Finite element systems applications to upwinding

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Goals

- FEEC (Arnold-Falk-Winther 06) quite final:
  – general Hilbert complexes, de Rham (L^2),
  – all affine invariant spaces of polynomial DF on simplexes.
- But...:
  – Theory for *hp*-elements, tensor-products.
  – Polyhedral meshes,
  dual grid for dual elements (Buffa-C. 07).
  – Parameter dependent forms for singular perturbations,
  upwinding for convection diffusion.
- When does gluing together piecewise defined differential forms produce a good complex of spaces?
- Reference: §5 in Owren-Munthe-Kaas-C. 11.

3D discrete de Rham sequences

- Hilbert spaces of scalar and vector fields:

$$\mathrm{H}_{\mathsf{op}} = \{u \in \mathrm{L}^2 \ : \ \mathsf{op}\, u \in \mathrm{L}^2\}.$$

- de Rham sequence in $\mathbb{R}^3$, Hilbert style:

$$\mathrm{H}_{\mathsf{grad}} \xrightarrow{\mathsf{grad}} \mathrm{H}_{\mathsf{curl}} \xrightarrow{\mathsf{curl}} \mathrm{H}_{\mathsf{div}} \xrightarrow{\mathsf{div}} \mathrm{H}$$

- Goal: define good finite dimensional subcomplexes:

$$\begin{array}{ccccccc}
\mathrm{H}_{\mathsf{grad}} & \xrightarrow{\mathsf{grad}} & \mathrm{H}_{\mathsf{curl}} & \xrightarrow{\mathsf{curl}} & \mathrm{H}_{\mathsf{div}} & \xrightarrow{\mathsf{div}} & \mathrm{H} \\
\cup \uparrow & & \cup \uparrow & & \cup \uparrow & & \cup \uparrow \\
X_h^0 & \xrightarrow{\mathsf{grad}} & X_h^1 & \xrightarrow{\mathsf{curl}} & X_h^2 & \xrightarrow{\mathsf{div}} & X_h^3
\end{array}$$

One d to rule them all

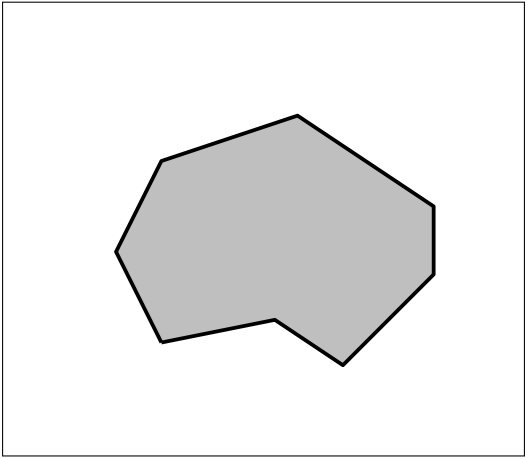
- On $\mathbb{R}^3$: scalar product g, determinant ω.
- Scalar/vector fields correspond to differential forms:

$$\begin{aligned} \Psi^0 u &= u \in \Omega^0 \\ \Psi^1 u &= g(u, \cdot) \in \Omega^1 \\ \Psi^2 u &= \omega(u, \cdot, \cdot) \in \Omega^2 \\ \Psi^3 u &= u\omega(\cdot, \cdot, \cdot) \in \Omega^3 \end{aligned}$$

