$$f_\varepsilon(t,x,v) = \int_M \rho_\varepsilon^\theta(t,x)\delta_{v=u_\varepsilon^\theta(t,x)}\mu(d\theta),$$

Derivation in Analytic regularity

This leads to the study of a **system of coupled Burgers eq.** :

$$\begin{cases} \partial_t \rho_\varepsilon^\theta + \nabla_x \cdot (\rho_\varepsilon^\theta u_\varepsilon^\theta) = 0, \\ \partial_t u_\varepsilon^\theta + u_\varepsilon^\theta \cdot \nabla_x u_\varepsilon^\theta = -\nabla_x U_\varepsilon, \\ U_\varepsilon - \varepsilon^2 \Delta_x U_\varepsilon = \int_M \rho_\varepsilon^\theta \, \mu(d\theta) - 1. \end{cases}$$

Theorem 3 (*[Grenier '96]*)

Assume that for f_0 with analytic regularity ($\|\cdot\|$ is a norm that is analytic in x, continuous in v)

$$\|f_{\varepsilon,0} - f_0\| \to 0.$$

Then there is $T > 0$ such that f_ε weakly converges on $[0, T]$ to a weak solution to Vlasov-Dirac-Benney with initial condition f_0.

In [DHK, Iacobelli, preprints] : still true for **exponentially small but rough** perturbations of such data ($d \leq 3$). Uses quantitative Wasserstein stability estimates [Loeper 2006], [Hauray 2013].

Derivation in stable cases ?

• Is it possible to say something under an assumption of **Penrose stability** on the initial condition ?

• The first result in this direction is due to [Brenier 2000] where the **Modulated Energy method** was introduced (see also [Yau '91], [Golse 2000]) .

For **monokinetic data**

$$f(t,x,v) = \rho(t,x)\delta_{v=u(t,x)},$$

note that f satisfies Vlasov-Dirac-Benney iff (ρ, u) satisfies the **Shallow Water equations** (isentropic compressible Euler with $\gamma = 2$) :

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t u + u \cdot \nabla_x u + \nabla_x \rho = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases}$$

Derivation in the zero–temperature regime

- Consider

$$f_{0,\varepsilon} \rightharpoonup \rho_0(x)\delta_{v=u_0(x)}$$

(zero-temperature limit, extremal case of a Maxwellian (**stable**)).

- Following [Brenier 2000], introduce

$$\mathcal{H}_\varepsilon(t) := \frac{1}{2}\int f_\varepsilon |v-u(t,x)|^2\,dvdx + \frac{1}{2}\int (U_\varepsilon - \rho(t,x))^2\,dx + \frac{\varepsilon^2}{2}\int |E_\varepsilon(t,x)|^2\,dx.$$

where (ρ, u) solves the **Shallow Water system** on $[0,T]$.

- One proves [DHK 2011] that

$$\frac{d}{dt}\mathcal{H}_\varepsilon(t) \lesssim \mathcal{H}_\varepsilon(t) + o(1)$$

so that roughly

$$f_{0,\varepsilon} \rightharpoonup \rho_0(x)\delta_{v=u_0(x)} \implies \forall t \in [0,T], f_\varepsilon(t) \rightharpoonup \rho(t,x)\delta_{v=u(t,x)}.$$

$\implies$ **Derivation of Shallow Water**.

May one generalize the modulated energy method ?

• A natural idea would be to adapt this method to handle other stable initial conditions.
• [DHK, Hauray 2015] : works for stationary $\mu(v)$ satisfying

$$\nearrow \text{ on } (-\infty, 0] \ , \quad \searrow \text{ on } [0, +\infty) \text{ and } \textbf{even}$$

• Fails to handle other stable initial data one would like to consider, for **symmetry** and **rigidity** reasons.
The modulated energy method requires that the solution of the limit system is the **minimizer of some entropy** and thus satisfies

$$f \equiv g(t, x, -|v - v(t, x)|^2).$$

We prove that such solutions of Vlasov-Dirac-Benney are necessarily of the form $g(-|v - \overline{v}|^2)$...
• Implies one can not hope to use the modulated energy method...

