Some Interesting Phenomena about Boltzmann Equation without Angular Cutoff

Tong YANG

Department of Mathematics
City University of Hong Kong

November 29, 2013

Gas Motions:

Many famous equations of motion, which have been derived by focusing on different aspects of gases and fluids in different physical scales. Most of them are classical, dating back to the 19th century or earlier.

In the macroscopic scale where the gas and fluid are regarded as a continuum, their motion is described by the macroscopic quantities such as macroscopic mass density, bulk velocity, temperature, pressure, stresses, heat flux and so on.

The Euler and Navier-Stokes equations, either compressible or incompressible, are the most famous equations among governing equations proposed so far in fluid dynamics.
Both systems of Euler and Navier-Stokes consist of conservation of mass, momentum and energy which are the typical examples of conservation laws.

The extreme contrary is the microscopic scale where the gas, fluid, and hence any matter, are viewed as a many-body system of microscopic particles (atom/molecule). Thus, the motion of the system is governed by the coupled Newton equations, within the framework of the classical mechanics.

Note that the number of the involved equations is $6N$ if the total number of the microscopic particles is N.

Although the Newton equation is the first principle of the classical mechanics, it is not of practical use because the number of the equations is so enormous ($N \sim$ the Avogadro number 6×10^{23}) that it is hopeless to specify all the initial data, and we must rely on statistics.

On the other hand, the macroscopic (fluid dynamical) quantities mentioned above are related to statistical average of quantities depending on the microscopic state.

Thus, the kinetic theory that gives the mesoscopic description of the gas and fluid is a key theory that links the microscopic and macroscopic scales.

The Boltzmann equation, which was derived by Boltzmann in statistical physics in 1872, is the most classical and fundamental equation in the mesoscopic kinetic theory.

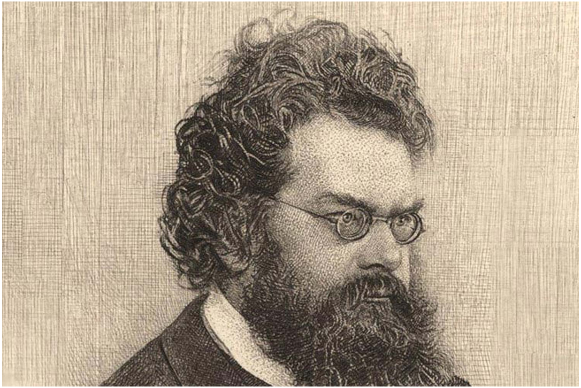

Figure 1 : Boltzmann

Boltzmann equation for non-equilibrium gas is about the time evolution of

$$f = f(t, x, v) \quad t \in \mathbb{R}^+, x \in \mathbb{R}^3, v \in \mathbb{R}^3,$$

which stands for the number density function of particles having position x and velocity v at time t:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f)$$

Here Q, *the collision operator* describes the binary elastic collision of molecules.

$$Q(g,f)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B\left(v-v_*,\sigma\right)\left\{g(v_*')f(v')-g(v_*)f(v)\right\}d\sigma dv_*,$$

where

$$v'=\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma,\ \ v_*'=\frac{v+v_*}{2}-\frac{|v-v_*|}{2}\sigma,\ \ \sigma\in\mathbb{S}^2,$$

satisfying the conservation of momentum and energy.

$$\frac{v-v_*}{|v-v_*|} \cdot \sigma = \cos\theta, \quad \theta \in [0, \pi/2],$$

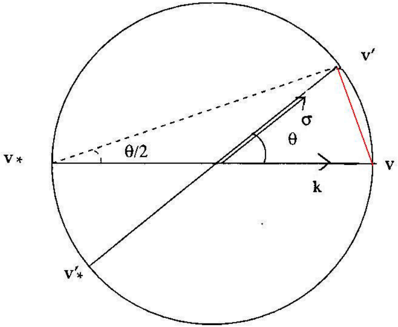

Figure 2 : post and pre collisional velocities

Two classical models:

- Hard sphere gas:

$$B(v - v_*, \theta) = q_0 |v - v_*| |\cos\theta|,$$

- Potential of inverse power law with $U(r) \sim r^{-\rho}$:

$$B(v - v_*, \theta) \sim |v - v_*|^\gamma |\theta|^{-2-2s} b_0(\theta),$$
$$\gamma = 1 - \frac{4}{\rho}, \quad s = \frac{1}{\rho},$$

hard potential: $\rho > 4$, Maxwellian molecule: $\rho = 4$, and soft potential: $1 < \rho < 4$.

