The logarithmic KPP front time delay in a periodic medium

Lenya Ryzhik, Stanford,

with F. Hamel (Marseille), J. Nolen (Duke),
and J.-M. Roquejoffre (Toulouse)

Outline

Fronts in homogeneous media

KPP front time delay

Generalized transition waves: KPP

Open questions

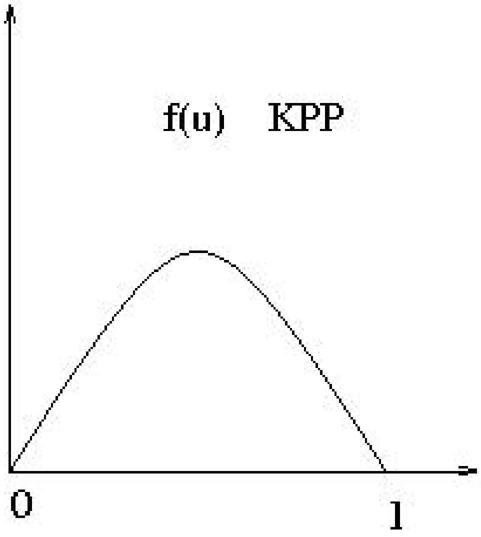
Reaction-diffusion fronts in homogeneous media

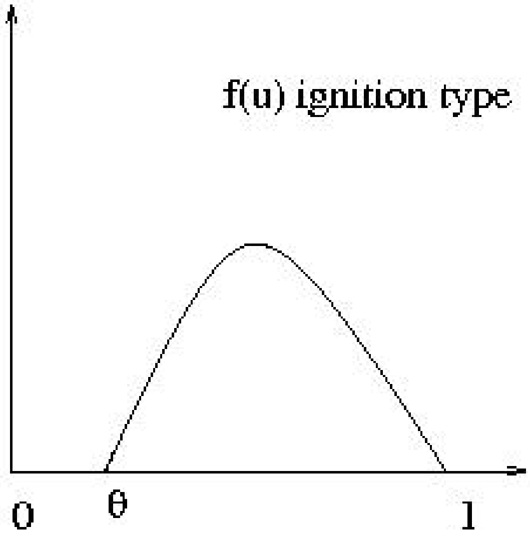
$$\frac{\partial u}{\partial t} = \Delta u + f(u)$$

