On the lack of compactness in 2D critical Sobolev embedding

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LJLL, February 17 2012

Lack of compactness

P. -L. Lions 1985

- Sobolev embedding in Lebesgue spaces
- Sobolev embedding in Orlicz spaces

Applications

- Nonlinear PDE
- Geometric problems

Defect of compactness related to a concentration phenomenon

Defect Measures

Sobolev embedding in Lebesgue spaces

$$\dot{H}^s(\mathbb{R}^d) \longrightarrow L^p(\mathbb{R}^d)$$

$0 \leq s < d/2$ and $p = 2d/(d-2s)$.

Non-compact

- $(\tau_{y_n} u)$, $(y_n) \to \infty$
- $\delta_{h_n} u(\cdot) = \frac{1}{h_n^{\frac{d}{p}}} u(\frac{\cdot}{h_n})$, $(h_n) \to \infty$ or $(h_n) \to 0$

Invariance by translation and by scaling

$$\|\tau_{y_n} u\|_{L^p} = \|u\|_{L^p}, \quad \|\delta_{h_n} u(\cdot)\|_{L^p} = \|u\|_{L^p}.$$

Concentration phenomenon

- P. -L. Lions 1985, defect measures

- P. Gérard 1996, microlocal defect measures

- P. Gérard 1998, profile decomposition

- S. Jaffard 1999, wavelet (L^q frame)

- H. Bahouri, A. Cohen and G. Koch 2011, wavelet (abstract frame)

Adimurthi, H. Bahouri-I. Gallagher, H. Brezis-J.-M. Coron, O. Druet, I. Gallagher-P. Gérard, E. Hebey, S. Ibrahim, S. Keraani, C. Laurent, M. Majdoub, G. Mancini, K. Sandeep, I. Schindler, S. Solimini, M. Struwe, T. Tao, K. Tintarev, ...

Profile decomposition, P. Gérard 1998

$$u_n(x) = \varphi^0(x) + \sum_{j=1}^{\ell} \frac{1}{(h_n^j)^{\frac{d}{p}}} \varphi^j\left(\frac{x - x_n^j}{h_n^j}\right) + \psi_n^\ell(x),$$

- \- φ^0 : weak limit

\- (h_n^j) : scales, (x_n^j) : cores, (φ^j) : profiles

- Orthogonality : $j \neq k$

\- $h_n^j / h_n^k \to 0$ or $h_n^j / h_n^k \to \infty$
\- $h_n^j = h_n^k$ and in this case $|x_n^j - x_n^k| / h_n^j \to \infty$.

- ψ_n^ℓ remainder term small in L^p

- Version within the Heisenberg group (J. Ben Ameur)

Stability

$$\|u_n\|^2_{\dot{H}^s} = \sum_{j=1}^{\ell} \|\psi^{(j)}\|^2_{\dot{H}^s} + \|\mathrm{r}_n^{(\ell)}\|^2_{\dot{H}^s} + \circ(1), \quad n \to \infty.$$

L^p norm

$$\|u_n\|^p_{L^p} \to \sum_{j \geq 1} \|\psi^{(j)}\|^p_{L^p}.$$

Applications

- Qualitative study (Bahouri-Gérard)

$$\partial_t^2 u - \Delta u + u^5 = 0$$

- Sharp estimate of the span time life (Kenig-Merle)

$$\partial_t^2 u - \Delta u - u^5 = 0$$

Critical Sobolev embedding

H. Bahouri, A. Cohen and G. Koch

Use of nonlinear wavelet approximation theory (inspired by the method of S. Jaffard) to treat a larger range of examples of critical embedding of functions spaces

$$X \hookrightarrow Y$$

Sobolev, Besov, Lorentz, BMO,...

Identification of two generic properties

- Existence of a nonlinear projector best approximation : $Q_M u_n$

- Stability of wavelet expansions in X : Fatou's lemma

- Nonlinear projector

$$u_n = Q_M u_n + R_M u_n$$

- Construction of approximate profiles

$$Q_M u_n = \sum_{m=1}^{M} d_{m,n}\psi_{\lambda(m,n)} = \sum_{m=1}^{M} d_m\psi_{\lambda(m,n)} + t_{n,M}$$

$$\sum_{m=1}^{M} d_m\psi_{\lambda(m,n)} = \sum_{l=1}^{L} \phi^{l,M}_{\lambda_l(n)} = \sum_{l=1}^{L} \sum_{m\in E(l,M)} d_m\psi_{\lambda(m,n)}.$$

- Construction of the exact profiles

- ϕ^l : limits in X of $\phi^{l,M}$ as $M \to +\infty$

- stability of the decomposition Fatou's lemma

- Conclusion

$$Q_M f = \sum_{\lambda \in E_M} d_\lambda \psi_\lambda$$

$E_M = E_M(f)$ is of cardinality M and satisfies $E_M(f) \subset E_{M+1}(f)$

In the particular case,

$$\dot{B}^s_{p,p} \hookrightarrow \dot{B}^t_{q,q}, \quad d/p - d/q = s - t$$

$E_M = E_M(f)$ the subset of cardinality M corresponding to the M largest values of $|d_\lambda|$ is appropriate.

$$\begin{aligned} \|f - Q_M f\|_{\dot{B}^t_{q,q}} &\leq C(\textstyle\sum_{m>M} |d_m|^q)^{\frac{1}{q}} \\ &\leq |d_M|^{1-\frac{p}{q}} (\textstyle\sum_{m>M} |d_m|^p)^{\frac{1}{q}} \\ &\leq (M^{-1} \textstyle\sum_{m=1}^{M} |d_m|^p)^{\frac{1}{p}-\frac{1}{q}} (\textstyle\sum_{m>M} |d_m|^p)^{\frac{1}{q}} \\ &\leq M^{-(\frac{1}{p}-\frac{1}{q})} (\textstyle\sum_{m>0} |d_m|^p)^{\frac{1}{p}} \\ &\leq CM^{-\frac{s-t}{d}} \|f\|_{\dot{B}^s_{p,p}}. \end{aligned}$$

Orlicz spaces

$$H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2),$$

$2 \leq p < \infty$.

$$H^1(\mathbb{R}^2) \hookrightarrow \mathrm{BMO}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2),$$

BMO : functions locally integrable,

$$\|f\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |f - f_B| \, dx < \infty \quad \text{with} \quad f_B = \frac{1}{|B|} \int_B f \, dx.$$

$$H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L},$$

$\mathcal{L}$ et $\mathrm{BMO} \cap L^2$ are not comparable.

