Some methods in the identification of immersed obstacles in a fluid

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Outline

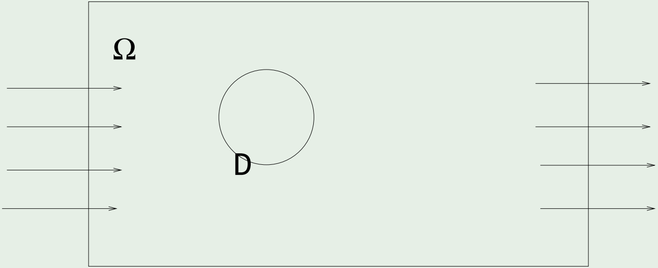
This lecture is organized as follows :

- Setting of the problem
- Identifiability and stability results for a static unknown obstacle
- Identifiability results for a moving unknown obstacle in a perfect fluid
- Identifiability results for a moving unknown obstacle in a viscous fluid

Problem statement

for Both static and moving cases

- To recover geometrical information (position and shape) about an a-priori unknown, static or moving body D immersed in an incompressible fluid.



- To this end, we perform measurements (on velocity and stress forces) along the boundary of the cavity Ω fulfilled by the fluid or liquid.

Identifiability of a static obstacle

- Let Ω be a smooth bounded set in $\mathbb{R}^N$ and let $D \subset\subset \Omega$ be an unknown rigid body immersed in the liquid.
- Let $\vec{\phi} \in H^{\frac{1}{2}}(\partial\Omega)^N$ be a non homogeneous Dirichlet boundary data satisfying the standard flux condition $\int_{\partial\Omega} \vec{\phi} \cdot \mathbf{n} ds = 0$, and let $(\mathbf{v}, p)$ be the solution of Stokes equations in $\Omega^* := \Omega \setminus \overline{D}$

$$
(\mathrm{P}) \qquad \begin{cases} \mathrm{div}\,(\sigma(\mathbf{v}, p)) = 0 & \text{in} \quad \Omega^* \\ \mathrm{div}\ \mathbf{v} = 0 & \text{in} \quad \Omega^* \\ \mathbf{v} = \vec{\phi} & \text{on} \quad \partial\Omega \\ \mathbf{v} = 0 & \text{on} \quad \partial D \end{cases}
$$

where

$$
\sigma(\mathbf{v}, p) = -pI + 2\nu \mathrm{e}(\mathbf{v}) \qquad \mathrm{e}(\mathbf{v}) = \frac{(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)}{2}
$$

(σ and e are the stress and strain tensors, respectively).

The Dirichlet to Neumann operator

- Admissible bodies

$$\mathscr{U}_{ad} = \{D \subset\subset \Omega \quad : \; D \text{ is a smooth, convex, open set, such that } \Omega \setminus \overline{D} \text{ is connected}\}$$

- Let Λ be the following boundary map, velocity to stress tensor operator

$$\Lambda \; : \; D \in \mathscr{U}_{ad} \longrightarrow \Lambda_D$$

defined as follows

$$\Lambda_D(\vec{\phi}) = \sigma(\mathbf{v}, p)\mathbf{n} \quad \text{on} \quad \Gamma_m \subset \partial\Omega;$$

$(\mathbf{v}, p)$ being the solution of the Stokes system (P).

Identifiability result

C^2, L. Friz, O. Kavian & J. Ortega, *Inverse Problems* **21**, 2005

Theorem 1

Let Ω be a smooth open bounded domain in $\mathbb{R}^N$ and $D_0, D_1 \in \mathscr{U}_{ad}$. Let Γ_m be an open non empty subset of $\partial\Omega$ and $\phi \in H^{\frac{1}{2}}(\partial\Omega)$, $\phi \not\equiv 0$, $\int_{\partial\Omega} \phi = 0$. Let $(\mathbf{v}_i, p_i)$ for $i = 0, 1$ be solutions of

$$(P) \quad \left\{ \begin{array}{rclcc} \mathrm{div}\,(\sigma(\mathbf{v}_i, p_i)) & = & 0 & in & (\Omega \setminus \overline{D}_i) \\ \mathrm{div}\ \mathbf{v}_i & = & 0 & in & (\Omega \setminus \overline{D}_i) \\ \mathbf{v}_i & = & \phi & on & \partial\Omega \\ \mathbf{v}_i & = & 0 & on & \partial D_i \end{array} \right.$$

such that

$$\sigma(\mathbf{v}_0, p_0)\mathbf{n} = \sigma(\mathbf{v}_1, p_1)\mathbf{n} \quad on \quad \Gamma_m,$$

then

$$D_0 \equiv D_1$$

Stability issue

weak stability or directional continuity (hemicontinuity)