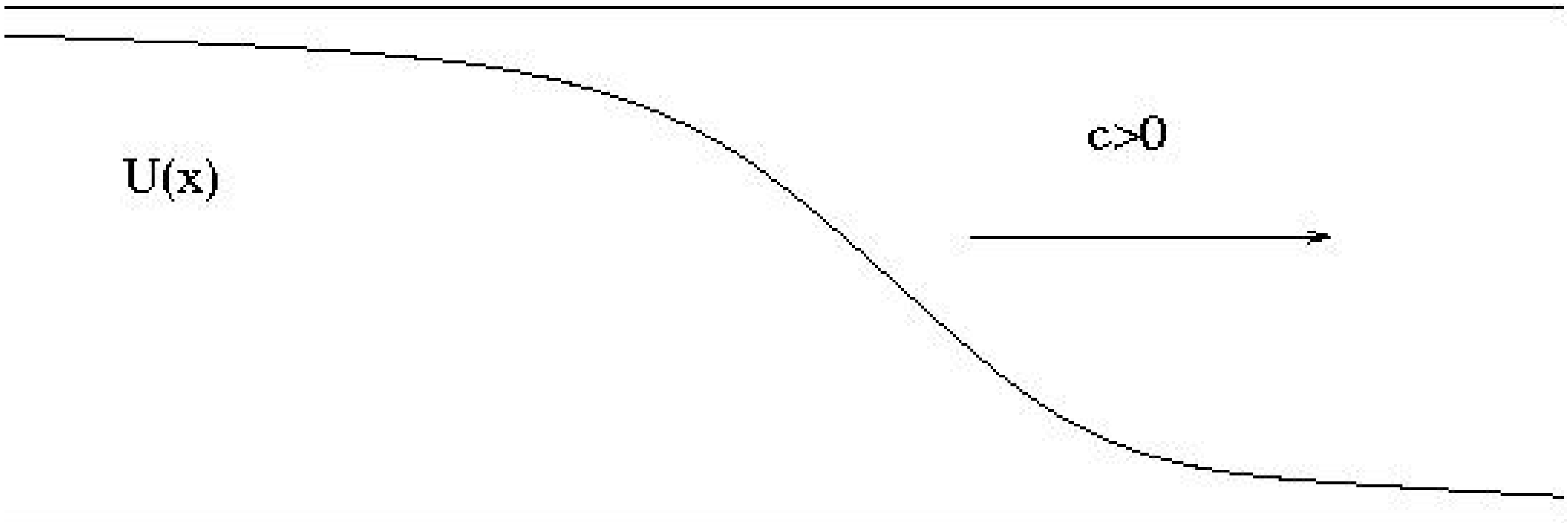
Reaction types: $f(u) \geq 0$

1. KPP: $f(u) \leq f'(0)u$, $f'(0) \neq 0$ $(f(u) = u(1-u))$

2. Ignition: $f(u) = 0$ for $u \in [0, \theta]$ – NOT TODAY.







Planar fronts: $u(t,x) = U_c(x-ct)$, $U_c(-\infty) = 1$, $U_c(+\infty) = 0$,

exist for $c \geq c_* = 2\sqrt{f'(0)}$ in the KPP case.

TW equation: $-cU_c' = U_c'' + f(U_c)$

Existence and stability of fronts

KPP: Fisher; Kolmogorov, Petrovski and Piskunov (1937):

traveling waves exist for any $c \geq c_*$.

Supercritical: exponentially stable – if $u_0(x) \sim e^{-\lambda_c x}$,

$\lambda_c^2 - c\lambda_c + f'(0) = 0$, then $|u(t,x) - U_c(x - c_* t - \xi)| \leq Ce^{-\alpha t}$

Minimal speed: $|u(t,x) - U_{c^*}(x - c^* t - m(t))| \to 0$ as $t \to +\infty$,

if $u_0(x) \in C_c(\mathbb{R})$.

How does $m(t)$ behave?

$m(t) = o(t)$ but does it grow in time?

History of the shift

Level set $E_s(t) = \{x : \; u(t,x) = s\}$, $0 < s < 1$.

KPP (1937), key observation: start with a step function,

then $t \mapsto |u_x(t,.)|$ decreases on each $E_s \Rightarrow m(t)$ exists.

Uchiyama'78: $m(t) = -3/(2\lambda^*) \ln t + O(\ln \ln t)$

Bramson'78,'83: $m(t) = -3/(2\lambda^*) \ln t + x_m + o(1)$

Lau'85 – a PDE proof based on the intersection number,

$f'(s) \le f'(0)s$, one-dimensional.

Brunet-Derrida – effect of finite system size

How to understand this in simple PDE terms?

Higher dimensions?

What happens in heterogeneous media?

Pulsating fronts in periodic media

Pulsating front: $U(x \cdot e - ct, x)$, periodic in the second variable

– periodic picture in the moving frame

$u_t = \Delta u + f(x, u)$

Xin (1992-93): ignition, periodic BC:

$U(x \cdot e - c_* t, x, y)$, c_* unique, stability.

H. Berestycki, F. Hamel (2002): KPP: $c \geq c_*$;

Ignition, bistable: c_* unique, periodic boundaries.

Fronts in general media

What is a front? (i) Self-replication, (ii) Togetherness.

Why do we want a front? (i) Curiosity, (ii) Stability.

Media inhomogeneities:

(i) Variable reaction-rates, diffusion coefficients, drifts etc.

(ii) Non-flat, non-periodic boundaries: crooked pipes, holes, fronts around an obstacle ... Draw lots of pictures!

(iii) Species persistence/extinction – H. Berestycki, F. Hamel, L. Roques

Matano's definition (90's). Replication: front shape is a continuous function of the environment – homogeneous, (almost) periodic...

Berestycki-Hamel. Togetherness: a transition front with interface Γ_t is a global in time solution which stays together: $u(t,x) \to p_\pm$ uniformly in $t \in \mathbb{R}$ as $d(x,\Gamma_t) \to +\infty$.

BH fronts are not always M-fronts – even in the uniform case!

Berestycki and Hamel'07:

In the homogenous KPP case there exist

(i) transition fronts which have no speed (different speed as $t \pm \infty$), and

(ii) There exist fronts turning direction and changing speed in $\mathbb{R}^n$, $n > 1$.

Inhomogeneous case: R., Nolen'09, Mellet, Roquejoffre'09, Mellet, Nolen, Roquejoffre, R.'11, Zlatos'11.