Two classical properties of Q:

(i) $Q(f,f) = 0$ if and only if

$$f = \frac{\rho}{(2\pi T)^{3/2}} \exp\Big(-\frac{|v-u|^2}{2T}\Big),$$

(ii) A function $\phi(\xi)$ is called a *collision invariant* of Q if

$$\langle \phi, Q(f,f)\rangle = 0, \qquad \text{for all } F,$$

Q has five collision invariants

$$\phi_0 = 1, \quad \phi_i = v_i \ (i = 1,2,3) \quad \phi_4 = |v|^2,$$

H-Theorem

$$H(t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \log f dx dv$$

Then

(1) $$\frac{dH(t)}{dt} \leq 0$$

(2) $$f \to \frac{\rho}{(2\pi RT)^{n3/2}} \exp\left(-\frac{|v-u|^2}{2RT}\right) \qquad (t \to \infty)$$

• Many important mathematical thoeries are based on the Grad's angular cutoff assumption: B integrable and gain and loss parts in Q can be separately considered.

• Recent progress has been made on the well-posedness theories without angular cutoff assumption

We will focus on:

I. Regularizing effect without angular cutoff
II. Large time behavoir
III. Spectrum analysis on the linearized collision operator

Boltzmann equation without angular cutoff

• Recall the well-posedness theories for perturbative solutions around a global Maxwellian

Normalized Maxwellian

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Since $Q(\mu, \mu) = 0$, by setting $f = \mu + \sqrt{\mu} g$, we have

$$\begin{aligned} & Q(\mu + \sqrt{\mu} g,\, \mu + \sqrt{\mu}\, g) \\ = & \, Q(\mu,\, \sqrt{\mu}\, g) + Q(\sqrt{\mu}\, g,\, \mu) + Q(\sqrt{\mu}\, g,\, \sqrt{\mu}\, g). \end{aligned}$$

Define

$$\Gamma(g,\ h) = \mu^{-1/2} Q(\sqrt{\mu}\, g,\ \sqrt{\mu}\, h).$$

Then the linearized Boltzmann collision operator:

$$Lg = \Gamma(\sqrt{\mu}\,,\ g) + \Gamma(g,\ \sqrt{\mu}\,),$$

Equation for the perturbation g:

$$g_t + v \cdot \nabla_x g = Lg + \Gamma(g,\ g), \quad t > 0\,.$$

Theorem (Alexandre-Morimoto-Ukai-Xu-Y., 2011,12)
[Existence]
For $0 < s < 1$ and $\gamma > \max\{-3, -3/2 - 2s\}$:
1. When $\gamma + 2s \leq 0$, wellposed in $X = \tilde{\mathcal{H}}^6_6$;
2. When $\gamma + 2s > 0$, well-posed in $X = H^6_l$ with $l > 3/2 + 2s + \gamma$.

$$\|f\|^2_{H^k_\ell} = \sum_{|\alpha|+|\beta|\leq k} \|W_\ell \partial^\alpha_\beta f\|^2_{L^2}, \quad \|f\|^2_{\tilde{\mathcal{H}}^k_\ell} = \sum_{|\alpha|+|\beta|\leq k} \|\tilde{W}_{\ell-|\beta|}\partial^\alpha_\beta f\|^2,$$

with $W_\ell = (1+|v|^2)^{\ell/2}$, $\tilde{W}_\ell = (1+|v|^2)^{|s+\gamma/2|\,\ell/2}$, $\partial^\alpha_\beta = \partial^\alpha_x \partial^\beta_v$.

Remark. Similar result was obtained by Gressman-Strain by using a different approach in the same period.

I. Regularizing effect without angular cutoff

Recall the potential of inverse power laws:

$$B(v - v_*, \theta) \sim |v - v_*|^{\gamma} |\theta|^{-2-2s} b_0(\theta),$$
$$-3 < \gamma < 1, \quad 0 < s < 1,$$

- It is believed that

$$Q(f, f) \sim -C_f(-\Delta_v + ...)^s f + \text{more regular terms}$$

For the precise structure, see the works by Bobylev, Lerner-Morimoto-Pravda Starov-Xu on the linearized operator that show the relation to the harmonic oscillator operator.