Bahouri-Cohen-Koch : $\dot{H}^1(\mathbb{R}^2) \hookrightarrow \mathrm{BMO}(\mathbb{R}^2)$

Profile decomposition

Orlicz spaces

If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex increasing function,

$$\phi(0) = 0 \quad lim_{s\to\infty} \phi(s) = \infty,$$

then $u \in L^\phi$ if there exists $\lambda > 0$,

$$\int \phi(\frac{|u(x)|}{\lambda})\, dx < \infty.$$

$$\|u\|_{L^\phi} = inf \left\{ \lambda > 0, \quad \int_{\mathbb{R}^d} \phi(\frac{|u(x)|}{\lambda})\, dx \leq 1 \right\}$$

- 1 or constant

- $\phi(s) = s^p$, $1 \leq p < \infty$, $\quad L^\phi = L^p$.

- $\phi(s) = \mathrm{e}^{s^2} - 1$, $\quad L^\phi = \mathcal{L}$.

Sobolev embedding in Orlicz space

$$H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L} \hookrightarrow \bigcap_{2\le p<\infty} L^p(\mathbb{R}^2).$$

$$\|u\|_{\mathcal{L}} \le \frac{1}{\sqrt{4\pi}}\|u\|_{H^1}.$$

- Translation invariance

- Non invariance by scaling nor oscillations

Moser-Trudinger

$$\sup_{\|u\|_{H^1}\le 1} \int_{\mathbb{R}^2} (\mathrm{e}^{4\pi|u|^2} - 1)\, dx := \kappa < \infty$$

- Sharpness : the estimate is false for $\alpha > 4\pi$.

- If $\alpha \in [0, 4\pi[$, then there exists c_α

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\alpha|u|^2} - 1\right) dx \le c_\alpha \|u\|^2_{L^2(\mathbb{R}^2)}, \quad u \in H^1(\mathbb{R}^2) \quad \text{with} \quad \|\nabla u\|_{L^2(\mathbb{R}^2)} \le 1$$

Lack of compactness in Orlicz spaces

The embedding $H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L}$ is non-compact

- Lack of compactness at infinity

$u_k(x) = \varphi(x + x_k)$, $0 \neq \varphi \in \mathcal{D}$ and $|x_k| \to \infty$.

- P.-L. Lions (J. Moser)

$$f_\alpha(x) = \begin{cases} 0 & \text{if} \quad |x| \geq 1, \\ -\frac{\log |x|}{\sqrt{2\alpha\pi}} & \text{if} \quad \mathrm{e}^{-\alpha} \leq |x| \leq 1, \\ \sqrt{\frac{\alpha}{2\pi}} & \text{if} \quad |x| \leq \mathrm{e}^{-\alpha}, \end{cases}$$

with $\alpha > 0$.

Straightforward computations show that

- $\|f_\alpha\|^2_{L^2(\mathbb{R}^2)} = \frac{1}{4\alpha}(1 - \mathrm{e}^{-2\alpha}) - \frac{1}{2}\mathrm{e}^{-2\alpha}$

- $\|\nabla f_\alpha\|_{L^2(\mathbb{R}^2)} = 1$

- $f_\alpha \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$ when $\alpha \to \infty$ or $\alpha \to 0$.

- $\|f_\alpha\|_{\mathcal{L}} \to \frac{1}{\sqrt{4\pi}}, \qquad \alpha \to \infty\,.$

- $\|f_\alpha\|_{\mathcal{L}} \to 0, \qquad \alpha \to 0.$

If $\displaystyle\int (\mathrm{e}^{\frac{|f_\alpha(x)|^2}{\lambda^2}} - 1)\,dx \le \kappa$, then

$$2\pi \int_0^{\mathrm{e}^{-\alpha}} \left(\mathrm{e}^{\frac{\alpha}{2\pi\lambda^2}} - 1\right)\, r\,dr \le \kappa,$$

therefore

$$\lambda^2 \ge \frac{\alpha}{2\pi \log\left(1 + \frac{\kappa \mathrm{e}^{2\alpha}}{\pi}\right)}.$$

when $\alpha \to \infty$, we deduce that

$$liminf_{\alpha\to\infty} \|f_\alpha\|_{\mathcal{L}} \ge \frac{1}{\sqrt{4\pi}}\,.$$

In other respects, thanks to Moser-Trudinger, we have $\varepsilon > 0$ $(\|f_\alpha\|_{L^2} \to 0)$

$$\int \left(\mathrm{e}^{(4\pi-\varepsilon)|f_\alpha(x)|^2} - 1\right)\,dx \le C_\varepsilon \|f_\alpha\|_{L^2}^2 \le \kappa, \quad \text{for} \quad \alpha \ge \alpha_\varepsilon\,.$$

Then, $limsup_{\alpha\to\infty} \|f_\alpha\|_{\mathcal{L}} \le \frac{1}{\sqrt{4\pi-\varepsilon}}$.

Essential contribution on $|x| \le \mathrm{e}^{-\alpha}$

For $\alpha \to 0$

$$\begin{aligned}\int (\mathrm{e}^{\frac{|f_\alpha(x)|^2}{\alpha^{1/2}}} - 1)\, dx &= 2\pi \int_0^{\mathrm{e}^{-\alpha}} \left(\mathrm{e}^{\frac{\sqrt{\alpha}}{2\pi}} - 1\right) r\, dr + 2\pi \int_{\mathrm{e}^{-\alpha}}^1 \left(\mathrm{e}^{\frac{\log^2 r}{2\pi\alpha^{3/2}}} - 1\right) r\, dr \\ &\leq \pi \left(\mathrm{e}^{\frac{\sqrt{\alpha}}{2\pi}} - 1\right) \mathrm{e}^{-2\alpha} + 2\pi \left(1 - \mathrm{e}^{-\alpha}\right) \mathrm{e}^{\frac{\alpha^{1/2}}{2\pi}} .\end{aligned}$$

Then for α small enough,

$$\|f_\alpha\|_{\mathcal{L}} \leq \alpha^{1/4}$$

$$\|f_\alpha\|_{\mathcal{L}} \to 0, \qquad \alpha \to 0.$$

The difference between the behavior of $\|f_\alpha\|_{\mathcal{L}}$ in Orlicz space when $\alpha \to \infty$ or $\alpha \to 0$ comes from the fact that the concentration effect is only displayed when $\alpha \to \infty$