M. Brenner: who cares about global in time solutions!

$u_t = u_{xx} + f(u), \qquad u(0,\cdot) = u_0 \in C_c(\mathbb{R})$.

The critical front $U_{c^*}(x - c^* t)$: $U_{c^*}(\xi) \sim A\,\xi\,e^{-\lambda^* \xi}$ as $\xi \to +\infty$, with $\lambda^* = c^*/2 = \sqrt{f'(0)}$.

Theorem. (Bramson'78)

If $u_0 \in C_c(\mathbb{R})$ then the solution $u(t,x)$ looks like a traveling wave at $X(t) = c_* t - 3/(2\lambda^*) \log t + O(1)$.

Possible generalizations – linear vs. nonlinear or transition fronts.

The periodic case

$u_t = u_{xx} + f(x,u), \quad u(0,x) = u_0(x) \in C_c(\mathbb{R})$.

Pulsating fronts exist for all $c \geq c_*$: $U_{c_*}(\xi, x) \sim \xi e^{-\lambda_* \xi}\psi(x)$.

Theorem. (Hamel, Nolen, Roquejoffre, R.'12)

If $u_0 \in C_c(\mathbb{R})$ then the solution $u(t,x)$ looks like a pulsating front at $X(t) = c_* t - 3/(2\lambda^*) \log t + O(1)$.

How many PDE'ists does it take to reprove a 25 year old probabilistic result? (Hamel, Nolen, Roquejoffre, R.'12)

Key (new?) observation: KPP behaves as the linearized equation with the Dirichlet BC:

$$z_t = z_{xx} + f'(0)z, \quad t > 0, \quad x > \xi(t), \quad z(t, \xi(t)) = 0.$$

Main claim: if $\xi(t) = c^* t - (3/2\lambda^*) \ln(t + t_0)$ then $z(t, x)$ remains bounded from above and below in a certain region.

Two proofs: explicit formulas and with tied hands.

No shame: use explicit formulas

New variables $x' = x - \left(c^*t - \frac{r}{\lambda^*}\ln(t+t_0)\right)$ (general $r > 0$):

$$z_t - z_{xx} - \left(c^* - \frac{r}{\lambda^*(t+t_0)}\right) z_x - f'(0)z = 0, \quad z(t,0) = 0,$$

$z_0 \in C_c(\mathbb{R})$.

Lemma. If $r = 3/2$, then there is a constant $t_0 > 0$ that depends on z_0 such that, for all $0 < a \le b < +\infty$, we have

$$0 < \inf_{t \ge 1,\, a \le x \le b} z(t,x) \le \sup_{t \ge 1,\, a \le x \le b} z(t,x) < +\infty.$$

Introduce $z(t,x) = e^{-\lambda^* x} v(t,x)$:

$$v_t - v_{xx} + \frac{r}{\lambda^*(t+t_0)}(v_x - \lambda^* v) = 0, \quad v(t,0) = 0$$

Self-similar variables $\tau = \ln(t+t_0)/t_0$, $y = \dfrac{x}{\sqrt{t+t_0}}$:

$$v_\tau + Lv = (r-1)v - \varepsilon e^{-\tau/2} v_y, \quad \tau > 0,\ y > 0,$$

with $Lv = -v_{yy} - \dfrac{y}{2} v_y - v$, and $\varepsilon = r/(t_0^{1/2}\lambda^*)$.

A perturbative estimate:

$$v(\tau,y) = e^{(r-1)\tau} y \Big(\frac{e^{-y^2/4}}{2\sqrt{\pi}} \int_0^{+\infty} \xi v_0(\xi) d\xi + O(\varepsilon) + O(e^{-\tau/2}) \Big).$$

$O(\varepsilon)$ and $O(e^{-\tau/2})$ – functions of τ and y which are of that order for $\tau > 0$, and y in any fixed compact set.

For $z(t,x)$ (still in the logarithmically shifted frame):

$$z(t,x) = \frac{(t+t_0)^{r-3/2}}{t_0^{r-1}} x e^{-\lambda^* x} \left[C e^{-x^2/4(t+t_0)} + h(t,x) \right],$$

with $\limsup_{t \to +\infty} \sup_{0 \le x \le \sigma\sqrt{t+1}} |h(t,x)| < \frac{C}{2}$.

Hence: $r = 3/2 \Rightarrow z(t,x)$ bounded from above and below

away from zero on the interval $1 \le x \le 2$ for all $t > 1$.

Without the shift we get a solution that decays as $t^{-3/2}$.

