Stochastic homogenization of interfaces moving by oscillatory normal velocity

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Outline

1. Average behavior of moving interfaces
   - Moving interfaces and Hamilton Jacobi equations
   - Known results and main contribution

2. Homogenization of oscillating fronts
   - The macroscopic metric problem
   - Homogenization of the metric problem
   - Effective Hamiltonian

3. Open questions and future work
Average behavior of moving interfaces

Figure 1: Oscillating interface (red) and its average behavior (black).

\[
\Gamma_t^\varepsilon : V(x/\varepsilon, t) = a(x/\varepsilon)n^\varepsilon(x/\varepsilon, t) + \kappa(x/\varepsilon, t)
\]

Question

Understand the average behavior, as \(\varepsilon \to 0\), of \(\Gamma_t^\varepsilon\),

\[
\Gamma_t^\varepsilon := \{ x \in \mathbb{R}^n : u^\varepsilon(x, t) = 0 \}.
\]
Homogenization of oscillating fronts

Problem

**Average behavior of solutions of Hamilton-Jacobi equations of the form**

\[
\begin{cases}
    u_t^\varepsilon + a\left(\frac{x}{\varepsilon}, \omega\right) |Du^\varepsilon| = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
    u^\varepsilon(x, \omega, 0) = u_0(x) & \text{in } \mathbb{R}^n.
\end{cases}
\]  

\((HJ_\varepsilon)\)

The moving interface is \(\Gamma^\varepsilon(\omega, t) = \{x \in \mathbb{R}^n; u^\varepsilon(x, \omega, t) = 0\} \).

Assumptions

1. \(\omega\) is an element of a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) and describes a stationary-ergodic environment.

2. \(a(\cdot, \omega)\) changes sign, hence the corresponding Hamiltonian

\[H(x, p, \omega) = a(x, \omega)|p|\]

is non-coercive, and non-convex.
Assumptions on the velocity

Probability space \((\Omega, \mathcal{F}, \mathbb{P})\), endowed with an *ergodic group of measure preserving transformations* \((\tau_z)_{z \in \mathbb{R}^n}\). Assume \(a(\cdot, \cdot)\) satisfies

(A1) is stationary with respect to the group \((\tau_z)_{z \in \mathbb{R}^n}\), that is, for every \(y, z \in \mathbb{R}^n\) and \(\omega \in \Omega\),

\[
a(y, \tau_z \omega) = a(y + z, \omega).
\]

(A2) is bounded and equi-Lipschitz continuous, with Lipschitz constant \(L > 0\), that is, for every \(y, z \in \mathbb{R}^n\) and \(\omega \in \Omega\),

\[
|a(y, \omega) - a(z, \omega)| \leq L|y - z|.
\]

(A3) for any \(\omega \in \Omega\) the function \(a(\cdot, \omega) : \mathbb{R}^n \to \mathbb{R}\) changes signs.
Periodic environments

**Figure 2:** Periodic configurations of environments: the velocity $a(\cdot)$ is positive inside white regions and negative on black regions.
Random environments

**Figure 3:** Random configurations of environments: the velocity $a(\cdot)$ is positive inside white regions and negative on black regions.
The above Hamiltonian is coercive and convex if

$$\inf a(\cdot, \omega) = a_0 > 0.$$
Known results II

- Coercive Hamiltonians: Lions-Papanicolau-Varadhan ’88, Evans ’89, Ishii ’99
- Non-coercive, non-convex Hamiltonians: Bhattacharya-Craciun ’03, Cardaliaguet-Lions-Souganidis ’09.
Homogenization of oscillating fronts

Theorem (Main result)

There exists an event of full probability $\tilde{\Omega} \subseteq \Omega$ such that, for each $\omega \in \tilde{\Omega}$, the unique solution $u^\varepsilon = u^\varepsilon(\cdot, \omega)$ of $(HJ^\varepsilon)$, satisfies that, for each $R, T > 0$,

$$u^\varepsilon(\cdot, \omega) \overset{*}{\rightharpoonup} \bar{u} \text{ as } \varepsilon \to 0 \text{ in } L^\infty(B_R \times (0, T)) \text{ a.s. in } \Omega, \quad (2)$$

where the weak limit $\bar{u}$ is given by the convex combination

$$\bar{u} = \theta_0 u_0 + \sum_{i \in I} \theta_i \bar{u}_i. \quad (3)$$

with

$$\begin{cases} \bar{u}_{i,t} + H_i(D\bar{u}_i) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ \bar{u}_i(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n. \end{cases} \quad (HJ_i)$$
Average behavior of moving interfaces

Figure 4: Oscillating interface (red) and its average behavior (black).