- $\alpha \to \infty$

$$|\nabla f_\alpha(x)|^2 dx \rightharpoonup \delta_0$$

- $\alpha \to 0$

$$\|f_\alpha\|_{\mathcal{L}} \sim \|f_\alpha\|_{L^2}$$

In $H^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$

$$\|\cdot\|_{\mathcal{L}} \sim \|\cdot\|_{L^2}$$

P. -L. Lions

- $(u_n) \in H^1(\mathbb{R}^2)$
- $u_n \rightharpoonup 0 \quad \text{in} \quad H^1$
- $liminf_{n\to\infty} \|u_n\|_{\mathcal{L}} > 0$
- $lim_{R\to\infty}\ limsup_{n\to\infty} \int_{|x|>R} |u_n(x)|^2\, dx = 0$
- $|\nabla u_n(x)|^2 dx \rightharpoonup \mu,$

Then, there exists $x_0 \in \mathbb{R}^2$ and a constant $C > 0$ such that

$$\mu \geq C\delta_{x_0}$$

Case of $H^1_{rad}(\mathbb{R}^2)$

H. Bahouri, M. Majdoub and N. Masmoudi

- Our strategy is based on the control of the L^∞ norm away from the origin

$$|u(x)| \leq \frac{C}{r^{\frac{1}{2}}} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}}$$

- No lack of compactness away from the origin

- The control of the L^∞ norm in the non radial case is not valid

$$u_n(x) = \sum \frac{1}{k^2} f_{\alpha_n}(x - x_k),$$

with $\alpha_n \to \infty$ and (x_k) a sequence of $\mathbb{R}^2$.

Starting point

$s := -\log r$, with $r = |x|$

$$\tilde{f}_\alpha(s) := f_\alpha(\mathrm{e}^{-s}) = \sqrt{\frac{\alpha}{2\pi}}\,\mathbf{L}\left(\frac{s}{\alpha}\right),$$

where

$$\mathbf{L}(t) = \begin{cases} 0 & \text{if} \quad t \leq 0, \\ t & \text{if} \quad 0 \leq t \leq 1, \\ 1 & \text{if} \quad t \geq 1. \end{cases}$$

α is the scale and $\mathbf{L}$ the profile.

Main goal

Characterize the lack of compactness of

$$H^1_{rad}(\mathbb{R}^2) \hookrightarrow \mathcal{L},$$

- Asymptotic decomposition by means of generalizations of Moser's examples

- Properties : orthogonality and stability

Theorem

Let (u_n) be a sequence in $H^1_{rad}(\mathbb{R}^2)$ such that

- $u_n \rightharpoonup 0$,
- $limsup_{n\to\infty}\|u_n\|_{\mathcal{L}} = A_0 > 0$,
- $lim_{R\to\infty}\ limsup_{n\to\infty} \displaystyle\int_{|x|>R} |u_n|^2\, dx = 0.$

Then (up to subsequence extraction), for all $\ell \geq 1$,

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\ \psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x),$$

$limsup_{n\to\infty}\ \|\mathrm{r}_n^{(\ell)}\|_{\mathcal{L}} \longrightarrow 0$, when $\ell \to \infty$.

- $(\alpha_n^{(j)})$ scales :

- $\alpha_n^{(j)} \to \infty$, when $n \to \infty$

- $(\alpha_n^{(j)} \perp \alpha_n^{(k)})$

$$\left| \log \left(\alpha_n^{(j)} / \alpha_n^{(k)} \right) \right| \to \infty.$$

- $\psi^{(j)}$ profile :

$$\mathcal{P} := \left\{ \psi \in L^2(\mathbb{R}, \mathrm{e}^{-2s} ds); \quad \psi' \in L^2(\mathbb{R}), \ \psi_{|]-\infty,0]} = 0 \right\}.$$

A profile ψ is a continuous function

$$g_n^{(j)} = \sqrt{\frac{\alpha_n^{(j)}}{2\pi}} \, \psi^{(j)} \left(\frac{-\log |x|}{\alpha_n^{(j)}} \right)$$

$$lim_{n\to\infty} \|g_n^{(j)}\|_{\mathcal{L}} = \frac{1}{\sqrt{4\pi}} \max_{s>0} \frac{|\psi^{(j)}(s)|}{\sqrt{s}}.$$

Denoting by $L = liminf_{n\to\infty} \|g_n\|_{\mathcal{L}}$, we have for fixed $\varepsilon > 0$ and n big enough (up to subsequence extraction)

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{|\frac{g_n(x)}{L+\varepsilon}|^2} - 1\right) dx \leq \kappa\,,$$

$$\alpha_n \int_0^\infty \mathrm{e}^{2\alpha_n t\left(\frac{1}{4\pi(L+\varepsilon)^2}\left(\frac{\psi(t)}{\sqrt{t}}\right)^2 - 1\right)} dt \leq C.$$

Since ψ is continuous and $\alpha_n \to \infty$, when $n \to \infty$

$$\frac{1}{\sqrt{4\pi}} \frac{|\psi(t)|}{\sqrt{t}} \leq L + \varepsilon,$$

$$\frac{1}{\sqrt{4\pi}} \max_{t>0} \frac{|\psi(t)|}{\sqrt{t}} \leq liminf_{n\to\infty} \|g_n\|_{\mathcal{L}}$$

To conclude, it suffices to prove that for any $\delta > 0$,

$$limsup_{n\to\infty} \|g_n\|_{\mathcal{L}} \leq \lambda = \frac{1+\delta}{\sqrt{4\pi}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}} \,.$$

We are then reduced to estimate

$$\alpha_n \int_0^\infty \mathrm{e}^{-2\alpha_n s\left(1-\frac{|\psi(s)|^2}{4\pi\lambda^2 s}\right)}\, ds - \alpha_n \int_0^\infty \mathrm{e}^{-2\alpha_n s}\, ds \,.$$