Derivation result for stable data

- Another method is needed to handle general stable data.

- We say that $\mathbf{f}(v)$ satisfies the c_0 **Penrose stability condition** if

$$\inf_{(\gamma,\tau,\eta)\in\mathbb{R}_*^+\times\mathbb{R}\times\mathbb{R}_*^d}\left|1-\int_0^{+\infty}e^{-(\gamma+i\tau)s}\,\frac{i\eta}{1+|\eta|^2}\cdot\int_{\mathbb{R}^d}e^{-is\eta\cdot v}(\nabla_v\mathbf{f})(v)\,dv\,ds\right|$$
$$\geq c_0.$$

Recall that this also appears for **Landau Damping** [Mouhot, Villani 2011].
In particular, this is satisfied by functions with the shape of one bump.

- Introduce also for $k\in\mathbb{N}$, $r\in\mathbb{R}$, the weighted Sobolev norms

$$\|f\|_{\mathcal{H}_r^k}:=\left(\sum_{|\alpha|+|\beta|\leq k}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}(1+|v|^2)^r|\partial_x^\alpha\partial_v^\beta f|^2\,dvdx\right)^{1/2}$$

and the regularity indices

$$k_0=4+d,\quad r_0=\max(d,2+\frac{d}{2}).$$

Derivation result for stable data

Theorem 4 (*[DHK, Rousset, to appear]*)

Let $2m > k_0$, $2r > r_0$. Let $M_0 > 0, c_0 > 0$. Assume that for all $\varepsilon \in (0,1]$, $\|f_{0,\varepsilon}\|_{\mathcal{H}^{2m}_{2r}} \leq M_0$ and for all $x \in \mathbb{T}^d$, $f_{0,\varepsilon}(x,\cdot)$ satisfies the c_0 Penrose stability condition. Assume that $f_{0,\varepsilon} \to f_0$ in L^2. Then there is $T > 0$ such that

$$\sup_{[0,T]} \|f_\varepsilon(t) - f(t)\|_{L^2} \to 0,$$

where $f(t)$ satisfies Vlasov-Dirac-Benney with initial data f_0.

An example of admissible initial data is given by smooth local Maxwellians

$$M(x,v) = \frac{\rho(x)}{(2\pi T(x))^{d/2}} \exp\left(-\frac{|v-u(x)|^2}{T(x)}\right).$$

As a by-product we get **well-posedness (i.e. existence + uniqueness) in the class of such data for Vlasov-Dirac-Benney**.

Sketch of the proof

We have $\|f_{0,\varepsilon}\|_{\mathcal{H}^{2m}_{2r}} \leq M_0$. Introduce

$$\boxed{\mathcal{N}_{2m,\,2r}(t, f_\varepsilon) := \|f_\varepsilon\|_{L^\infty((0,t),\mathcal{H}^{2m-1}_{2r})} + \|\rho_\varepsilon\|_{L^2((0,t),H^{2m})}}$$

with $\rho_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon \, dv$. The main task is to find $T > 0$, $R > 0$ such that

$$\forall \varepsilon \in (0,1], \quad \sup_{[0,T]} \mathcal{N}_{2m,\,2r}(t, f_\varepsilon) \leq R.$$

The proof is based on a bootstrap argument. By a standard energy estimate, we see that the key quantity to be controlled is actually $\|\rho_\varepsilon\|_{L^2((0,t),H^{2m})}$.