Theorem 2

Let $\mathbf{u_0} \in W_c^{3,\infty}(\Omega;\mathbb{R}^N)^N$ *be given. Assume that* $\mathbf{u} = t\mathbf{u_0}$*. Then there exists a strictly positive constant* $C = C(\mathbf{u_0},\Omega,D,\phi)$ *and an integer* $m = m(\mathbf{u_0},\Omega,D,\phi) \in \mathbb{N}$*, such that for some* $t_0 > 0$ *and all* $t \in [-t_0, t_0]$*, we have*

$$\|\Lambda_D(\phi) - \Lambda_{D+t\mathbf{u_0}}(\phi)\|_{H^{-\frac{1}{2}}(\partial\Omega)} \geq C\,|t|^m$$

where

$$\Lambda_D(\phi) = \sigma(\mathbf{v}_0, p_0)\mathbf{n} \quad \text{on} \quad \partial\Omega$$

and

$$\Lambda_{D+t\mathbf{u_0}}(\phi) = \sigma(\mathbf{v}_{t\mathbf{u_0}}, p_{t\mathbf{u_0}})\mathbf{n} \quad \text{on} \quad \partial\Omega$$

Log-log type stability for the Haussdorf distance between obstacles

A. Ballerini, *Inverse Problems* **26**, 2010

General domains

1. How many obstacles are immersed in the liquid ? (open)
2. Where is (are) located ? (position ?)
   - (partial answer) Estimates for the distance from a given point to the obstacle (Heck, Uhlmann & Wang, *Inverse Probl. Imaging*, 2007);
   - (methods) geometrical optics solutions (Sylvester & Uhlmann)
3. Which is its volume ?, shape ? (in progress)
   - Alves, Kress & Silvestre, *J. Inverse Ill-posed Probl.*, 2007
   - (methods) Shape and position determination is reduced to a system of non linear integral equations
4. Numerical reconstruction (in progress; several authors)

   (a) Fabien CAUBET, Thèse submitted, Université de Pau, 2012
   (b) (methods) based on shape and topological derivatives, and asymptotic analysis

A moving obstacle immersed in a perfect fluid

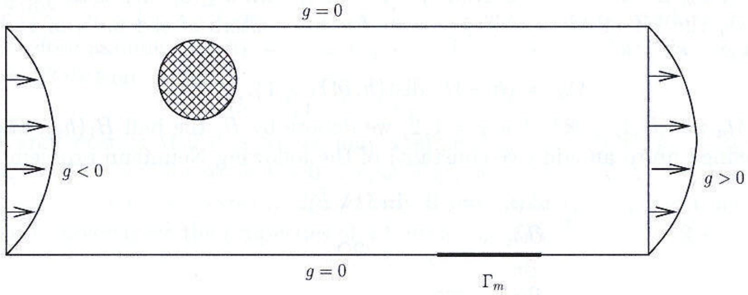


FIGURE 1. Moving obstacle in a pipeline.