Principal Contribution at the origin + $\frac{\psi(s)}{\sqrt{s}} \to 0,\ s \to 0$

$$\forall\, \varepsilon > 0,\ \exists \eta > 0 \text{ such that } \frac{|\psi(s)|^2}{4\pi\lambda^2 s} < \varepsilon, \text{ for } 0 \leq s < \eta \,.$$

$$\alpha_n \int_0^\eta \mathrm{e}^{-2\alpha_n s\left(1-\frac{|\psi(s)|^2}{4\pi\lambda^2 s}\right)}\, ds - \alpha_n \int_0^\eta \mathrm{e}^{-2\alpha_n s}\, ds \leq \frac{\varepsilon}{2(1-\varepsilon)} + \circ(1), \quad n \to \infty.$$

Stability

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|\psi^{(j)'}\|_{L^2}^2 + \|\nabla \mathrm{r}_n^{(\ell)}\|_{L^2}^2 + \circ(1), \quad n \to \infty.$$

Orlicz norm

$$\|u_n\|_{\mathcal{L}} \to \sup_{j\geq 1} \left(lim_{n\to\infty} \|g_n^{(j)}\|_{\mathcal{L}} \right)$$

P. Gérard

$$\|u_n\|_{L^p}^p \to \sum_{j\geq 1} \|\psi^{(j)}\|_{L^p}^p.$$

Comments

- No cores : radial case
- $\psi_{]-\infty,0]} = 0$: g_n is supported in the unit disc

Compactness away from the origin in the radial case

- $max_{s>0} \frac{|\psi(s)|}{\sqrt{s}} \leq \|\psi'\|_{L^2}$.

\- $max_{s>0} \frac{|\psi(s)|}{\sqrt{s}} = \|\psi'\|_{L^2}$: concentration of the total mass

\- $max_{s>0} \frac{|\psi(s)|}{\sqrt{s}} < \|\psi'\|_{L^2}$: involved in the qualitative study

- For any $\lambda > 0$, $g_{\underline{\alpha},\psi} = g_{\lambda\underline{\alpha},\psi_\lambda}$, where $\psi_\lambda(t) = \frac{1}{\sqrt{\lambda}}\psi(\lambda t)$.

Invariance of energy and Orlicz norms

- Concentration of the total mass : there exists $s_0 > 0$ such that

$$\psi(s) = \begin{cases} 0 & \text{if} \quad s \leq 0, \\ \frac{s}{\sqrt{s_0}} & \text{if} \quad 0 \leq s \leq s_0, \\ \sqrt{s_0} & \text{if} \quad s \geq s_0. \end{cases}$$

- The characterization for $\mathcal{L}$ is different from that of *BMO*

Idea of proof

- Diagonal subsequence extraction
- Crucial fact : under the assumptions of the theorem

- we can extract a scale (α_n)

- a profile ψ such that $\|\psi'\|_{L^2} \geq C\,A_0$.

- Strategy : r_n remainder

- $lim_{n\to\infty} \|r_n\|_{\mathcal{L}} = 0$, we stop the process

- $lim_{n\to\infty} \|r_n\|_{\mathcal{L}} = A_1 > 0$, we continued the process

- Orthogonality
- Convergence of the process

Key point

- Compactness at infinity $\longrightarrow \|u_n\|_{L^2} \to 0$
- Radial setting

$$\forall M \in \mathbb{R},\ \|v_n\|_{L^\infty(]-\infty,M[)} \to 0, \qquad n \to \infty, \quad \text{where} \quad v_n(s) = u_n(\mathrm{e}^{-s}).$$

- As a consequence

$$\forall \delta > 0,\ \sup_{s \geq 0} \left(\left| \frac{v_n(s)}{A_0 - \delta} \right|^2 - s \right) \to \infty, \quad n \to \infty.$$

if not $limsup_{n\to\infty} \|u_n\|_{\mathcal{L}} \leq A_0 - \delta$.

Lebesgue theorem

$$\int_{|x|<1} \left(\mathrm{e}^{\left| \frac{u_n(x)}{A_0-\delta} \right|^2} - 1 \right) dx = 2\pi \int_0^\infty \left(\mathrm{e}^{\left| \frac{v_n(s)}{A_0-\delta} \right|^2} - 1 \right) \mathrm{e}^{-2s}\, ds \to 0, \quad n \to \infty.$$

L^∞ estimate away from the origin

$$\int_{|x|\geq 1} \left(\mathrm{e}^{\left| \frac{u_n(x)}{A_0-\delta} \right|^2} - 1 \right) dx \leq C \|u_n\|_{L^2}^2 \to 0\,.$$

Extraction of the first scale : $\alpha_n^{(1)} \to \infty$

$$\frac{A_0}{2}\sqrt{\alpha_n^{(1)}} \leq |v_n(\alpha_n^{(1)})| \leq C\sqrt{\alpha_n^{(1)}} + \circ(1)$$

Extraction of the first profile : $\psi^{(1)} \in \mathcal{P}$

$$\psi_n(y) = \sqrt{\frac{2\pi}{\alpha_n^{(1)}}}\, v_n(\alpha_n^{(1)} y).$$

$$\psi_n' \rightharpoonup (\psi^{(1)})' \quad \text{in} \quad L^2(\mathbb{R}) \quad \text{with} \quad \|(\psi^{(1)})'\|_{L^2} \geq \frac{\sqrt{2\pi}}{2} A_0.$$

$$|\psi^{(1)}(1)| = \Big| \int_0^1 (\psi^{(1)})'(\tau)\, d\tau \Big| \leq \|(\psi^{(1)})'\|_{L^2}$$

$$\mathrm{r}_n^{(1)}(x) = \sqrt{\frac{\alpha_n^{(1)}}{2\pi}} \left(\psi_n \left(\frac{-\log|x|}{\alpha_n^{(1)}} \right) - \psi^{(1)} \left(\frac{-\log|x|}{\alpha_n^{(1)}} \right) \right).$$

$$limsup_{n\to\infty} \|\nabla \mathrm{r}_n^{(1)}\|_{L^2}^2 \leq limsup_{n\to\infty} \|\nabla u_n\|_{L^2}^2 - \|(\psi^{(1)})'\|_{L^2}^2 .$$

$A_1 = limsup_{n\to\infty} \|\mathrm{r}_n^{(1)}\|_{\mathcal{L}}$

- if $A_1 = 0$, we stop the process

- if not, we argue similarly to obtain a second scale and a second profile.