Harnack inequality:

$$0 < \inf_{t \ge 1} \left(\min_{[a,b]} z(t,\cdot) \right) \le \sup_{t \ge 1} \left(\max_{[a,b]} z(t,\cdot) \right) < +\infty$$

for all $0 < a \le b$.

An upper bound for u

Proposition.

(i) $\limsup_{t\to+\infty} u\Big(t, c^*t - \frac{3}{2\lambda^*}\ln t + y\Big) < 1$ for all $y \in \mathbb{R}$.

(ii) For every $\sigma > 0$, there is $\rho > 0$ such that

$$u\Big(t, c^*t - \frac{3}{2\lambda^*}\ln t + y\Big) \le \rho\,(y+1)\,e^{-\lambda^* y}$$

for all $t \ge 1$ and $0 \le y \le \sigma\sqrt{t}$.

Proof. Define $\bar{U}(t,x) = \begin{cases} 1, & \text{if } x \le A \\ \min(1, Bz_A(t,x)), & \text{if } x \ge A. \end{cases}$

The $3/(2\lambda^*)\ln t$ shift ensures that $U \equiv 1$ and Bz_A intersect

at a point $x(t)$ whose location is uniformly bounded in time.

A lower bound for u when f is linear at zero

Special case $f(s) = f'(0)s$ for $s \in [0, s_0)$ with some $s_0 > 0$, the same argument gives a lower bound on $u(t,x)$.

The function $\tilde{z} = \delta z_A$ is a subsolution for u if $\delta > 0$ is sufficiently small to ensure that $0 \leq \tilde{z} \leq s_0$, then

$$u\Big(t, c^* t - \frac{3}{2\lambda^*} \ln(t+1) + x\Big) \geq \tilde{z}(t,x) \text{ for all } t > 0 \text{ and } x \in \mathbb{R}.$$

The periodic case

$u_t = u_{xx} + f(x,u), \quad u(0,x) = u_0(x) \in C_c(\mathbb{R})$.

Pulsating fronts exist for all $c \geq c_*$: $U_{c_*}(\xi,x)" \sim "\xi e^{-\lambda_* \xi}\psi(x)$.

Theorem. (Hamel, Nolen, Roquejoffre, R.'11)

If $u_0 \in C_c(\mathbb{R})$ then the solution $u(t,x)$ looks like a pulsating front at $X(t) = c_* t - 3/(2\lambda^*)\log t + O(1)$.

PDE proof without explicit formulas

The linearized problem

$$v_t = v_{xx} + f'(0)v, \quad x > Y(t), \qquad v(t, Y(t)) = 0.$$

Choose $Y(t) = c_* t - 3/(2\lambda_*) \log t \Rightarrow v(t) = O(1)$ as $t \to +\infty$.

Morally equivalent:

$$w_t = w_{xx} + f'(0)w, \quad x > c_* t, \qquad w(t, c_* t) = 0.$$

Show $w(t, c_* t + 1) \geq C/t^{3/2}$ – without explicit formulas!

The homogeneous case: $w(t,x) = e^{-\lambda_*(x-c_*t)}p(t,x)$ then

$p_t = p_{xx} - c_* p_x, \;\; x > c_* t, \;\; p(t, c_* t) = 0.$

Lemma 1. There exist constants $T_0 > 0$, $\sigma > 0$, and $C_0 > 0$ such that $p(t, c^*t + \sigma\sqrt{t}) \geq C_0/t$ for all $t \geq T_0$.

Lemma 2. There exists $T_0 > 0$, and $c_0 > 0$, $\beta > 0$, and $N > 0$ that depend only on the initial data so that for any $t > T_0$ there exists a set $I_t \subset [c^*t + N^{-1}\sqrt{t}, c^*t + N\sqrt{t}]$ such that $|I_t| \geq \beta\sqrt{t}$ and $p(t,x) \geq \dfrac{c_0}{t}$ for all $x \in I_t$.