$g = \mathbf{v} \cdot \mathbf{n}$ denotes the input flow through the boundary Ω

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p &= 0, \quad \mathrm{div}\ \mathbf{v} = 0 \quad \text{in} \quad \Omega \setminus \overline{S(t)} \\
\mathbf{v}\cdot\mathbf{n} &= (\mathbf{h}' + \omega(\boldsymbol{x} - \mathbf{h})^{\perp})\cdot\mathbf{n} \quad \text{on} \quad \partial S(t) \\
\mathbf{v}\cdot\mathbf{n} &= g \quad \text{on} \quad \partial\Omega \\
m\mathbf{h}''(t) &= \int_{\partial S(t)} p\,\mathbf{n}\,ds + \boldsymbol{f}(t) \\
J\omega'(t) &= \int_{\partial S(t)} (\boldsymbol{x} - \mathbf{h}(t))^{\perp}\cdot p\,\mathbf{n}\,ds + T(t).
\end{aligned}$$

and for $t \geq 0$, where

- $\mathbf{h}(t)$: center of mass of $S(t)$
- ω : angular velocity
- $\mathbf{n}$: outward unit normal vector, $\boldsymbol{x}^{\perp} = (-x_2, x_1)$ if $\boldsymbol{x} = (x_1, x_2)$

Ghost solutions

A. Majda, *Comm. Pure Appl. Math.* **37**, 1984

- Euler equations do not exhibit any unique continuation property because of the existence of *ghost solutions* with compact support.
- A simple example is provided by the stationary solution $\mathbf{v}(x) = (\partial\psi/\partial x_2, -\partial\psi/\partial x_1)$, where the stream function ψ is given by

$$\psi(x) = -\int_1^{|x|} \frac{1}{r}\left(\int_1^r s\, w(s)\, ds\right) dr$$

  and the vorticity $w \in C^\infty(\mathbb{R}^+)$ is chosen so that $w(s) = 0$ for $s \geq 1$ and $\int_r^1 sw(s)ds = 0$ for $r \in (0, r_0)$, where $r_0 \in (0, 1)$.
- $\mathbf{v}(x)$ is supported in the set $\{r_0 \leq |x| \leq 1\}$. The identifiability property thus fails for Eulerian flows.

A simplified inverse problem
corresponding to a simplified model

- Let us fix $t = t_0$ and focus on the determination of the position and velocity of $S(t_0)$ from one boundary measurement of the velocity at time t_0.
- We will hence ignore Newton's laws for $S(t)$ in our analysis.

- (**Moreover**) we will restrict ourselves to potential flows, i.e. flows for which

$$\mathbf{v} = \nabla \varphi$$

- and to spherical obstacles, say

$$S(t) = B_1(\mathbf{h}(t)) = \text{ ball of radius 1 and centered at } \mathbf{h}(t)$$

Potential flow model

Plugging $\mathbf{v} = \nabla\varphi$ in Euler's system results in a Laplace type-like system

$$\begin{aligned} \Delta\varphi &= 0 \quad \text{in} \quad \Omega \setminus S(t) \\ \frac{\partial\varphi}{\partial n} &= (\mathbf{h}' + \omega(\boldsymbol{x} - \mathbf{h})^{\perp}) \cdot \mathbf{n} \quad \text{on} \quad \partial S(t) \\ \frac{\partial\varphi}{\partial n} &= g \quad \text{on} \quad \partial\Omega \end{aligned}$$

and for all $t \geq 0$.

Clearly,

measuring the normal component of the velocity $\mathbf{v}$ on one part of the boundary amounts to measuring the function φ itself.

Assume that $S(t) = B_1(\mathbf{h}(t))$. Then

$$\boldsymbol{x} - \mathbf{h} = -\mathbf{n}, \quad \forall x \in \partial B_1(\mathbf{h}(t)),$$

and hence $(\boldsymbol{x} - \mathbf{h})^{\perp} \cdot \mathbf{n} = 0$. Setting $\ell = \mathbf{h}'$, the system reads

$$\begin{cases} \Delta\varphi = 0 & \text{in} \quad \Omega \setminus \overline{B_1(\mathbf{h}(t))} \\ \dfrac{\partial\varphi}{\partial n} = \ell \cdot \mathbf{n} & \text{on} \quad \partial B_1(\mathbf{h}(t)) \\ \dfrac{\partial\varphi}{\partial n} = g & \text{on} \quad \partial\Omega \end{cases}$$

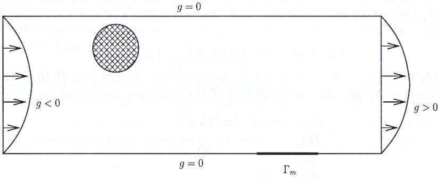


FIGURE 1. Moving obstacle in a pipeline.