Ansatz

\[ u^\varepsilon(x, t, \omega) = \overline{u}(x, t) + \varepsilon w \left( x, \frac{x}{\varepsilon}, \omega \right) + o(\varepsilon^2). \]

where:
- \( \overline{u}(x, t) \) is the averaged profile
- \( w \left( x, \frac{x}{\varepsilon}, \omega \right) \) is the corrector at scale \( \varepsilon \)
Formal Asymptotic Expansion

Ansatz

\[ u^\varepsilon(x, t) = \bar{u}(x, t) + \varepsilon w \left( x, \frac{x}{\varepsilon}, \omega \right) + o(\varepsilon^2). \]

Plugging the expansion in \((HJ_\varepsilon)\) and identifying the terms in front of powers of \(\varepsilon\)

\[ \bar{u}_t(x, t) + a(x/\varepsilon, \omega) |D_x \bar{u}(x, t) + D_y w(x/\varepsilon, t, \omega)| = 0. \]

Problem (Step 1 - Effective Hamiltonian / Macroscopic problem)

*For each* \(p \in \mathbb{R}^n\) *there exists a constant* \(\mu_i = \bar{H}_i(p)\) *such that*

\[ a(y, \omega)|p + Dw_i| = \mu_i \text{ in } U_i = cc\{a > 0\}. \]

Problem (Step 2 - Convergence / Sublinear decay at infinity)

\[ \lim_{|y| \to \infty} \frac{w_i(y, \omega; p)}{|y|} = \lim_{|y| \to \infty} \frac{w_i(y, \omega; 0) - p \cdot y}{|y|} = 0. \]
The metric problem

Fix $\omega \in \Omega$. Let $U(\omega)$ be an unbounded, connected component of $\{a(\cdot, \omega) > 0\}$. For each $z \in U(\omega)$, we consider the metric problem

\[
\begin{cases}
  a(y, \omega)|Dm| = 1 & \text{in } U(\omega) \setminus \{z\}, \\
  m(\cdot, z, \omega) = 0 & \text{at } \{z\}.
\end{cases}
\]

(4)

Proposition (Well-posedness)

The metric problem has a maximal subsolution

\[
m(y, z, \omega) := \sup \left\{ w(y, \omega) : w(\cdot, \omega) \in L, \ w(z, \omega) = 0, \ a(y, \omega)|Dw| \leq 1 \text{ in } U(\omega) \right\},
\]

(5)

where $L$ is the class of globally Lipschitz functions.
A subadditive quantity

**Lemma (Pseudo-Metric)**

\[ m(\cdot, \cdot, \omega) : U(\omega) \times U(\omega) \rightarrow [0, \infty) \text{ is symmetric, and subadditive, i.e. for all } x, y, z \in U(\omega) \]

\[ m(x, z, \omega) \leq m(x, y, \omega) + m(y, z, \omega), \]

**and for each fixed** \( z \in U(\omega), \)

\[ \lim_{d(y, \partial U(\omega)) \to 0} m(y, z, \omega) = +\infty. \]

**Proof.** Define the barrier

\[ b^\delta(y, \omega) = -1/L \log a(y, \omega) + c \]

Then \( b^\delta \) is an admissible test function for \( m \), with \( \| Db^\delta(\cdot, \omega) \|_{L^\infty(U^\delta(\omega))} \leq 1/\delta: \)

\[ |Db^\delta(\cdot, \omega)| = \left| \frac{1}{L} \frac{Da(\cdot, \omega)}{a(\cdot, \omega)} \right| \leq \frac{1}{\delta}. \]
Behavior of the maximal subsolution

**Lemma (Lipschitz Continuity)**

There exists a constant $C(\omega)$ s.t. for each $\delta \in (0, \delta_0)$, for all $y_1, y_2 \in U^\delta(\omega)$,

$$|m(y_1, z, \omega) - m(y_2, z, \omega)| \leq \frac{C(\omega)}{\delta} |y_1 - y_2|,$$

(6)

**Proof.** Let $w$ be an admissible function. Observe that

$$\delta |Dw| \leq a(y, \omega) |Dw| \leq 1 \quad \text{in} \quad U^\delta(\omega) = \{ y \in U(\omega) : a(y, \omega) > \delta \}.$$