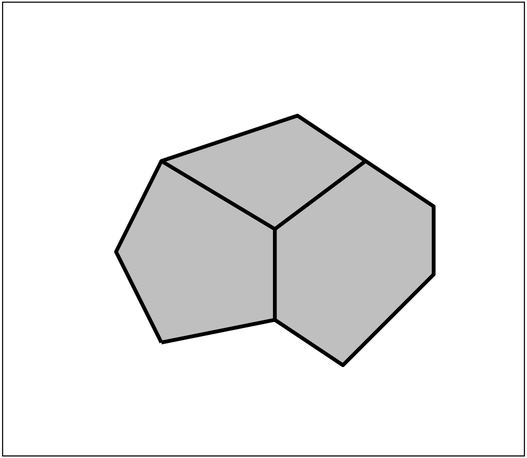
- Commuting diagram with vertical isomorphisms:

$$\begin{array}{ccccccc} \mathcal{F} & \xrightarrow{\text{grad}} & \mathcal{V} & \xrightarrow{\text{curl}} & \mathcal{V} & \xrightarrow{\text{div}} & \mathcal{F} \\ \downarrow{\scriptstyle \Psi^0} & & \downarrow{\scriptstyle \Psi^1} & & \downarrow{\scriptstyle \Psi^2} & & \downarrow{\scriptstyle \Psi^3} \\ \Omega^0 & \xrightarrow{\mathrm{d}} & \Omega^1 & \xrightarrow{\mathrm{d}} & \Omega^2 & \xrightarrow{\mathrm{d}} & \Omega^3 \end{array}$$

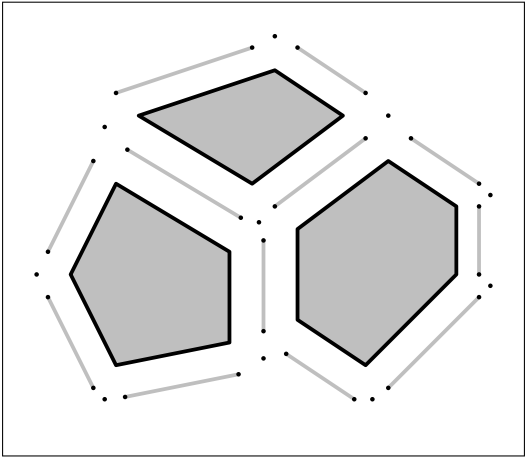
Cellular complexes



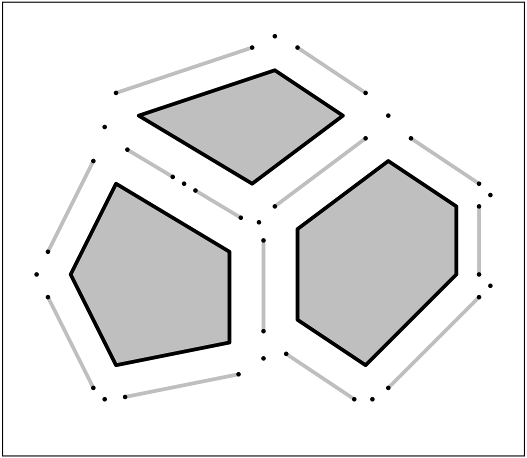
Cellular complexes



Cellular complexes



Cellular complexes



Cellular complexes: definition

- Cells are Lipschitz isomorphic to a ball of some dimension.
- Cellular complex $\mathcal{T}$ is a collection of cells such that:
  – cover the physical domain,
  – disjoint interiors,
  – boundary of cell is a union of cells.
- Intersection of two cells is a union of cells,
  but can consist of several cells and be non-connected.

Finite element systems

- Fix a cellular complex $\mathcal{T}$.
- A finite element system A is
  $A^k(T) \subseteq \Omega^k(T)$ for $k \in \mathbb{N}$ and $T \in \mathcal{T}$ of all dimensions.
  – exterior derivative, $\mathrm{d} : A^k(T) \to A^{k+1}(T)$.
  – pullback, $i : T' \subseteq T$ gives $i^\star : A^k(T) \to A^k(T')$.
- Inverse system of complexes.
  Global spaces from inverse limit:

$$A^k(\mathcal{T}) = \{u \in \bigoplus_{T \in \mathcal{T}} A^k(T) \ : \ T' \subseteq T \Rightarrow u_T|_{T'} = u_{T'}\}. \quad (1)$$

Gives continuity appropriate for exterior derivative.