Natural input flows g

Standard Neumann compatibility condition

$$\int_{\partial\Omega} g \, ds = 0$$

Linear input flows are excluded

Clearly, $g(x) = \boldsymbol{\ell} \cdot \mathbf{n}(x)$, with $\boldsymbol{\ell} \in \mathbb{R}^2$ a given vector, has to be excluded, for it may lead to the situation where the ball, which is surrounded by a fluid flowing at the same velocity ($\varphi(x,t) = \boldsymbol{\ell} \cdot x$), is not identifiable.

Theorem of identifiability

C^2, P. Cumsille, J.H. Ortega & L. Rosier, *Inverse Problems* **24**, 2008

Theorem 3

- *Define* $V = \mathrm{span}\,\{\mathbf{e}_1 \cdot \mathbf{n}, \mathbf{e}_2 \cdot \mathbf{n}\} \subset L^\infty(\partial\Omega)$.
- *Assume that* $g \in H^s(\partial\Omega) \setminus V$ *with* $s > 1/2$. *For* $i = 1, 2$, *choose* $(\boldsymbol{h}_i, \boldsymbol{\ell}_i) \in \Omega_{ad} \times \mathbb{R}^2$ *and let* φ_i *denote the solution to*

$$(P) \quad \begin{cases} \Delta\varphi_i = 0 & \text{in} \quad \Omega \setminus \overline{B_i} \\ \dfrac{\partial\varphi_i}{\partial n} = g & \text{on} \quad \partial\Omega \\ \dfrac{\partial\varphi_i}{\partial n} = \boldsymbol{\ell}_i \cdot \mathbf{n} & \text{on} \quad \partial B_i, \qquad \displaystyle\int_{\Omega\setminus B_i} \varphi_i = 0 \end{cases}$$

Then problem (P) is identifiable in the following sense :

$$\varphi_1 = \varphi_2 \quad \text{on} \quad \Gamma_m \quad \Rightarrow \boldsymbol{h}_1 = \boldsymbol{h}_2 \quad \text{and} \quad \boldsymbol{\ell}_1 = \boldsymbol{\ell}_2.$$

Here, $\Omega_{ad} = \{\boldsymbol{h} \in \Omega \mid \mathrm{dist}(\boldsymbol{h}, \partial\Omega) > 1\}$.

Sketch of the proof (I)

- $\varphi_i \in H^{s+\frac{3}{2}}(\Omega \setminus \overline{B_i}) \subset C^1(\overline{\Omega} \setminus B_i)$ (standard elliptic regularity)
- Assume that $\varphi_1 = \varphi_2$ on Γ_m
- Since also $\frac{\partial \varphi_1}{\partial n} = g = \frac{\partial \varphi_2}{\partial n}$ on Γ_m, classical unique continuation property for the Laplace equation yields

$$\varphi_1 = \varphi_2 \quad \text{on} \quad \Omega \setminus \overline{B_1 \cup B_2}$$

- Define $\varphi \colon \Omega \setminus \overline{B_1 \cap B_2} \to \mathbb{R}$ by

$$\varphi(x) \overset{\text{(def)}}{=} \begin{cases} \varphi_1(x) & \text{if} \quad x \in \Omega \setminus \overline{B_1} \\ \varphi_2(x) & \text{if} \quad x \in \Omega \setminus \overline{B_2} \end{cases}$$

-

$$\begin{aligned} \Delta \varphi &= 0 \quad \text{in} \quad \Omega \setminus \overline{B_1 \cap B_2} \\ \frac{\partial \varphi}{\partial n} &= g \quad \text{on} \quad \partial \Omega \end{aligned}$$

$$\frac{\partial \varphi}{\partial n} = \ell_1 \cdot n \quad \text{on } \partial B_1 \quad , \quad \frac{\partial \varphi}{\partial n} = \ell_2 \cdot n \quad \text{on } \partial B_2$$