- $(\alpha_n^{(2)} \perp \alpha_n^{(1)})$ by contradiction arguments

Iteration

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)} \left(\frac{-\log|x|}{\alpha_n^{(j)}} \right) + \mathrm{r}_n^{(\ell)}(x),$$

$$limsup_{n\to\infty} \|\nabla\, \mathrm{r}_n^{(\ell)}\|_{L^2}^2 \leq limsup_{n\to\infty} \|\nabla\, u_n\|_{L^2}^2 - \sum_{j=1}^{\ell} \|(\psi^{(j)})'\|_{L^2}^2 \,.$$

with $\|(\psi^{(j)})'\|_{L^2}^2 \geq CA_{j-1}$, where C is an absolute constant.

It follows that

$$limsup_{n\to\infty} \|\mathrm{r}_n^{(\ell)}\|_{H^1}^2 \leq limsup_{n\to\infty} \|\nabla\, u_n\|_{L^2}^2 - C(A_0^2 + A_1^2 + \cdots + A_{\ell-1}^2)\,.$$

Hence $A_\ell \to 0$ as $\ell \to \infty$.

Lack of compactness in Orlicz space, the general case

H. Bahouri, M. Majdoub and N. Masmoudi

Theorem

Let (u_n) be a sequence in $H^1(\mathbb{R}^2)$ such that

- $u_n \rightharpoonup 0$,
- $limsup_{n\to\infty} \|u_n\|_{\mathcal{L}} = A_0 > 0$,
- $\lim_{R\to\infty} \ \limsup_{n\to\infty} \ \|u_n\|_{\mathcal{L}(|x|>R)} = 0$.

Then (up to subsequence extraction), for all $\ell \geq 1$,

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}} \, \psi^{(j)} \left(\frac{-\log|x - x_n^{(j)}|}{\alpha_n^{(j)}} \right) + \mathrm{r}_n^{(\ell)}(x),$$

$limsup_{n\to\infty} \ \|\mathrm{r}_n^{(\ell)}\|_{\mathcal{L}} \longrightarrow 0$, when $\ell \to \infty$.

- $(\alpha_n^{(j)})$ scales : $\alpha_n^{(j)} \to \infty$, when $n \to \infty$

- $\psi^{(j)}$ profile :

$$\mathcal{P} := \Big\{ \psi \in L^2(\mathbb{R}, \mathrm{e}^{-2s} ds); \quad \psi' \in L^2(\mathbb{R}),\ \psi_{|]-\infty,0]} = 0 \Big\}.$$

- $(x_n^{(j)})$ cores

Orthogonality

$(\underline{\alpha}, \underline{x}, \psi) \perp (\underline{\tilde{\alpha}}, \underline{\tilde{x}}, \tilde{\psi})$

- either

$$\Big| \log \left(\tilde{\alpha}_n / \alpha_n \right) \Big| \longrightarrow \infty$$

- or $\tilde{\alpha}_n = \alpha_n$ and $-\frac{\log |x_n - \tilde{x}_n|}{\alpha_n} \longrightarrow a \geq 0$ with ψ or $\tilde{\psi}$ null for $s < a$.

Comments

• Elements involved in the general case are, up to cores, similar to those that characterize the lack of compactness in the radial case.

Idea behind

Let

$$g_n(x) := \sqrt{\frac{\alpha_n}{2\pi}}\, \psi\Big(\frac{-\log|\varphi(x)|}{\alpha_n}\Big)$$

where $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a global diffeomorphism, $\varphi(0) = 0$

then

$$g_n(x) \sim \sqrt{\frac{\alpha_n}{2\pi}}\, \psi\Big(\frac{-\log|x|}{\alpha_n}\Big) \quad \text{in} \quad H^1 .$$

• $g_n^{(j)} := \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)} \left(\frac{-\log|x - x_n^{(j)}|}{\alpha_n^{(j)}} \right)$ are completely different from the profiles involved in the characterization of the lack of compactness of $\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$

without scales in the point of view of frequencies

For any $0 < a < b$ and any sequence (h_n) of nonnegative real numbers

$$\int_{a<h_n|\xi|<b} |\mathcal{F}(\nabla g_n^{(j)})(\xi)|^2 \, d\xi \to 0, \quad n \to \infty.$$

$$\|g_n^{(j)}\|_{\dot{B}^1_{2,\infty}} \to 0, \quad \text{as} \quad n \to \infty.$$

• In the context of Orlicz space, the scales $\alpha_n^{(j)}$ correspond to values taken by the functions $g_n^{(j)}$ in consistent sets of size. Actually, saying that α_n is a scale for u_n means that

$|u_n| \geq c\sqrt{\alpha_n}$ on a set E_n of Lebesgue measure greater than $\mathrm{e}^{-2\alpha_n}$.

• Orthogonality hypothesis means that the interaction between the elementary concentrations is negligible in the energy space :

it requires in the case where the cores are not sufficiently distant, the cancelation of one of the profiles in the area $s < a$ $\left(-\frac{\log|x_n - \tilde{x}_n|}{\alpha_n} \to a > 0\right)$.

• This assumption is unavoidable because the parts of the elementary concentrations respectively around the cores x_n and $\tilde{x}_n$ resulting from the profiles for the values $s < a$ interact.

Strategy of proof

Strikingly different from the one conducted in the radial case

The control of the L^∞ norm in the non radial case is not valid

$$u_n(x) = \sum \frac{1}{k^2} f_{\alpha_n}(x - x_k),$$

with $\alpha_n \to \infty$ and (x_k) a sequence of $\mathbb{R}^2$.

Main difficulty : extract cores

Capacity arguments

Essential ingredient consists to demonstrate by contradiction that if the mass responsible for the lack of compactness is scattered, then the energy used exceeds that of the starting sequence.