Examples

- Start with forms on cells with max dim, add exterior derivatives, restrict to each subcell, take linear spans. Not always good!
- Raviart-Thomas 77, Nédélec 80, Hiptmair 99, Arnold-Falk-Winther 06.
  Koszul operator (or Poincaré operator):

  $$(\kappa u)_x(\xi_1, \ldots, \xi_k) = u_x(x, \xi_1, \ldots, \xi_k). \tag{2}$$

  Fix $p \in \mathbb{N}^*$. For all simplexes T:

  $$A^k(T) = \{u \in \mathbb{PA}^k_p(T) \ : \ \kappa u \in \mathbb{PA}^{k-1}_p(T)\} \tag{3}$$

Interpolators, DoFs, Extensions

- Interpolator is a collection of projections $\Omega^k(T) \to A^k(T)$ which commutes with restrictions.
- A system of DoFs is a collection of spaces $Z^k(T) \subseteq \Omega^k(T)^\star$. Unisolvent if isomorphism:

$$\Phi^k : A^k(\mathcal{T}) \ni u \mapsto \langle \cdot, u|_T \rangle_{T \in \mathcal{T}} \in \bigoplus_{T \in \mathcal{T}} Z^k(T)^\star. \qquad (4)$$

- Equivalent properties:
  - A admits an interpolator.
  - A admits unisolvent degrees of freedom.
  - Pullback to boundary $A^k(T) \to A^k(\partial T)$ is onto.
  - $\dim A^k(T) = \sum_{T' \subseteq T} A_0^k(T')$.

Commuting interpolators

- Suppose A admits an interpolator and $A^0(T)$ contains $\mathbb{R}$.
- Equivalent:
  - A admits an interpolator commuting with exterior derivative.
  - For each T the following sequence is exact:

$$0 \to \mathbb{R} \to A^0(T) \to A^1(T) \to \cdots \to A^{\dim T}(T) \to 0. \tag{5}$$

  - For each T the following sequence is exact:

$$0 \to A^0_0(T) \to A^1_0(T) \to \cdots \to A^{\dim T}(T) \to \mathbb{R} \to 0. \tag{6}$$

- Then de Rham map induces isomorphisms on cohomology:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^0(\mathcal{T}) & \xrightarrow{\mathrm{d}} & \cdots & \xrightarrow{\mathrm{d}} & A^k(\mathcal{T}) & \xrightarrow{\mathrm{d}} & \cdots \\ & & \downarrow \rho^0 & & \downarrow & & \downarrow \rho^k & & \\ 0 & \longrightarrow & \mathcal{C}^0(\mathcal{T}) & \xrightarrow{\delta} & \cdots & \xrightarrow{\delta} & \mathcal{C}^k(\mathcal{T}) & \xrightarrow{\delta} & \cdots \end{array} \tag{7}$$

Old and new

- Ciarlet's definition:

Motivated by the previous examples, we are now in a position to give a general definition, first proposed by CIARLET [1975, p. 61]: A *finite element* in $\mathbb{R}^n$ is a triple

(T, P, Σ) where:

(i) T is a closed subset of $\mathbb{R}^n$ with a nonempty interior and a Lipschitz-continuous boundary;

(ii) P is a finite-dimensional space of real-valued functions defined over the set T; we let $N = \dim P$;

(iii) Σ is a set of N linear forms ϕ_i, $1 \leqslant i \leqslant N$, defined over the space P and, *by definition*, it is assumed that the set Σ is *P-unisolvent*, in the following sense: given any real scalars α_i, $1 \leqslant i \leqslant N$, there exists a unique function $p \in P$ that satisfies

$$\phi_i(p) = \alpha_i, \quad 1 \leqslant i \leqslant N. \tag{10.1}$$

- New definition:

– (i) T is a cell of any dimension, in a complex.
– (ii) P_T is a complex of differential forms, stable under pullbacks (inverse system).
– (iii) DoFs not part of the definition.
Compatibility guarantees commuting interpolator.