In the moving frame

Ingredient 1: $\int_0^\infty xp(t,x)dx = \int_0^\infty xp_0(x)dx$

Ingredient 2: $|p(t,x)| \le \frac{Cx}{(t+1)^{3/2}}\int_0^\infty yp_0(y)dy$

Ingredient 3:

$$\left(\int_0^\infty \frac{e^{2\alpha x}-e^{-2\alpha x}}{2\alpha x}p^2(t,x)dx\right)^{1/2} \le \frac{Ce^{\alpha^2 t}}{t^{3/4}}\int_0^\infty \frac{e^{\alpha x}-e^{-\alpha x}}{\alpha}p_0(x)dx.$$

To prove Lemma 2: take $\alpha = 1/\sqrt{t}$:

$$\left(\int_{N\sqrt{t}}^{\infty} \frac{e^{2x/\sqrt{t}}}{x} p^2(t,x)dx\right)^{1/2} \le \frac{C}{t}\int_0^{\infty} xp_0(x)dx.$$

Cauchy-Schwartzing: $\int_{N\sqrt{t}}^{\infty} xp(t,x)dx \le \frac{1}{4}\int_0^{\infty} xp_0(x)dx = \frac{I(0)}{4}$.

for $N > N_0$ is large enough (but independent of t). As $I(t) = I(0)$, it follows that

$$\int_0^{N\sqrt{t}} xp(t,x)dx \ge \frac{3I_0}{4}.$$

Now, combine with $p(t,x) \le x/t^{3/2}$ and show that $p(t,x)$ can not be too small too often for $x \sim c_* t + O(\sqrt{t})$.

Linear Dirichlet to a nonlinear subsolution: $\bar{w}(t,x) = a(t)w(t,x)$

$\bar{w}_t \leq \bar{w}_{xx} + g(x)f'(0)\bar{w} - g(x)q(\bar{w})$, with $q(w) = f'(0)w - f(w)$,

provided that

$$\frac{da}{dt}w(t,x) \leq -g(x)q(aw).$$

As $q(w) \leq mw^2$, enough:

$$\frac{da}{dt}w(t,x) \leq -Ma^2w^2.$$

But: $p(t,x) \leq \frac{C_0(x-c_*t)}{(t+1)^{3/2}} \Rightarrow w(t,x) \leq \dfrac{C_0}{(t+1)^{3/2}}$.

Hence, enough:

$$\frac{da}{dt} \le -\frac{M}{(t+1)^{3/2}} a^2,$$

and we may take

$$a(t) = \frac{a(0)}{1 + 2Ma(0)(1 - (t+1)^{-1/2})},$$

$a(0)/(1 + 2Ma(0)) \le a(t) \le a(0)$ for all $t \ge 0$.

The comparison principle:

$$u(t,x) \ge \bar{w}(t,x) = a(t)w(t,x) \ge Cw(t,x).$$

and $u(t, ct + \sigma\sqrt{t}) \ge Ct^{-1}e^{-\lambda\sigma\sqrt{t}}$

Last step: this implies

$$u(t, c^*t - (\frac{3}{2\lambda^*}) \log t - h_\varepsilon) \geq 1 - \varepsilon,$$

Set $\tilde{U}(t,x) = U(t - r(t), x)$:

$$\tilde{U}_t - \tilde{U}_{xx} - f(\tilde{U}) = U_t - r'(t)U_t - U_{xx} - f(U) = -r'(t)U_t \leq 0,$$

provided that $r'(t) \geq 0$. Then

$$\tilde{U}(t,x) \leq u(t,x) \quad \text{for all} \quad x \leq c^*t + \sigma\sqrt{t}, \;\; t \geq T_0,$$

if $\tilde{U}(t, c^*t + \sigma\sqrt{t}) \leq Ct^{-1}e^{-\lambda\sigma\sqrt{t}}$,

which is true with $r(t) = \left(\frac{3}{2\lambda^* c^*}\right) \log t + L$,

if L is sufficiently large $\Rightarrow X(t) \geq c^*t - \left(\frac{3}{2\lambda^*}\right) \log t$.

For the upper bound: consider

$$v_t = v_{xx} + f'(0)v,\ x \geq c_* t - \frac{3}{2}\log t,\ v(t, c_* t - \frac{3}{2}\log t) = 0.$$

Show:

(i) $v(t, c_* t - \frac{3}{2}\log t + 1) \geq \alpha_0 > 0$,

(ii) $\lim_{y \to +\infty} v(t, c_* t - \frac{3}{2}\log t + y) = 0$ uniformly in t.

The nonlinear super-solution is

$$\bar{v}(t,x) = \min(1, Av(t,x)) \Rightarrow X(t) \leq c_* t - \frac{3}{2}\log t + O(1).$$

Existence of the transition fronts in heterogeneous media

Ignition nonlinearity:

Mellet, Nolen, Roquejoffre, Ryzhik'09-'11: transition fronts exist, are unique and are exponentially stable.