Entropy dissipation;

$$\|\sqrt{f}\|^2_{H^s_{\frac{\gamma}{2}}(\mathbb{R}^3)} \leq C_f(\|f\|_{L^1_{s+\gamma/2}} + D(f)^{\frac{1}{2}}).$$

where

$$D(f) = -\int_{\mathbb{R}^3} Q(f,f)\log f dv,$$

• The above estimate was first derived by P.-L. Lions (1990s), improved by Alexandre-Desvillettes-Villani-Wennberg(2000), Alexandre-Morimoto-Ukai-Xu-Y.(2012).

• Around 2000s, the regularizing effect analyzed by the entropy production was developed by Alexandre, Bouchut, Desvillettes, Golse, P.-L. Lions, Mouhot, Villani, Wennberg, ...

Coercivity estimates in Dirichlet form:

$$\Big(- Q(g,f), f \Big) \geq c_g ||f||^2_{H^s_{\gamma/2}} - C_g \|f\|^2_{H^{s'}_{\gamma/2}}, \quad s' < s.$$

• Alexandre-Desvillettes-Villani-Wennberg, Alexandre-Morimoto-Ukai-Xu-Y., cf. also Pao, Mouhot, ...

• How to use the coercivity estimate to show the gain of regularity, such as H^∞ in each variable?

Theorem (Alexandre-Morimoto-Ukai-Xu-Y., 2008)
Consider a kinetic equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = g.$$

For $0 < s < 1$, if $f \in L^\infty(]0, T[; L^2 \cap L^\infty(\mathbb{R}^3_x; L^1_2(\mathbb{R}^3_v))$ and

$$\Lambda^s_v f \in L^2_{loc}(]0, T[\times\mathbb{R}^6), \ \Lambda^{-s}_v g \in L^2(]0, T[\times\mathbb{R}^6).$$

Then

$$\Lambda_{t,x}^{\frac{s(1-s)}{1+s}} f \in L^2_{loc}(\mathbb{R}^7).$$

Remark Similar hypoelliptic regularity estimate was obtained by Bouchut(2002).

Theorem (Alexandre-Morimoto-Ukai-Xu-Y. 2012)[Regularizing effect]
For the solution f obtained in the well-posedness theorem, we have

$$f \in C^\infty((0,\infty) \times \mathbb{R}^3_x; \mathcal{S}(\mathbb{R}^3_v)) .$$

- Holds for other Sobolev spaces with smaller indices.

- Different from the well-known velocity averaging lemma,
Golse-Perthame-Sentis(1985),
Golse-Lions-Perthame-Sentis(1988),
DiPerna-Lions-Meyer(1991).

Villani conjectured (in a private communication, 2008):

Any weak solution with measure valued initial datum except a single Dirac mass acquires C^∞ regularity in the velocity variable for any positive time.

• True for the Maxwellian molecule type cross section and spatially homogeneous Boltzmann equation.

• Other cases are still unsolved.

Spatially homogeneous Boltzmann Equation

$$\frac{\partial f}{\partial t} = Q(f, f), \;\; f(0, v) = f_0(v),$$

Assume

$$B = \Phi(|v - v_*|) b(\cos\theta), \quad \cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma\,, \;\; 0 \le \theta \le \frac{\pi}{2},$$

with

$$\Phi(|v - v_*|) = |v - v_*|^{\gamma}\,, \;\; \gamma > -3,$$

and

$$b(\cos\theta) = K\theta^{-2-2s}\;\;, \;\; 0 < s < 1.$$

Note that many results on the weighted L^p solutions

Works by Desvillettes-Wennberg, Alexandre-Safadi, Huo-MUY, Chen-He, Alexandre-Morimoto-Ukai-Xu-Y ....

Measure valued solutions [existence]:

• Maxwellian molecule type: probability theory, Tanaka '78;
· Fourier transform by Bobylev formula '75;
· Pulvirenti-Toscani '96; Toscani-Villani '99;
• Infinite energy, Cannone-Karch 2010; Morimoto, 2012;
· self-similar solution with infinite energy, Bobylev-Cercignani 2002;

• Hard potential: Lu-Mouhot 2012.