Sketch of the proof

• **Natural idea** : up to commutators, $\partial_{\mathbf{x}}^{\mathbf{2m}} f_\varepsilon$ evolves according to the linearized equation around f_ε, that is

$$\partial_t \partial_{\mathbf{x}}^{\mathbf{2m}} f_\varepsilon + v \cdot \nabla_x \partial_{\mathbf{x}}^{\mathbf{2m}} f_\varepsilon + \partial_{\mathbf{x}}^{\mathbf{2m}} E_\varepsilon \cdot \nabla_v f_\varepsilon + E_\varepsilon \cdot \nabla_v \partial_{\mathbf{x}}^{\mathbf{2m}} f_\varepsilon = S,$$

where S should involve remainder terms only.

• When $f_\varepsilon \equiv \mu(v)$ does not depend on t and x, then $E_\varepsilon = 0$ and the linearized equation reduces to

$$\begin{aligned} &(\partial_t + v \cdot \nabla_x)\, g + E_g \cdot \nabla_v \mu(v) = S, \\ &E_g = -\nabla_x (I - \varepsilon^2 \Delta)^{-1} \rho_g. \end{aligned}$$

This case was studied in [Mouhot, Villani 2011] in view of Landau Damping.

Sketch of the proof

By the method of characteristics,

$$g(t,x,v) = g_0(x - tv, v) - \int_0^t E_g(x-(t-s)v)\cdot\nabla_v\mu(v)\,ds + \mathcal{S}$$

and thus, integrating in v, we obtain an integral equation for $\rho_g = \int_{\mathbb{R}^d} g\,dv$:

$$\rho_g(t,x) = \int_0^t \int_{\mathbb{R}^d} \nabla_x (I-\varepsilon^2\Delta)^{-1}\rho_g(x-(t-s)v)\cdot\nabla_v\mu(v)\,dvds + \mathcal{S} + \mathcal{S}_0$$

By **Fourier analysis** one may estimate ρ_g in $L^2_{t,x}$, **under a Penrose stability condition** for $\mu(v)$.

Sketch of the proof

• **However**, when applying this strategy, there are subprincipal terms which involve $2m$ derivatives of f :

$$\partial_x E_\varepsilon \cdot \nabla_v \partial_x^{2m-1} f_\varepsilon .$$

• Applying more general vector fields would also generate bad subprincipal terms. Instead : Consider powers of relevant **second order operators** (with coefficients depending on f_ε itself), yielding a family $(f_{I,J})_{I,J \in \{1,\cdots,d\}^m}$ that satisfy two key properties.

- We have the control

$$\|\rho_\varepsilon\|_{H^{2m}} \lesssim \sum_{I,J} \| \int_{\mathbb{R}^d} f_{I,J} \, dv \|_{L^2} + R$$

  where R is a good remainder.

- $f_{I,J}$ roughly satisfies

  $$\partial_t f_{I,J} + v \cdot \nabla_x f_{I,J} + E_\varepsilon \cdot \nabla_v f_{I,J} + E_{f_{I,J}} \cdot \nabla_v f_\varepsilon + \text{ (zero order terms)} = S_{I,J},$$

  where $S_{I,J}$ is a good remainder.

Sketch of the proof

In dimension one, only one operator is needed :

$$L := \partial_{xx} + \varphi \partial_x \partial_v + \psi \partial_{vv},$$

with (φ, ψ) satisfying the system

$$\begin{cases} (\partial_t + v \cdot \nabla_x + E \cdot \nabla_v)\varphi = \partial_x E + \text{ (zero order terms)}, & \varphi|_{t=0} = 0 \\ (\partial_t + v \cdot \nabla_x + E \cdot \nabla_v)\psi = \varphi \partial_x E + \text{ (zero order terms)}, & \psi|_{t=0} = 0 \end{cases}$$

this system being designed to kill the bad subprincipal term.

In dimension d, we obtain in a similar way some relevant operators $(L_{i,j})_{1 \leq i,j \leq d}$, and we define

$$f_{I,J} := L_{i_1,j_1} \cdots L_{i_m,j_m} f.$$

Sketch of the proof

We thus study

$$(\partial_t + v \cdot \nabla_x + E_\varepsilon \cdot \nabla_v)\, g + E_g \cdot \nabla_v f_\varepsilon = S.$$

As f_ε depends on x, the force field E_ε is not trivial.