Several possible situations

If $B_1 \cap B_2 = \emptyset$

- then φ is, as φ_2, defined and harmonic in B_1.
- Since $\frac{\partial \varphi}{\partial n} = \boldsymbol{\ell}_1 \cdot \boldsymbol{n} \quad \text{on} \quad \partial B_1$, uniqueness of the Neumann problem in B_1, yields

$$\varphi(x) = \boldsymbol{\ell}_1 \cdot \boldsymbol{x} + \text{constant} \quad \text{on} \quad B_1,$$

- and by unique continuation, the same is true in Ω.
- This implies $g = \boldsymbol{\ell}_1 \cdot \boldsymbol{n}$ on $\partial\Omega$, and hence $g \in V$, which is a contradiction.

Assume $B_1 \cap B_2 \neq \emptyset$ from now on

Of course, if $B_1 = B_2$ (i.e., $\boldsymbol{h}_1 = \boldsymbol{h}_2$)

then

$$\boldsymbol{\ell}_1 \cdot \boldsymbol{n} = \frac{\partial \varphi}{\partial n} = \boldsymbol{\ell}_2 \cdot \boldsymbol{n} \quad \forall \boldsymbol{n}$$

and hence $\boldsymbol{\ell}_1 = \boldsymbol{\ell}_2$.

If $B_1 \neq B_2$ but $\boldsymbol{\ell}_1 = \boldsymbol{\ell}_2$,

then set $D_1 \overset{(\text{def})}{=} B_1 \setminus \overline{B_2}$ ("croissant"). Clearly, φ solves

$$\begin{aligned} \Delta\varphi &= 0 && \text{in} \quad D_1 \\ \frac{\partial \varphi}{\partial n} &= \boldsymbol{\ell}_1 \cdot \boldsymbol{n} && \text{on} \quad \partial D_1, \end{aligned}$$

which gives again $\varphi \equiv \boldsymbol{\ell}_1 \cdot \boldsymbol{x} + \textit{constant}$ in D_1 and $g \in V$, which is a contradiction.

1st step: $(\boldsymbol{h}_1 - \boldsymbol{h}_2)$ and $(\boldsymbol{\ell}_1 - \boldsymbol{\ell}_2)$ must be orthogonal

- Without loss of generality, full symmetry with respect to both axes can be assumed (see figure)
- Since φ is harmonic in the croissant-shaped domain D_1; integration by parts yields

$$\int_{\gamma_1} \boldsymbol{\ell}_1 \cdot (\boldsymbol{x} - \boldsymbol{h}_1) ds = \int_{\gamma_2} \boldsymbol{\ell}_2 \cdot (\boldsymbol{x} - \boldsymbol{h}_2) ds$$

- Here, both boundary contributions can be easily calculated using polar coordinates; the desired result follows.

2nd step: Complex variable analysis

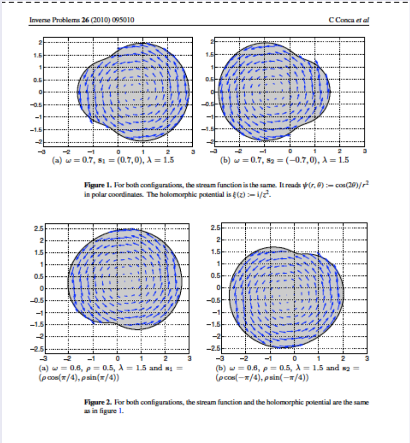
φ is analytically extended to the lentil-shaped region $B_1 \cap B_2$ and a similar contradiction can be achieved.

ill-posedness for general domains

C^2, M. Malik & A. Munnier, *Inverse Problems* **26**, 2010

Main issues

- Counterexamples shows that a same potential velocity may correspond to different positions and velocities of a same solid :



(a) $\omega = 0.7$, $\mathbf{s}_1 = (0.7, 0)$, $\lambda = 1.5$

(b) $\omega = 0.7$, $\mathbf{s}_2 = (-0.7, 0)$, $\lambda = 1.5$

Figure 1. For both configurations, the stream function is the same. It reads $\psi(r,\theta) := \cos(2\theta)/r^2$ in polar coordinates. The holomorphic potential is $\xi(z) := \mathrm{i}/z^2$.