Bobylev formula:

Denote $\psi(t,\xi) = \mathcal{F}_v(f(t,v))$ and $\psi_0(\xi) = \mathcal{F}_v(f_0(v))$

$$
\begin{cases}
\partial_t \psi(t,\xi) = \\
\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\Big(\psi(t,\xi^+)\psi(t,\xi^-) - \psi(t,\xi)\psi(t,0)\Big)\, d\sigma, \\
\psi(0,\xi) = \psi_0(\xi), \quad \text{where } \ \xi^{\pm} = \frac{\xi}{2} \pm \frac{|\xi|}{2}\sigma \,.
\end{cases}
$$

Notations and definitions

For $0 \leq \alpha < \infty$, denote by P_α the class of all probability measure F such that

$$\int_{\mathbb{R}^3} |v|^\alpha dF(v) < \infty \,.$$

Definition: A function $\psi : \mathbb{R}^3 \to \mathbb{C}$ is called a characteristic function if there is a probability measure Ψ such that

$$\psi(\xi) = \int_{\mathbb{R}^3} e^{-iv\cdot\xi} d\Psi(v).$$

Denote the set of all characteristic functions by $\mathcal{K}$.

Function spaces introduced by Cannone-Karch

Motivated by the works by Carlen-Gabetta-Toscani, Gabetta-Toscani-Wennberg, Toscani-Villani, for $\alpha \geq 0$, Cannone-Karch introduce:

$$\mathcal{K}^\alpha = \{\varphi \in \mathcal{K}\,;\, \|\varphi - 1\|_\alpha < \infty\}\,,$$

where

$$\|\varphi - 1\|_\alpha = \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - 1|}{|\xi|^\alpha}.$$

The space $\mathcal{K}^\alpha$ endowed with the distance

$$\|\varphi - \tilde{\varphi}\|_\alpha = \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^\alpha}$$

is a complete metric space. $\mathcal{K}^\alpha = \{1\}$ for all $\alpha > 2$ and

$$\{1\} \subset \mathcal{K}^\alpha \subset \mathcal{K}^\beta \subset \mathcal{K}^0 = \mathcal{K} \quad \text{for all } 2 \geq \alpha \geq \beta \geq 0\ .$$

Lemma. Let Ψ be a probability measure such that

$$\exists \alpha \in (0,2]; \int |v|^\alpha d\Psi(v) < \infty, \text{ and}$$
$$\int v_j d\Psi(v) = 0, j = 1,2,3, \text{ when } \alpha > 1.$$

Then $\psi(\xi) = \int e^{-iv\cdot\xi} d\Psi(v)$ belongs to $\mathcal{K}^\alpha$.
The inverse of the lemma does not hold, in fact,

$$P_\alpha \subsetneq \tilde{P}_\alpha = \mathcal{F}^{-1}(\mathcal{K}^\alpha).$$

$\tilde{P}_\alpha$ is endowed also with the distance:

$$\|\varphi - \tilde{\varphi}\|_\alpha = \sup_{\xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^\alpha}.$$

Theorem (Cannone-Karch, 2010; Morimoto, 2012) [Existence and Stability]

Assume $\alpha \in (2s, 2]$, Then for $\forall \psi_0 \in \mathcal{K}^\alpha$, there exists a unique solution $\psi \in C([0, \infty), \mathcal{K}^\alpha)$.

Moreover, if $\psi(t, \xi)$, $\varphi(t, \xi) \in C([0, \infty), \mathcal{K}^\alpha)$ are two solutions with initial data $\psi_0, \varphi_0 \in \mathcal{K}^\alpha$, then

$$\|\psi(t) - \varphi(t)\|_\alpha \leq e^{\lambda_\alpha t} \|\psi_0 - \varphi_0\|_\alpha,$$

where

$$\lambda_\alpha = 2\pi \int_0^{\pi/2} b(\cos\theta)\{\cos^\alpha \frac{\theta}{2} + \sin^\alpha \frac{\theta}{2} - 1\} \sin\theta d\theta \,.$$

Furthermore, $\partial_t \psi(t, \xi)$ is continuous in $[0, \infty) \times \mathbb{R}^3$.

Theorem. (Morimoto-Y., 2012) [Regularizing effect]
Let $\alpha \in (2s, 2]$, if $F_0 \in \tilde{P}_\alpha$ is not a single Dirac mass and $f(t, v)$ is a unique solution in $C([0, \infty), \tilde{P}_\alpha)$, then $f(t, \cdot) \in H^\infty(\mathbb{R}^3)$ for $t > 0$.

Ideas of the proof:

- A time degenerate coercivity estimate;
- Uniform bounds on moment of order $\alpha' < \alpha$ and entropy after some finite time.

Remarks:

1. Recall

$$P_\alpha \subsetneq \tilde{P}_\alpha = \mathcal{F}^{-1}(\mathcal{K}^\alpha).$$

2. Unlike the space P_α, the physical meaning of the space $\tilde{P}_\alpha$ is unclear.
3. How to characterize P_α in Fourier space?