However, we can use a **near identity change of variables to straighten the vector field** and come down to the equation

$$(\partial_t + \Phi(t,x,v) \cdot \nabla_x)\, g + E_g \cdot \nabla_v f_\varepsilon = S$$

where $\Phi(t,x,v)$ satisfies the **Burgers equation**

$$\partial_t \Phi + \Phi \cdot \nabla_x \Phi = E, \quad \Phi|_{t=0} = v,$$

and is close to v for small times.

Integrating along characteristics and integrating in v, we end up with the study, for **small times**, of...

Sketch of the proof

...the integral equation

$$h = K_{\nabla_v f_{0,\varepsilon}} (I - \varepsilon^2 \Delta)^{-1} h + R,$$

with

$$K_{\nabla_v f_{0,\varepsilon}}(G) = \int_0^t \int_{\mathbb{R}^d} (\nabla_{\mathbf{x}} G)(s, x - (t-s)v) \cdot \nabla_v f_{0,\varepsilon}(x, v) \, dv \, ds.$$

Note that $K_{\nabla_v f_{0,\varepsilon}}$ may seem to feature a loss of derivative. However, we have

Proposition 1

$K_{\nabla_v \mu}$ is a bounded operator on L^2 if μ is smooth and decaying in v.

This is an effect in the spirit of **averaging lemmas** ([Golse, Lions, Perthame, Sentis '88]).

$$h = K_{\nabla_v f_{0,\varepsilon}} (I - \varepsilon^2 \Delta)^{-1} h + R$$

Let $\gamma > 0$ to be chosen. We can relate $e^{-\gamma t} K_{\nabla_v f_{0,\varepsilon}} (e^{\gamma t} (I - \varepsilon^2 \Delta)^{-1} \cdot)$ to a **semi-classical pseudodifferential operator**, of symbol

$$a(\gamma, \tau, x, \eta) := \int_0^{+\infty} e^{-(\gamma + i\tau)s} \, \frac{i\eta}{1 + |\eta|^2} \cdot \int_{\mathbb{R}^d} e^{-is\eta \cdot v} \nabla_v f_{0,\varepsilon}(x, v) \, dv \, ds,$$

and we rewrite the integral equation as

$$Op^{\gamma,\varepsilon}_{(1-a)}(e^{-\gamma t} h) = \mathcal{R}.$$

The c_0 Penrose condition is precisely $\inf |1 - a| \geq c_0$ and therefore implies the **ellipticity of the symbol** associated to the operator we want to invert.

Sketch of the proof

$$Op^{\gamma,\varepsilon}_{(1-a)}(e^{-\gamma t}h) = \mathcal{R}.$$

We can finally use a **semi-classical pseudodifferential calculus with parameter** γ in order to invert $Op^{\gamma,\varepsilon}_{(1-a)}$ up to a **small remainder**.
More precisely we have the general estimate

$$\|Op^{\gamma,\varepsilon}_b Op^{\gamma,\varepsilon}_c u - Op^{\gamma,\varepsilon}_{bc} u\|_{L^2} \lesssim \frac{1}{\gamma}\|b\|\|c\|\|u\|_{L^2}.$$

that we apply to

$$b = \frac{1}{1-a}, \quad c = 1-a, \quad u = e^{-\gamma t}h,$$

which roughly gives

$$\|h\|_{L^2} \leq \|Op^{\gamma,\varepsilon}_{\frac{1}{1-a}}\mathcal{R}\|_{L^2} + \frac{1}{\gamma}\left\|\frac{1}{1-a}\right\|\|1-a\|\|h\|_{L^2}$$

Choosing $\gamma \gg 1$ yields an estimate for h, in $L^2_{t,x}$.
This allows to close the bootstrap argument.

Merci pour votre attention !