(a) $\omega = 0.6$, $\rho = 0.5$, $\lambda = 1.5$ and $\mathbf{s}_1 = (\rho\cos(\pi/4), \rho\sin(\pi/4))$

(b) $\omega = 0.6$, $\rho = 0.5$, $\lambda = 1.5$ and $\mathbf{s}_2 = (\rho\cos(-\pi/4), \rho\sin(-\pi/4))$

Figure 2. For both configurations, the stream function and the holomorphic potential are the same as in figure 1.

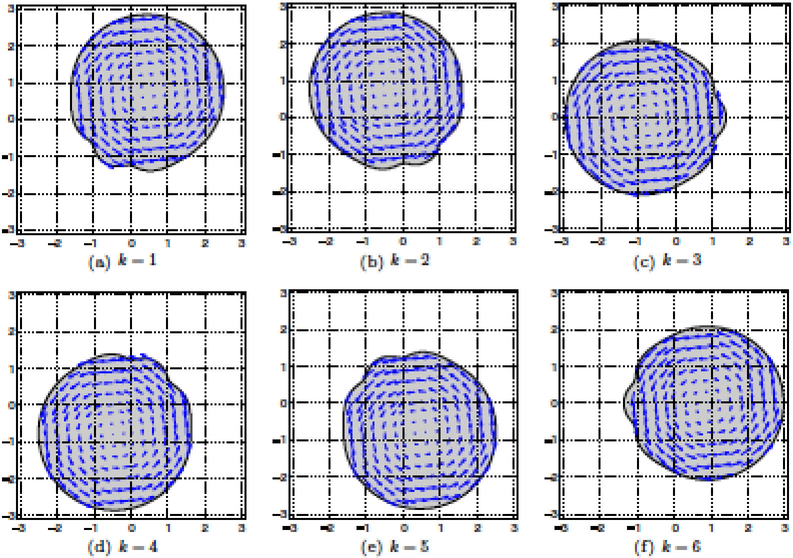


Figure 3. The stream function is $\psi(r,\theta) = \cos(6\theta)/r^6$, the holomorphic potential is $\xi = i/z^6$ and $\omega = 0.7$, $\rho = 0.9$, $\lambda = -2.5$ and $s_1 = (\rho\cos(2k\pi/6), \rho\sin(2k\pi/6))$ for $k = 1, \ldots, 6$.

- However, for specific shapes (moving ellipses for instance), the problem of detection has a unique solution.
- Solids enjoying symmetries can be partially identified.
- Continuous measurements of the fluid potential over a time interval allows a continuous tracking of the solid :

Theorem 4 (Tracking)

For any solid S_0 if its position at $t = 0$ is known and continuous measurements of the potential over $[0, T]$, $T > 0$ are performed, then the configuration of the solid at any time $t \in [0, T]$ can be deduced.

Methods

Complex analysis (conformal mappings)

A moving obstacle immersed in a (stationary) viscous fluid

$$
\begin{aligned}
\operatorname{div}(\sigma(\mathbf{u},p)) &= 0 \quad \text{in} \quad F(t) = \Omega \setminus \overline{S(t)} \subset \mathbb{R}^{2,3} \\
\operatorname{div}\, \mathbf{u} &= 0 \quad \text{in} \quad F(t) \\
\mathbf{u} &= \mathbf{h}' + \boldsymbol{\omega} \times (\boldsymbol{x} - \mathbf{h}) \quad \text{on} \quad \partial S(t) \\
\mathbf{u} &= \mathbf{g} \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

$$
\int_{\partial S(t)} \sigma(\mathbf{u},p)\mathbf{n}ds = 0, \qquad \int_{\partial S(t)} (\boldsymbol{x} - \mathbf{h}) \times \sigma(\mathbf{u},p)\mathbf{n}ds = 0
$$

$$
S(0) = S_0, \qquad \mathbf{h}(0) = \mathbf{h}_0, \ \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0
$$

and for $t \in (0, T)$, where

$\mathbf{h}(t)$: center of mass of $S(t)$

$\boldsymbol{\omega}(t)$: angular velocity

$\mathbf{g}$ is a given velocity satisfying the compatibility condition

$$
\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}ds = 0
$$

A simplified inverse geometrical problem

Low Reynold's number regime so inertial forces are neglected

- Assume Γ_m is a non empty open subset of $\partial\Omega$ where we can measure $\sigma(u,p)\mathbf{n}_{|\Gamma}$ at some time $t_0 > 0$.
- Is it possible to recover S_0?