A new function space: (Morimoto-Wang-Y.)

- Observations:

1. Let $0 < \alpha < 2$ and $\alpha \neq 1$. If $\Psi \in P_\alpha$ and satisfies when $\alpha > 1$,

$$\int v_j d\Psi(v) = 0 \text{ for } j = 1, 2, 3,$$

then its Fourier transform $\psi(\xi) = \int e^{-iv\cdot\xi} d\Psi(v)$ satisfies

$$\int_{\mathbb{R}^d} \frac{|1 - \psi(\xi)|}{|\xi|^{d+\alpha}} d\xi < \infty \,. \tag{1}$$

When $\alpha = 1$, in addition, we need

$$\int \langle v \rangle \log\langle v \rangle d\Psi(v) < \infty, \quad \langle v \rangle = \sqrt{1 + |v|^2}.$$

2. If a probability measure Ψ satisfies (1) for $\alpha \in (0, 2)$, then Ψ belongs to P_α.

Hence, for each $0 < \alpha < 2$, introduce a function space

$$\mathcal{M}^\alpha = \{\varphi | \varphi \in \mathcal{K} \text{ and } \int_{\mathbb{R}^3} \frac{|\varphi(\xi) - 1|}{|\xi|^{3+\alpha}} d\xi < \infty\},$$

endowed with the metric

$$||\varphi - \tilde{\varphi}||_{\mathcal{M}^\alpha} = \int_{\mathbb{R}^3} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^{3+\alpha}} d\xi.$$

For any $\beta \in (0, \alpha]$, introduce the distance

$$dis_{\alpha,\beta}(\varphi, \tilde{\varphi}) = \|\varphi - \tilde{\varphi}\|_{\mathcal{M}^\alpha} + \|\varphi - \tilde{\varphi}\|_\beta \,.$$

Note that:

• If $0 < \beta \leq \alpha < 2$ then the space $\mathcal{M}^\alpha$ is a complete metric space endowed with the distance $dis_{\alpha,\beta}(\varphi, \tilde{\varphi})$.

• If β, β' are in $(0, \alpha)$, both distances $dis_{\alpha,\beta}(\cdot, \cdot)$ and $dis_{\alpha,\beta'}(\cdot, \cdot)$ are equivalent in the following sense:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } dis_{\alpha,\beta'}(\varphi, \tilde{\varphi}) < \delta \Rightarrow dis_{\alpha,\beta}(\varphi, \tilde{\varphi}) < \varepsilon.$$

Furthermore, we have

$$\mathcal{K}^\beta \subset \mathcal{M}^\alpha \text{ if } \alpha < \beta \text{ and } \alpha \in (0,2),$$
$$\mathcal{M}^\alpha \subset \mathcal{F}(P_\alpha(\mathbb{R}^d)) \left(\subsetneq \mathcal{K}^\alpha \right) \text{ for } \alpha \in (0,2),$$
$$\mathcal{M}^\alpha = \mathcal{F}(P_\alpha(\mathbb{R}^d)), \text{ furthermore if } \alpha \neq 1,$$

Theorem. (Morimoto-Wang-Y., 2013) [Existence]
Let $\alpha \in (2s, 2]$ with $\alpha \neq 1$ and $0 < s < 1$, If $F_0 \in P_\alpha$ satisfying

$$\int v_j dF_0(v) = 0\,, j = 1, 2, 3, \text{ if } \alpha > 1,$$

then there exists a unique measure valued solution $F_t \in C([0, \infty), P_\alpha(\mathbb{R}^3))$.

Remark When $F_0 \in P_1(\mathbb{R}^3)$, for any $T > 0$ there exists a $C_T > 0$ such that

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^3} \langle v \rangle dF_t(v) \le C_T.$$

Moreover, if $F_0 \in P_1(\mathbb{R}^3)$ satisfies $\int v_j dF_0(v) = 0\,, j = 1, 2, 3,$ and $\int_{\mathbb{R}^3} \langle v \rangle \log \langle v \rangle dF_0(v) < \infty$ then the solution $F_t \in C([0, \infty), P_1(\mathbb{R}^3))$.