For all $y_1, y_2 \in U^\delta(\omega)$,

$$w(y_1, \omega) - w(y_2, \omega) \leq \inf_{\gamma} \int_0^1 |Dw(\gamma(s))||\gamma'(s)| \, ds$$

$$\leq \frac{1}{\delta} d_{U^\delta(\omega)}(y_1, y_2) \leq \frac{C(\omega)}{\delta} |y_1 - y_2|.$$
Behavior of the maximal subsolution

Lemma (Lipschitz Continuity)

There exists a constant $C(\omega)$ s.t. for each $\delta \in (0, \delta_0)$, for all $y_1, y_2 \in U^\delta(\omega)$,

$$|m(y_1, z, \omega) - m(y_2, z, \omega)| \leq \frac{C(\omega)}{\delta} |y_1 - y_2|,$$

and, there exists a deterministic constant $C > 0$ such that, a.s. in $\omega$, for all $y_1 \in U^\delta(\omega)$,

$$\limsup_{y_2 \in U^\delta(\omega)} \frac{|m(y_1, z, \omega) - m(y_2, z, \omega)|}{|y_1 - y_2|} \leq \frac{C}{\delta}.$$

Idea.

$$\limsup_{y_2 \in U^\delta(\omega)} \frac{d_{U^\delta(\omega)}(y_1, y_2)}{|y_1 - y_2|} \leq \frac{C}{\delta}.$$
Theorem

There exist an event of full probability $\tilde{\Omega} \subseteq \Omega$ and $\bar{m} : \mathbb{R}^n \rightarrow \mathbb{R}$, such that, for every $\omega \in \tilde{\Omega}$, $y, z \in \mathbb{R}^n$ and every $\delta \in (0, \delta_0)$,

$$\limsup_{t \to \infty} \frac{m(ty, tz, \omega)}{t} = \liminf_{t \to \infty} \frac{m(ty, tz, \omega)}{t} = \bar{m}(y - z).$$
Homogenization around the origin I

Lemma

Suppose that $0 \in U(\omega)$ a.s. in $\omega$. There exists a set of full probability $\Omega_{y, \delta}^\mu, \delta \in \mathcal{F}$ and a Lipschitz extension of $m$, $\tilde{m}^\delta(\cdot, \cdot, \omega) : \mathbb{R}^+ y \times \mathbb{R}^+ y \to \mathbb{R}$ s.t., for $\omega \in \Omega_{y, \delta}^\mu, \delta$,

$$
\lim_{t \to \infty} \frac{1}{t} \tilde{m}^\delta(ty, sy, \omega) =: \overline{m}^\delta(y, \omega).
$$

Proof. For each $t \geq 0$, consider the entrance and exit times in a hole

$$
t_* := \sup \left\{ 0 \leq s \leq t : sy \in \overline{U}^\delta(\omega) \right\} \quad \text{and} \quad t^* := \inf \left\{ s \geq t : sy \in \overline{U}^\delta(\omega) \right\}.
$$

Let, for $t = (1 - \alpha) t_* + \alpha t^*$, $s = (1 - \beta)s_* + \beta s^*$,

$$
\tilde{m}^\delta(ty, sy, \omega) = (1 - \alpha) (1 - \beta) m(t_* y, s_* y, \omega) + \beta m(t_* y, s^* y, \omega) + \alpha (1 - \beta) m(t^* y, s_* y, \omega) + \beta m(t^* y, s^* y, \omega).
$$

(10)
The random process \( Q : \mathcal{I} \rightarrow L^1(\Omega, \mathbb{P}) \) defined given by

\[
Q([s, t])(\omega) = \tilde{m}^\delta(ty, sy, \omega).
\]

is continuous and subadditive on \((\Omega, \mathcal{F}, \mathbb{P})\) w.r. to the semigroup \( (\tau_{ty})_{t \geq 0} \).

In view of the subadditive ergodic theorem, there exists an event \( \tilde{\Omega}^\mu,\delta \) of full probability such that, for all \( \omega \in \tilde{\Omega}^\mu,\delta \), there exists

\[
\overline{m}^\delta(y, \omega) := \lim_{t \to \infty} \frac{1}{t} \tilde{m}^\delta(ty, 0, \omega) = \lim_{t \to \infty} \frac{1}{t} m(ty, 0, \omega).
\]
**Lemma**

There exists a set of full probability $\Omega^{\mu,\delta} \in \mathcal{F}$ and $\overline{m}^{\delta} : \mathbb{R}^n \to \mathbb{R}$ such that, for every $\omega \in \Omega^{\delta}$ and $y \in \mathbb{R}^n$,

$$
\lim_{t \to \infty} \frac{1}{t} m(ty, 0, \omega) = \overline{m}^{\delta}(y).
$$  \hspace{1cm} (11)

$m^{\delta}$ is 1–positively homogeneous, C-Lipschitz continuous and subadditive.