Degrees of Freedom: examples

- Most important example of FES:

$$A^k(T) = \{u \in \mathbb{PA}_p^k(T) \ : \ \kappa u \in \mathbb{PA}_p^{k-1}(T)\} \qquad (8)$$

- Standard degrees of freedom:

$$Z^k(T) = \{u \mapsto \int v \wedge u \ : \ v \in \mathbb{PA}_{p-\dim T+k-1}^{\dim T-k}\}. \qquad (9)$$

- Projection based interpolation (Demkowicz-Buffa 05):
  Choose scalar products a on cells (e.g. $\mathrm{L}^2(T)$ or $\mathrm{H}^{-1/2}(T)$).

$$Z^k(T) = \{a(\cdot, v) \ : v \in \mathrm{d}A_0^{k-1}(T)\} \oplus \qquad (10)$$

$$\{a(\mathrm{d}\cdot, v) \ : v \in \mathrm{d}A_0^k(T)\} \quad \text{or} \quad \int . \qquad (11)$$

- with F. Rapetti: extend Lagrange interpolation to complex.

Locally harmonic forms

- Start with compatible FES (e.g. from simplicial subgrid).
- (C. 08) In PBI only "$\int$" DoFs allowed to be non-zero.
- – Galerkin version of $\mathrm{d}^\star \mathrm{d} u = 0$ and $\mathrm{d}^\star u = 0$.
  – Recovers Whitney forms from any constant metric.
  – dR map is isomorphism: one DoF per k-cell for k-forms.
  – Generalizes Kuznetzov-Repin 05 (Hdiv).
  – Gives dual spaces (Hdiv-Hrot) in 2D (Buffa-C. 07).
  – Can be applied recursively (Pasciak-Vassilevski 08).

Minimal compatible FES

- A a FES, B a compatible ES, $A^k(T) \subseteq B^k(T)$.
- There is a minimal compatible FES C such that:

$$A^k(T) \subseteq C^k(T) \subseteq B^k(T). \tag{12}$$

  Comes with construction recipy! C. 10.
- Examples:
  – RTN_r is a minimal compatible FES containing $\mathcal{P}^{r-1}\Lambda^k$.
  – Locally harmonic: minimal containing ... 0.

Tensor products

- Two FES: $A^k(U)$ on $(M, \mathcal{U})$, and $B^l(V)$ on $(N, \mathcal{V})$, gives FES C on $M \times N$ (with product cells), defined by:

$$C^k(U \times V) = \sum_{l=0}^{k} A^l(U) \otimes B^{k-l}(V). \tag{13}$$

- Preserves compatibility (local exactness from Kunneth), C. 09.
- Preserves also:
  – local harmonicity w.r.t. Riemannian metrics,
  – minimality.

Upwinding

- V a vector field, gives covariant operators:

$$\operatorname{grad} u + Vu, \quad \operatorname{curl} u + V \times u, \quad \operatorname{div} u + V \cdot u. \tag{14}$$

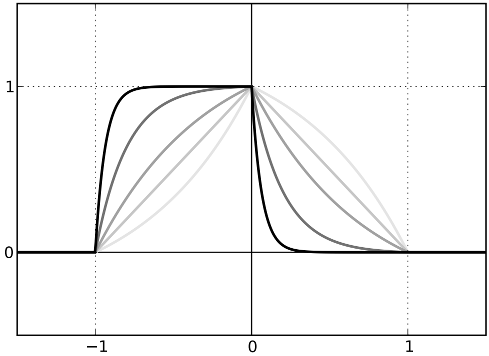
- Scalar convection diffusion:

$$-\epsilon \Delta u + V \cdot \operatorname{grad} u = f \tag{15}$$

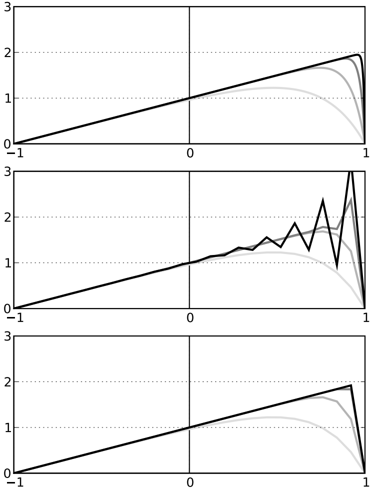
$$-\epsilon(\operatorname{div} - V/\epsilon) \operatorname{grad} u = f. \tag{16}$$

- Petrov Galerkin method with standard trial functions, and test functions that satisfy the homogeneous adjoint equation piecewise.