Theorem. (Morimoto-Wang-Y., 2013) [Regularizing Effect] Let $\alpha \in (2s, 2]$ with $0 < s < 1$. If $F_0 \in P_\alpha(\mathbb{R}^3)$ is not a single Dirac mass, then $f(t, \cdot)$ belongs to $L^1_\alpha(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3)$ for any $t > 0$.
Moreover when $\alpha \neq 1$ we have
$f(t, v) \in C((0, \infty), L^1_\alpha(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3))$.

Remark The solution in the function space P_α is completely analyzed.

II. Large time behavior

• Finite energy

For $f_0 \in P_2$, define a global Maxwellian $\mu(v)$ with the same total mass, momentum and energy. Tanaka showed that the solution $f(v,t)$ converges to $\mu(v)$ weakly in distribution sense.

Theorem (Morimoto-Y.-Zhao, 2013)[Uniform bound in Sobolev norm]
For any positive integer N, there exists C_N independent of time such that

$$\|f(v,t)\|_{H^N(\mathbb{R}^3)} \leq C_N.$$

Hence

$$\|f(v,t) - \mu(v)\|_{H^N(\mathbb{R}^3)} \to 0, \quad t \to \infty.$$

Remarks

• Almost exponential decay can be obtained following the work by Mouhot on Maxwellian lower bound, and the hypercoercivity estimate by Desvillettes-Villani.

• Note that the uniform in time bound in $H^N(\mathbb{R}^3)$-norm follows from the conservation of the energy and the uniform coercivity estimate. And this is not true for the self-similar solution with infinite energy.

• Infinite energy

• Existence of self-similar solution, Bobylev-Cercignani (2002)
For $\alpha \in (2s, 2)$ with $0 < s < 1$, denote $\mu_\alpha = \frac{\lambda_\alpha}{\alpha}$. For each $K > 0$, there exists a radially symmetric nonnegative functon $\Psi_{\alpha,K} \in L^1(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3)$ satisfying

$$\hat{\Psi}_{\alpha,K} \in \mathcal{K}^\alpha, \quad \lim_{|\xi|\to 0} \frac{1 - \hat{\Psi}_{\alpha,K}(\xi)}{|\xi|^\alpha} = K,$$

such that

$$f_{\alpha,K} = e^{-3\mu_\alpha t}\Psi_{\alpha,K}(ve^{-\mu_\alpha t})$$

is a solution of Boltzmann equation.

Central Limit Theorem(Carnnone-Karch, 2013)
For $\alpha \in (2s, 2)$ with $0 < s < 1$, if an initial datum $f_0 \in \tilde{P}_\alpha$ satisfying

$$\int |f_0(v) - \Psi_{\alpha,K}(v)||v|^\alpha < \infty,$$

then

$$\int (f(v,t) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(ve^{-\mu_\alpha t}))g(e^{-\mu_\alpha t}v)dv \to 0, \quad t \to \infty,$$

for any $g(v) \in \mathcal{S}(\mathbb{R}^3)$. On the other hand, if $\int f_0(v)|v|^\alpha dv < \infty$, then

$$\int f(v,t)g(e^{-\mu_\alpha t}v)dv \to g(0), \quad t \to \infty.$$

Uniform stability (Morimoto-Y.-Zhao, 2013)

Assume

$$\int |f_0 - \Psi_{\alpha,K}(v)||v|^2 dv = C_0.$$

Then

$$\|f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(ve^{-\mu_\alpha t})\|_2 \lesssim C_0.$$

Note that

$$\|f(t,v) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(ve^{-\mu_\alpha t})\|_2 = \sup_{|\xi|\neq 0} \frac{|\hat{f}(t,\xi) - \hat{\Psi}_{\alpha,K}(\xi e^{\mu_\alpha t})|}{|\xi|^2},$$

provided that $\max\{6/5, 2s\} < \alpha < 2$. If f_0 is radially symmetric, then $\alpha > 2s$ is enough.

Slightly stronger convergence result

Central Limit Theorem (Morimoto-Y.-Zhao, 2013)
For $\alpha \in (\max\{6/5, 2s\}, 2)$ with $0 < s < 1$, if an initial datum $f_0 \in \tilde{P}_\alpha$ satisfying

$$\int |f_0(v) - \Psi_{\alpha,K}(v)||v|^2 < \infty,$$

then

$$\int (f(v,t) - e^{-3\mu_\alpha t}\Psi_{\alpha,K}(ve^{-\mu_\alpha t}))g(H(t)v)dv \to 0, \quad t \to \infty,$$

for any $g(v) \in \mathcal{S}(\mathbb{R}^3)$. Here, $H(t) > 0$ is any continuous function satisfying $\lim_{t\to\infty} H(t) = 0$.
On the other hand, if $\int f_0(v)|v|^\alpha dv < \infty$, then

$$\int f(v,t)g(e^{-\beta t}v)dv \to g(0), \quad t \to \infty.$$

Remark We still can not show weak convergence.