Existence and uniqueness

T. Takahashi & M. Tucsnak, *J. Math. Fluid Mech.* **6**, 2004; M. Boulakia, *J. Math. Fluid Mech.* **9**, 2007.

Theorem 5

Assume $\mathbf{g} \in \boldsymbol{H}^{3/2}(\partial\Omega)$, S_0 *to be a smooth non empty domain and assume* $(\mathbf{h}_0, \omega_0) \in \mathbb{R}^N \times \mathbb{R}^N$ *(N=2 or 3) be given.*
Then there exists a maximal time $T_* > 0$ *and a unique solution*

$$\mathbf{h} \in \boldsymbol{C}^1([0, T_*); \mathbb{R}^3), \quad (\boldsymbol{h}', \omega) \in \boldsymbol{C}([0, T_*); \mathbb{R}^3 \times \mathbb{R}^3),$$

$$(\boldsymbol{u}, p) \in \boldsymbol{C}\left([0, T_*); \boldsymbol{H}^2(F(t)) \times H^1(F(t))/\mathbb{R}\right)$$

satisfying the quasi-stationary Stokes system. Moreover one of the following alternatives holds:

- $T_* = +\infty$;
- $\lim_{t \to T_*} \operatorname{dist}\left(S(t), \partial\Omega\right) = 0.$

Preliminaries (Identifiability result)

- Let us take two smooth non empty open sets $S_0^{(1)}$, $S_0^{(2)}$
- Let us also consider $\left(\boldsymbol{h}_0^{(i)}, \omega_0^{(i)}\right)$, $i = 1, 2$ such that the corresponding solid structures are such that

$$\overline{S_0^{(1)}} \subset \Omega \quad \text{and} \quad \overline{S_0^{(2)}} \subset \Omega$$

- Applying the above existence result, we deduce that for any $\mathbf{g} \in \boldsymbol{H}^{3/2}(\partial\Omega)$, there exists $T_*^{(1)} > 0$ (respectively $T_*^{(2)} > 0$) and a unique solution $\left(\boldsymbol{h}^{(1)}, \omega^{(1)}, \mathbf{u}^{(1)}, p^{(1)}\right)$ (respectively $\left(\boldsymbol{h}^{(2)}, \omega^{(2)}, \mathbf{u}^{(2)}, p^{(2)}\right)$) in $\left[0, T_*^{(1)}\right)$ (respectively in $\left[0, T_*^{(2)}\right)$)

Theorem 6 (Identifiability)

Assume $\mathbf{g}$ *is not the trace of a rigid velocity on* Γ*, and that* $S_0^{(1)}$*,* $S_0^{(2)}$ *are convex. If there exists* $0 < t_0 < \min\left(T_*^{(1)}, T_*^{(2)}\right)$ *such that*

$$\sigma\left(\mathbf{u}^{(1)}(t_0), p^{(1)}(t_0)\right)\ \mathbf{n}_{|\Gamma} = \sigma\left(\mathbf{u}^{(2)}(t_0), p^{(2)}(t_0)\right)\ \mathbf{n}_{|\Gamma}$$

then there exists $R \in SO_3(\mathbb{R})$ *such that*

$$RS_0^{(1)} = S_0^{(2)}$$

and

$$\boldsymbol{h}_0^{(1)} = \boldsymbol{h}_0^{(2)}, \qquad \omega_0^{(1)} = \omega_0^{(2)}.$$

In particular, $T_*^{(1)} = T_*^{(2)}$ *and* $S^{(1)}(t) = S^{(2)}(t) \quad \forall t \in \left[0, T_*^{(1)}\right)$

Perspectives

Several deformable unknown moving obstacles