Steps of the proof

- Extraction of the scales :

Schawrz symmetrization we are reduced to the study of u_n^* the symmetric decreasing rearrangement of u_n

$u_n^* \in H^1_{rad}(\mathbb{R}^2)$

$$u_n^*(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \varphi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x), \ \limsup_{n\to\infty} \|\mathrm{r}_n^{(\ell)}\|_{\mathcal{L}} \stackrel{\ell\to\infty}{\longrightarrow} 0$$

- Reduction to one scale (truncation)

$$u_n^*(x) = \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\, \varphi^{(1)}\left(\frac{-\log|x|}{\alpha_n^{(1)}}\right)$$

- Extraction of the first core $x_n^{(1)}$

  Capacity arguments

- Extraction of the first profile $\psi^{(1)}$

$$\|{\psi^{(1)}}'\|_{L^2} \geq C\, A_0.$$

- Strategy : r_n remainder

- $lim_{n\to\infty} \|r_n\|_{\mathcal{L}} = 0$, we stop the process

- $lim_{n\to\infty} \|r_n\|_{\mathcal{L}} = A_1 > 0$, we continued the process

- Orthogonality

- Convergence of the process

Main step : extraction of the cores

$$u_n^*(x) = \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\, \varphi^{(1)}\left(\frac{-\log|x|}{\alpha_n^{(1)}}\right) = g_n^{(1)}(x)$$

$$A_0 = \lim_{n\to\infty} \left\| \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\, \varphi^{(1)}\left(\frac{-\log|x|}{\alpha_n^{(1)}}\right) \right\|_{\mathcal{L}} = \frac{1}{\sqrt{4\pi}} \max_{s>0} \frac{\varphi^{(1)}(s)}{\sqrt{s}} = \frac{1}{\sqrt{4\pi}}\, \varphi^{(1)}(1)$$

If

$$E_n^* := \left\{ x \in \mathbb{R}^2;\; g_n^{(1)}(x) \geq \sqrt{\frac{\alpha_n^{(1)}}{2\pi}} \left(1 - \frac{\varepsilon_0}{10}\right) \sqrt{4\pi}\, A_0 \right\},$$

then, there exists an integer N_0 such that for $n \geq N_0$

$$|E_n^*| \geq \mathrm{e}^{-2\alpha_n^{(1)}}$$

- x belongs to E_n^* is equivalent to

$$\varphi^{(1)}\left(\frac{-\log|x|}{\alpha_n^{(1)}}\right) \geq \left(1-\frac{\varepsilon_0}{10}\right)\sqrt{4\pi}\,A_0.$$

- Since $\varphi^{(1)}$ is continuous, there exists $\eta > 0$ such that

$$|s-1| \leq \eta \Rightarrow \left|\varphi^{(1)}(s) - \varphi^{(1)}(1)\right| \leq \frac{\varepsilon_0}{10}\sqrt{4\pi}\,A_0.$$

- Knowing that $\varphi^{(1)}(1) = \sqrt{4\pi}\,A_0$, we deduce that for

$$\mathrm{e}^{-\alpha_n^{(1)}(1+\eta)} \leq |x| \leq \mathrm{e}^{-\alpha_n^{(1)}(1-\eta)}$$

$$g_n^{(1)}(x) \geq \sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\left(1-\frac{\varepsilon_0}{10}\right)\sqrt{4\pi}\,A_0.$$

Key lemma

$$E_n := \left\{ x \in \mathbb{R}^2;\ |u_n(x)| \geq \sqrt{2\alpha_n^{(1)}} \left(1 - \frac{\varepsilon_0}{10}\right) A_0 \right\}$$

There exists $\delta_0 > 0$ such that for all n large enough there exists x_n

$$\frac{|E_n \cap B(x_n, \mathrm{e}^{-(1-2\varepsilon_0)\,\alpha_n^{(1)}})|}{|E_n|} \geq \delta_0 A_0^2$$

- $|E_n| = |E_n^*|$
- $|E_n^*| \geq \mathrm{e}^{-2\alpha_n^{(1)}(1-\eta)}$
- $|E_n^*| \leq \mathrm{e}^{-2\alpha_n^{(1)}(1-\frac{\varepsilon_0}{5})}$ $(\varphi^{(1)}(s) \leq \sqrt{s}\varphi^{(1)}(s))$

$$|B(x_n, \mathrm{e}^{-(1-2\varepsilon_0)\alpha_n^{(1)}})| \gg |E_n|$$

Sketch of proof of the key lemma

- Contradiction : we assume that for any $\delta > 0$, $n \in \mathbb{N}$ and $x \in \mathbb{R}^2$

$$\frac{|E_n \cap B(x, \mathrm{e}^{-(1-2\varepsilon_0)\,\alpha_n^{(1)}})|}{|E_n|} \le \delta A_0^2 \,. \tag{1}$$

- In particular, (1) holds for any ball in

$$\mathcal{B}_n := \Big\{ B(x, \mathrm{e}^{-(1-2\varepsilon_0)\,\alpha_n^{(1)}}),\ x \in \mathbf{T}_n \Big\}$$

where $\mathbf{T}_n := (\mathrm{e}^{-(1-2\varepsilon_0)\,\alpha_n^{(1)}}\mathbb{Z}) \times (\mathrm{e}^{-(1-2\varepsilon_0)\,\alpha_n^{(1)}}\mathbb{Z})$.

- But

$$\|\nabla u_n\|^2_{L^2(\mathbb{R}^2)} \ge \frac{1}{4} \sum_{\mathbf{B} \in \mathcal{B}_n} \|\nabla u_n\|^2_{L^2(\mathbf{B})} \,.$$

- covering of $\mathbb{R}^2$

- each point of $\mathbb{R}^2$ belongs at most to four balls among $\mathcal{B}_n$

• Now, the idea is to get a contradiction by proving that for δ small enough the sum $\frac{1}{4}\sum_{\mathbf{B}\in\mathcal{B}_n}\|\nabla u_n\|^2_{L^2(\mathbf{B})}$ exceeds the energy of u_n.

• For this purpose, we estimate the energy of u_n on each ball $\mathbf{B}\in\mathcal{B}_n$, making use of capacity arguments.