III. Spectrum analysis

Recall Spectrum of Boltzmann equation for hard potential with angular cutoff, Nicolaenko(1971), Ellis-Pinsky (1975), Ukai(1974):

For

$$\hat{B}(k) = L - i(v \cdot k) = \mathcal{F}_x(L + v \cdot \nabla_x)$$

there exists $\nu_1 > 0$ such that $\sigma(\hat{B}) \cap \{\lambda | Re\lambda > -\nu_1\}$ consists of only five eigenvalues $\lambda_j(k) < 0$ when $s = |k|$ is small in a small neighborhood of $\lambda = 0$.

$$\Phi(t,k) = e^{t\hat{B}(k)} = \frac{1}{2\pi}\int_{\lambda_0 - i\infty}^{\lambda_0 + i\infty} e^{\lambda t}(\lambda - \hat{B}(k))^{-1} d\lambda$$

$$= \sum_{j=-1}^{3} e^{\lambda_j(k)t} P_j(k) + U,$$

Moreover,

$$\lambda_j(k) = \lambda_j(s) = i\lambda_j^{(1)} s - \lambda_j^{(2)} s^2 + o(1)s^2,$$

with corresponding eigenprojection

$$P_j(k) = P_j^{(0)}(\omega) + sP_j^{(1)}(k), \qquad \omega = \frac{k}{s}.$$

Here

$$\sum_{j=-1}^{3} P_j^{(0)}(k) = \mathbf{P}_0,$$

$\mathbf{P}_0$ is the orthorgonal projection on the null space of L, and

$$\lambda_1(s) = \overline{\lambda}_{-1}(s) = ics - \frac{2}{3}(\mu + \frac{2}{5}\kappa)s^2 + o(1)s^2$$

$$\lambda_0(s) = -\frac{3}{5}\kappa s^2 + o(1)s^2, \qquad \kappa = -(L^{-1}\mathbf{P}_1(v_1\chi_4), v_1\chi_4)$$

$$\lambda_2(s) = \lambda_3(s) = -\mu s^2 + o(1)s^2, \qquad \mu = (L^{-1}\mathbf{P}_1(v_1\chi_2), v_1\chi_2),$$

with $\mathbf{P}_1 = \mathbf{I} - \mathbf{P}_0$, c: sound speed, μ: viscosity coefficient, κ: heat conductivity coefficient.

Key point in the cutoff case:

The gain part and loss part can be considered separately in the collision operator so that

$$L = -\nu(|v|) + K,$$

where $\nu(|v|)$ is the collision frequency and K is a compact operator with moment gain property.

Corresponding theory for angular non-cutoff?

An observation in a general framework: (Y.-H.J. Yu)

As usual, set $L = -A + K$ with

$$A(k)f = (-A - ik \cdot v)f, \quad B(k)f = (L - ik \cdot v)f.$$

The spectrum structure can be shown as the same as the cutoff hard potential Boltzmann equation under four assumptions:

H1. The operator A is a self-adjoint operator with $D(A) = \{f \in L^2;\ Af \in L^2\}$. For any $k \in \mathbb{R}^3$, $D(A(k))$ is dense in L^2. And there exists some constant $\nu_0 > 0$ such that for any $f \in D(A)$

$$\langle Af, f \rangle \geq \nu_0 \|f\|^2.$$

H2. The operator K is a bounded, self-adjoint and $A(k)$-compact in L^2 and for any $k \in \mathbb{R}^3$.