**Proof.** To show that $\overline{m}^{\delta}(y, \cdot)$ is deterministic, we check that, for all $z \in \mathbb{R}^n$,

$$
\overline{m}^{\delta}(y, \omega) = \overline{m}^{\delta}(y, \tau_z \omega).
$$  \hspace{1cm} (12)
The effective Hamiltonian

The effective Hamiltonian $\overline{H}(p, \omega)$ corresponding to $U(\omega)$ is given by

$$\overline{H}(p, \omega) = \inf \left\{ \mu > 0 : \exists w(\cdot, \omega) \in S^+ \text{ s.t. } a(y, \omega) |p + Dw| \leq \mu \text{ in } U(\omega) \right\}.$$ 

where $S^+ := \left\{ w \in \mathcal{L}_{loc} : \liminf_{|y| \to \infty} \frac{w(y)}{|y|} \geq 0 \right\}$.

Proposition

There exists a set of full probability $\tilde{\Omega} \subseteq \Omega$ such that, for every $\omega \in \tilde{\Omega}$ and $p \in \mathbb{R}^n$,

$$\overline{H}(p) = \overline{H}(p, \omega) = \inf_{w \in S^+} \left[ \text{ess sup}_{y \in U(\omega)} \left( a(y, \omega) |p + Dw| \right) \right].$$
The sublinear decay

Proposition (The deterministic effective Hamiltonian)

For each $p \in \mathbb{R}^n$ and $\omega \in \tilde{\Omega}$,

$$\overline{H}(p) = \inf \left\{ \mu > 0 : \liminf_{|y| \to \infty, y \in U(\omega)} \frac{m(y, z, \omega) - p/\mu \cdot (y - z)}{|y|} \geq 0 \right\}. \quad (13)$$
Main Homogenization Result

**Theorem (Convergence)**

\[
\lim_{\varepsilon \to 0} \sup_{x \in U_{i,\varepsilon}^\delta(\omega) \cap B_R} |u^\varepsilon(x, \omega; p) + \bar{u}_i(x)| = 0.
\]

where \( \bar{u}_i \) solves \((HJ_i), \) and \( u^\varepsilon \rightharpoonup \bar{u} := \theta_0 u_0 + \sum_{i \in I} \theta_i \bar{u}_i \)

**Step 1.** The unique solution \( w^\varepsilon = w^\varepsilon(\cdot, \omega; p) \) of

\[
w^\varepsilon + a \left( \frac{x}{\varepsilon}, \omega \right) |p + Dw^\varepsilon| = 0 \quad \text{in} \quad \mathbb{R}^n \times \Omega \tag{14}
\]

satisfies, for all \( \delta > 0 \) sufficiently small and \( R > 0, \)

\[
\lim_{\varepsilon \to 0} \sup_{x \in U_{i,\varepsilon}^\delta(\omega) \cap B_R} |w^\varepsilon(x, \omega; p) + \bar{H}_i(p)| = 0.
\]

Conclude using perturbed test function argument.

**Step 2.**

\[
w^\varepsilon \rightharpoonup \bar{w} := \sum_{i \in I} \theta_i \bar{H}_i(p).
\]
Open questions and future work

- **State constraint HJs - perforated domains** (in progress - with P. Cardaliaguet and P.E. Souganidis)
  \[
  \begin{aligned}
  u^\varepsilon + H(x, Du, \omega) &= 0 \quad \text{in } U(\omega) \\
  u^\varepsilon &= g \quad \text{on } \partial U(\omega).
  \end{aligned}
  \]

- **Viscous case** (in progress - with I. Kim)
  \[
  \begin{aligned}
  u_t^\varepsilon - \varepsilon \Delta u + a \left( \frac{x}{\varepsilon}, \omega \right) |Du^\varepsilon| &= 0 \quad \text{in } \mathbb{R}^n \times (0, \infty) \\
  u^\varepsilon &= u_0 \quad \text{in } \mathbb{R}^n.
  \end{aligned}
  \]