Transition fronts and bumps for heterogenous KPP

$u_t = u_{xx} + a(x)f(u), \quad x \in \mathbb{R}, \quad f'(0) = 1$

$0 < a_0 \le a(x) \le a_1 < +\infty$, and $a(x) \equiv 1$ for $|x| \ge M_0$.

A global in time $0 < u(t,x) < 1$ is "bump-like" if

$u(t,\cdot) \in L^1(\mathbb{R})$ – they do not exist if $a(x) \equiv$ const.

Nonexistence of transition fronts: $\lambda_M \uparrow \bar{\lambda}$ – the principle

Dirichlet eigenvalue of $\psi_{xx} + a(x)\psi = \lambda_M \psi, \quad \psi(x) > 0$,

in $(-M, M), \quad \psi(-M) = \psi(M) = 0$.

It all depends on $\bar{\lambda}$.

Theorem. (Nolen, Roquejoffre, R., Zlatos'10)

If $\bar{\lambda} > 2$ then there is unique (up to a time shift) global in time solution $u(t,x)$ such that $0 < u(t,x) < 1$. Moreover, $u(t,x)$ is "bump-like", and satisfies $\lim_{t\to-\infty} e^{-\bar{\lambda}t} u(t,x) = \psi(x)$. In particular, there exists no transition front.

A localized perturbation can kill transition fronts, and create a global in time bump-like solution.

A. Zlatos'11: vast generalization.

Where does nonexistence come from?

The linearized problem

$v_t = v_{xx} + a(x)v.$

Homogeneous: $a(x) \equiv 1$, exponential solutions

$v(t,x) = e^{\omega t - rx}$, growth rate $\omega = r^2 + 1$, speed $c = \omega / r$,

$r^2 - rc + 1 = 0$, $r \leq r_* = 1$.

Inhomogeneous: $\bar{v}_\omega(t,x) = e^{\omega t}\eta(x)$, with

$\eta_{xx} + a(x)\eta = \omega\eta, \quad \eta(x) \sim e^{-rx}$ as $x \to +\infty$,

also with $r \leq r_* = 1$. When is that possible?

$\psi_{xx} + a(x)\psi = \lambda_M \psi, \quad \psi(x) > 0$, in $(-M, M)$,

$\psi(-M) = \psi(M) = 0$, M large enough

Consider

$v_t = v_{xx} + a(x)v$,

$v(-T, x) = C_T \psi_M(x)$, $x \in (-M, M)$

$v(-T, x) = 0$ for $|x| \geq M$.

Choose C_T:

$v(-T, x) \leq \bar{v}_\omega(-T, x)$ but $v(-T, x_0) = \bar{v}_\omega(-T, x_0)$

$\Rightarrow C_T \sim e^{-\omega T}$.

The maximum principle:

$v(0,x) \leq \bar{v}_\omega(0,x) = \eta(x)$, and also

$v(0,x) \geq e^{\lambda_M T} C_T \psi_M(x)$.

This gives a contradiction for T large if $\lambda_M > \omega$.

Pass to $M \to +\infty \Rightarrow \eta_\omega(x)$ exists only if $\omega > \bar{\lambda}$.

For fronts: $\omega = r^2 + 1$, $r^2 - rc + 1 = 0$, $r \leq r_* = 1$.

Informally: no fronts if $\bar{\lambda} > 2$.

Fronts moving too fast should not exist if $\bar{\lambda} > 1$:

$c \leq \bar{\lambda}/\sqrt{\bar{\lambda} - 1}$.

Existence of not too fast KPP transitions fronts

A transition front $u(t,x)$ with speed c: $X(t)$ − the rightmost point x such that $u(t,x) = \frac{1}{2}$, then $\lim_{t-s\to+\infty} \frac{X(t)-X(s)}{t-s} = c$.

Homogeneous case: traveling fronts exist for all $c \geq 2$

Theorem. (Nolen, Roquejoffre, R., Zlatos'11)

If $\bar{\lambda} \in (1,2)$, then for each $c \in (2, \bar{\lambda}/\sqrt{\bar{\lambda}-1})$ there is a transition front solution moving with the speed c. No fronts move with $c > \bar{\lambda}/\sqrt{\bar{\lambda}-1}$. The bump-like solution also exists.

Zlatos'11: vast generalization.

Open problems

1. Front time delay in heterogeneous media – Zeitouni mysteries
2. Classification of global in time solutions for KPP
3. Higher dimensions