H3. The linear operator L is non-positive in L^2 and is invariant with respect to the rotation $\mathcal{R}$ of $v \in \mathbf{R}^3$. Moreover, it has a null space

$$\mathcal{N}(L) = \text{span}\{\sqrt{M}, \quad v_j\sqrt{M}, \quad |v|^2\sqrt{M}\}, \qquad 1 \le j \le 3.$$

H4. Under the assumptions (**H1**-**H2**), if $\text{Re}\lambda \ge -\nu_0 + \delta$ for any fixed $\delta > 0$, we have

$$\|K(\lambda - A(k))^{-1}\| \to 0, \quad \text{as} \quad |\text{Im}\lambda| + |k| \to \infty.$$

Application to the Boltzmann equation with $\gamma \geq 0$:

$$-Lg \equiv -\Gamma_1 g - \Gamma_2 g = -\Gamma(\sqrt{M}, g) - \Gamma(g, \sqrt{M}) = \mathcal{N}g + \mathcal{K}g.$$

Here

$$\mathcal{N}g = -\Gamma_1 g + \nu_{\mathcal{K}}(v)g(v), \qquad \mathcal{K}g = -\nu_{\mathcal{K}}(v)g(v) - \Gamma_2 g,$$

where $\nu_{\mathcal{K}}(v) = C(1+|v|^2)^{\gamma/2}$.
Set

$$Ag = \mathcal{N}g - \nu_{\mathcal{K}}(v)\mathbf{I}_{|v|\geq R}g(v) - (\Gamma_2 - \mathbf{I}_{|v|\leq R}\Gamma_2\mathbf{I}_{|v|\leq R})g(v),$$

$$Kg = \nu_{\mathcal{K}}(v)\mathbf{I}_{|v|\leq R}g(v) + \mathbf{I}_{|v|\leq R}\Gamma_2\mathbf{I}_{|v|\leq R}g(v).$$

A and K satisfy H_1-H_4 when R is large enough.

Apply to Landau equation with $\gamma \geq -2$

Landau collision operator

$$Q(f,g) = \sum_{i,j=1}^{3} \partial_{v_i} \int_{\mathbb{R}^3} \phi^{ij}(v-v')[f(v')\partial_{v_j} g(v) - g(v)\partial_{v'_j} f(v')]dv',$$

where

$$\phi^{ij}(v) = \{\delta_{ij} - \frac{v_i v_j}{|v|^2}\}|v|^{\gamma+2}, \quad \gamma \geq -2.$$

The linearized Landau operator takes the form

$$-Lg \equiv -\Gamma_1 g - \Gamma_2 g = -\Gamma(\sqrt{M}, g) - \Gamma(g, \sqrt{M}).$$

It is easy to see that

$$\Gamma(\sqrt{M}, g) = \partial_{v_i}[\sigma^{ij}\partial_{v_j} g] - \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2} g + \frac{1}{2}\partial_{v_i}\sigma^i g,$$

$$\Gamma(g, \sqrt{M})$$
$$= -\int_{\mathbb{R}^3} \partial_{v_i}[\phi^{ij}(v - v')M(v')M(v)]M^{-/2}(v)\partial_{v_j'}[M^{-/2}(v')g(v')]dv'.$$

Here σ^{ij} and σ^i are defined as

$$\sigma^{ij}(v) = \int \phi^{ij}(v-v')M(v')dv',\ \sigma^i(v) = \int \phi^{ij}(v-v')v_j' M(v')dv'.$$

Choose a smooth cut-off function $\chi_1(|v|)$ such that $\chi_1(|v|) = 1$ if $|v| < \epsilon$ and $\chi_1(|v|) = 0$ if $|v| > 2\epsilon$. Set

$$Kg = \frac{1}{2}\partial_{v_i}\sigma^i \mathbf{I}_{|v|\le R} g + \mathbf{I}_{|v|\le R} K_1 \mathbf{I}_{|v|\le R} g,$$

$$Ag = -\partial_{v_i}[\sigma^{ij}\partial_{v_j} g] + \sum_{i,j=1}^{3} \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2} g - \sum_{i=1}^{3}\frac{1}{2}\partial_{v_i}\sigma^i \mathbf{I}_{|v|\ge R} g$$
$$-A_1 g - (K_1 - \mathbf{I}_{|v|\le R} K_1 \mathbf{I}_{|v|\le R}) g.$$

Here

$$A_1 g$$
$$= -\int_{\mathbb{R}^3} \partial_{v_i}[\phi^{ij}(v-v')\chi_1(|v-v'|)M(v')M(v)]M^{-/2}(v)\partial_{v_j'}[M^{-/2}(v')g(v')]dv',$$

$$K_1 g = \Gamma(g, \sqrt{M}) - A_1 g.$$

• A and K satisfy the assumptions H_1-H_4 when R is large enough and $\epsilon > 0$ is small enough.

Remark This improves the result by Degond-Lemou (1997) for the case with $\gamma \geq 0$.

• The above spectrum structure leads to the optimal convergence rate estimates, detailed solution structure and comparison with the Navier-Stokes equations, etc.