Upwinded basis functions



Exponential fitting, Petrov-Galerkin, for $-\epsilon u'' + u' = 1$



Upwinding forms

- One-form α gives covariant exterior derivative:

$$u \mapsto \mathrm{d}_\alpha u = \mathrm{d}u + \alpha \wedge u. \tag{17}$$

- Convective harmonic functions: $\mathrm{d}_\alpha^\star \mathrm{d}u = 0$ and $\mathrm{d}_\alpha^\star u = 0$. adjoint taken with respect to standard L^2.
- If $\alpha = \mathrm{d}\beta$ equivalent conditions:

$$\mathrm{d}^\star \exp(-\beta)\mathrm{d}u = 0, \quad \mathrm{d}^\star \exp(-\beta)u = 0. \tag{18}$$

  Harmonic for weighted L^2 product!
- Use locally harmonic differential forms for this new scalar product. Gives upwinded compatible FES.
- Gauge theory: group $\mathbb{R}_+$ acting on $\mathbb{R}$.
  $\mathbb{U}(1)$ acting on $\mathbb{C}$ could be used for Helmholtz equation...

Smoothing

- Schöberl 08, C. 07, Arnold-Falk-Winther 06, C.-Winther 08.
- Scale-invariant spaces. Bramble-Hilbert:

$$\|u - I_h u\|_{\mathrm{L}^q(T)} \preceq \sum_{T' \subseteq T} h_T^{\ell + (\dim T - \dim T')/q} \|\nabla^\ell u\|_{\mathrm{L}^q(T')}, \quad (19)$$

- Smoothing by convolution on $\mathbb{R}^d$. Bell function ϕ, scaled ϕ^δ. For a cell T' of any dimension:

$$\|\phi^\delta * u\|_{\mathrm{L}^q(T')} \preceq \delta^{(\dim T' - d)/q} \|u\|_{\mathrm{L}^q(\mathcal{V}_\delta(T'))}, \quad (20)$$

- Fix ϵ. $u \mapsto u_h = I_h(\phi^{\epsilon h} * u)$ is:
  – $\mathrm{L}^q(M) \to \mathrm{L}^q(M)$ stable,
  – $\|u - u_h\| \preceq h^\ell \|\nabla^\ell u\|$, when $\phi *$ preserves $\mathcal{P}^{\ell - 1}$,
  – choose ϵ so that $\|u - u_h\| \leq 1/2 \|u\|$ when u discrete.

Applications

- Hilbert L^2 setting. Gives stable commuting projections therefore eigenvalue convergence for Hodge Laplace.
- Locality gives translation estimate (Karlsen-Karper 10):

$$\|v_h - \tau_\xi v_h\|_{\mathrm{L}^2} \preceq (|\xi| + |\xi|^{1/2} h^{1/2}) \|\mathrm{d} v_h\|_{\mathrm{L}^2}. \qquad (21)$$

  for $v_h \in X_h^k$, L^2 orthogonal to $\mathrm{d}X_h^{k-1}$.
- L^q estimates give Sobolev injection (Buffa-Ortner 08), C.-Scheid 11, for Maxwell-Klein-Gordon, e.g.

$$\|v_h\|_{\mathrm{L}^6} \preceq \|\operatorname{curl} v_h\|_{\mathrm{L}^2}, \qquad (22)$$

  for $v_h \in X_h^1$, Nédélec edge element, discrete divergence free.