• To do so, we take advantage of the fact that the values of $|u_n|$ on $\mathbf{B}$ varies at least

- from $\sqrt{2\alpha_n^{(1)}}\left(1-\frac{\varepsilon_0}{10}\right)A_0$ on $E_n\cap\mathbf{B}$

- to $\sqrt{2\alpha_n^{(1)}}\left(1-\frac{\varepsilon_0}{2}\right)A_0$ on a set of Lebesgue measure $\geq\frac{|\mathbf{B}|}{2}$

$|u_n|$ takes values less than

$\sqrt{2\alpha_n^{(1)}}\left(1-\frac{\varepsilon_0}{2}\right)A_0$ on a set of Lebesgue measure $\geq \frac{|\mathbf{B}|}{2}$

This is due to the fact that for $s \leq 1-\varepsilon_0$

$$\varphi^{(1)}(s) \leq (1-\frac{\varepsilon_0}{2})\sqrt{4\pi}\, A_0.$$

Indeed, this implies that the set

$$H_n^* = \left\{ x \in \mathbb{R}^2; \quad u_n^* \geq \sqrt{2\alpha_n^{(1)}}\left(1-\frac{\varepsilon_0}{2}\right)A_0 \right\}$$

is a subset of $B(0, \mathrm{e}^{-(1-\varepsilon_0)\alpha_n})$ and then

$$H_n = \left\{ x \in \mathbb{R}^2; \quad |u_n| \geq \sqrt{2\alpha_n^{(1)}}\left(1-\frac{\varepsilon_0}{2}\right)A_0 \right\}$$

is of Lebesgue measure $|H_n| = |H_n^*| \leq \pi \mathrm{e}^{-2(1-\varepsilon_0)\alpha_n}$.

- Finally using the fact that the values of $|u_n|$ on $\mathbf{B}$ varies at least

- from $\sqrt{2\alpha_n^{(1)}}\left(1 - \frac{\varepsilon_0}{10}\right) A_0$ on $E_n \cap \mathbf{B}$

- to $\sqrt{2\alpha_n^{(1)}}\left(1 - \frac{\varepsilon_0}{2}\right) A_0$ on a set of Lebesgue measure $\geq \frac{|\mathbf{B}|}{2}$

we get by standard capacity arguments that

$$\|\nabla u_n\|^2_{L^2(\mathbf{B})} \geq C\left(\left(\frac{\varepsilon_0}{2} - \frac{\varepsilon_0}{10}\right)\sqrt{2\alpha_n^{(1)}}\ A_0\right)^2 \frac{1}{\log \frac{\mathrm{e}^{-\alpha_n^{(1)}(1-2\varepsilon_0)}}{\sqrt{|E_n \cap \mathbf{B}|}}}$$

• In conclusion

$$\|\nabla u_n\|^2_{L^2(\mathbf{B})} \geq C\varepsilon_0^2\, A_0^2$$

• But

$$4\,\|\nabla u_n\|^2_{L^2(\mathbb{R}^2)} \geq \#\mathcal{B}_n\, C\varepsilon_0^2\, A_0^2,$$

where $\#\mathcal{B}_n$ denotes the cardinal of $\mathcal{B}_n$.

• Since by contradiction hypothesis

$$\#\mathcal{B}_n \geq \frac{1}{\delta\, A_0^2}$$

we obtain

$$4\,\|\nabla u_n\|^2_{L^2(\mathbb{R}^2)} \geq \frac{C\varepsilon_0^2}{\delta}$$

which yields a contradiction for δ small enough.

Extraction of the profiles

- Firstly, we consider

$$\psi_n(y,\theta) = \sqrt{\frac{2\pi}{\alpha_n^{(1)}}}\, v_n(\alpha_n^{(1)} y, \theta),$$

where $v_n(s,\theta) = \Big(\tau_{-x_n^{(1)}} u_n\Big)(\mathrm{e}^{-s}\cos\theta, \mathrm{e}^{-s}\sin\theta)$ and $x_n^{(1)}$ satisfying

$$\frac{|E_n \cap B(x_n^{(1)}, \mathrm{e}^{-(1-2\varepsilon_0)\,\alpha_n^{(1)}})|}{|E_n|} \geq \delta_0 A_0^2.$$

- Then, using the energy estimate

$$\|\nabla u_n\|_{L^2}^2 = \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^{2\pi} |\partial_y \psi_n(y,\theta)|^2 dy d\theta + \frac{(\alpha_n^{(1)})^2}{2\pi}\int_{\mathbb{R}}\int_0^{2\pi} |\partial_\theta \psi_n(y,\theta)|^2 dy d\theta$$

we deduce that

$$\begin{aligned} \partial_\theta \psi_n &\to 0 \quad \text{and} \\ \partial_y \psi_n &\rightharpoonup g, \end{aligned}$$

where the function g only depends on the variable y.

Our goal is to prove that

$$\frac{1}{2\pi}\int_0^{2\pi}\psi_n(y,\theta)\,d\theta \to \psi^{(1)}(y), \qquad (2)$$

as n tends to infinity, where $\psi^{(1)}(y) = \int_0^y g(\tau)\,d\tau$ and that there exists an absolute constant C so that

$$\|{\psi^{(1)}}'\|_{L^2} \geq C\,A_0\,. \qquad (3)$$

• Using the fact that $\partial_y\psi_n \rightharpoonup g$ in L^2 as n tends to infinity, we get for any $y_1 \leq y_2$

$$\langle \partial_y\psi_n, \mathbf{1}_{[y_1,y_2]}\rangle \to 2\pi\int_{y_1}^{y_2} g(y)\,dy.$$

As it turns that

$$\begin{aligned}\langle \partial_y\psi_n, \mathbf{1}_{[y_1,y_2]}\rangle &= \int_0^{2\pi}\int_{y_1}^{y_2}\partial_y\psi_n(y,\theta)\,dy\,d\theta\\ &= \int_0^{2\pi}(\psi_n(y_2,\theta)-\psi_n(y_1,\theta))\,d\theta,\end{aligned}$$

we get

$$\frac{1}{2\pi}\int_0^{2\pi}\Big(\psi_n(y_2,\theta)-\psi_n(y_1,\theta)\Big)\,d\theta \to \int_{y_1}^{y_2} g(y)\,dy$$

- But

$$\|u_n\|_{L^2}^2 = \frac{(\alpha_n^{(1)})^2}{2\pi}\int_{\mathbb{R}}\int_0^{2\pi}|\psi_n(y,\theta)|^2 \mathrm{e}^{-2\alpha_n^{(1)}y}\,dy\,d\theta$$
$$\geq \frac{(\alpha_n^{(1)})^2}{2\pi}\int_{-\infty}^{0}\int_0^{2\pi}|\psi_n(y,\theta)|^2\,dy\,d\theta$$

which implies that $\psi_n \to 0$ in $L^2(]-\infty,0]\times[0,2\pi])$.

- Thus g is null on $]-\infty,0]$ and one can easily prove the first point

$$\frac{1}{2\pi}\int_0^{2\pi}\psi_n(y,\theta)\,d\theta \to \psi^{(1)}(y) = \int_0^y g(\tau)\,d\tau$$

• To prove the last point (2), we use again capacity arguments and introduce the sets $\widetilde{E_n} \supset E_n$ defined by

$$\widetilde{E_n} := \Big\{x \in \mathbb{R}^2;\ |u_n(x)| \geq \frac{\sqrt{2\alpha_n^{(1)}}A_0}{2}\Big\}.$$

• We infer that ε_0 can be choosen so that

$$\left|\widetilde{E_n} \cap B(x_n^{(1)}, \mathrm{e}^{-(1-2\,\varepsilon_0)\,\alpha_n^{(1)}})\right| \geq \frac{1}{2}\left|B(x_n^{(1)}, \mathrm{e}^{-(1-2\,\varepsilon_0)\,\alpha_n^{(1)}})\right| \qquad (4)$$

Otherwise the values of u_n on the ball varies at least from the value $\sqrt{2\alpha_n^{(1)}}\left(1 - \frac{\varepsilon_0}{10}\right)A_0$ on $E_n \cap \mathbf{B}$ to the value $\frac{1}{2}\sqrt{2\alpha_n^{(1)}}\,A_0$ on a subset of Lebesgue measure $\geq \frac{|\mathbf{B}|}{2}$, which implies by capacity arguments that

$$\|\nabla u_n\|^2_{L^2(\mathbf{B})} \geq \frac{CA_0^2}{\varepsilon_0}\,.$$

• Since the ball $B(x_n^{(1)}, \mathrm{e}^{-(1+2\,\varepsilon_0)\,\alpha_n^{(1)}})$ is a negligible part of the ball $\mathbf{B}$, we can assume without loss of generality that

$$\widetilde{E_n} \cap B(x_n^{(1)}, \mathrm{e}^{-(1-2\,\varepsilon_0)\,\alpha_n^{(1)}}) \subset \Big\{ x \in \mathbb{R}^2; \quad |x - x_n^{(1)}| \geq \mathrm{e}^{-(1+2\,\varepsilon_0)\,\alpha_n^{(1)}} \Big\}$$

• This gives rise to the existence of $1 - 2\,\varepsilon_0 \leq y_0 \leq 1 + 2\,\varepsilon_0$ such that

$$\psi_n(y_0, \theta) \geq \sqrt{\pi}\, A_0$$

for θ varying over an interval of length at least π which ends the proof since ψ_n is positive values.

Qualitative study

- Behaviour of solutions of nonlinear Klein-Gordon equation

$$\partial_t^2 u - \Delta u + u + f(u) = 0, \quad u : \mathbb{R}_t \times \mathbb{R}_x^2 \to \mathbb{R},$$

with $f(u) = u \left(\mathrm{e}^{4\pi u^2} - 1\right)$.

- Conservation of energy

$$E(u,t) = \|\partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \frac{1}{4\pi} \|\mathrm{e}^{4\pi u(t)^2} - 1\|_{L^1} = E(u,0) := E_0.$$

The notion of criticality depends on the size of E_0 with respect to 1.

- subcritical case $E_0 < 1$.

- critical case $E_0 = 1$.

- supercritical case $E_0 > 1$.

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• We adopt the approach introduced by P. Gérard which consists to compare (u_n) and (v_n) the solutions of the nonlinear and linear Klein-Gordon equation with the same Cauchy data :

$(\varphi_n, \psi_n) \in H^1 \times L^2$ supported in a fixed compact

$$\varphi_n \rightharpoonup 0 \quad \text{in } H^1, \quad \psi_n \rightharpoonup 0 \quad \text{in } L^2$$

• We shall say that the sequence (u_n) is linearizable on $[0, T]$, if

$$\sup_{t\in[0,T]} E_c(u_n - v_n, t) \longrightarrow 0 \quad \text{quand} \quad n \to \infty$$

where $E_c(w, t) = \int_{\mathbb{R}^2} \left[|\partial_t w|^2 + |\nabla_x w|^2 + |w|^2 \right] (t, x)\, dx.$

Define

$$E^n = \|\psi_n\|_{L^2}^2 + \|\nabla\varphi_n\|_{L^2}^2 + \frac{1}{4\pi}\,\|\mathrm{e}^{4\pi\varphi_n^2} - 1\|_{L^1}$$

Theorem

- If $limsup_{n\to\infty}\, E^n < 1$, then (u_n) is linearizable

- If $limsup_{n\to\infty}\, E^n = 1$, then

(u_n) is linearizable on $[0, T]$ provided that

$$limsup_{n\to\infty}\, \|v_n\|_{L^\infty([0,T];\mathcal{L})} < \frac{1}{\sqrt{4\pi}}.$$

Non concentration of the total mass

Sketch of proof : subcritical case

$w_n = u_n - v_n$

$$\partial_t^2 w_n - \Delta w_n + w_n = -f(u_n)$$

Energy estimate

$$\|f(u_n)\|_{L^1([0,T],L^2(\mathbb{R}^2))} \longrightarrow 0 \quad \text{as} \quad n \to \infty,$$

$f(u_n) = -u_n \,(\mathrm{e}^{4\pi u_n^2} - 1).$

Strategy

- Convergence in measure to 0

- Bounded in L^q, $q > 1$.

Tools

- Measure theory
- Strichartz Estimates
- Logarithmic inequality
- Absorption arguments

$\|f(u_n)\|_{L^{1+\epsilon}([0,T],L^{2+\epsilon}(\mathbb{R}^2))}$ bounded

- Moser-Trudinger

$$\begin{aligned} \|f(u_n)\|_{L^{2+\epsilon}(\mathbb{R}^2)}^{2+\epsilon} &\leq C\mathrm{e}^{4\pi(1+\epsilon)\|u_n\|_{L^\infty}^2} \int_{\mathbb{R}^2} |u_n|^{2+\epsilon}(\mathrm{e}^{4\pi u_n^2} - 1)dx \\ &\leq C_\eta \mathrm{e}^{4\pi(1+\epsilon)\|u_n\|_{L^\infty}^2} \int_{\mathbb{R}^2} (\mathrm{e}^{(4\pi+\eta)u_n^2} - 1)dx \\ &\leq C_\eta \mathrm{e}^{4\pi(1+\epsilon)\|u_n\|_{L^\infty}^2} \end{aligned}$$

- Logarithmic inequality : $\lambda > \frac{2}{\pi}$

$$\mathrm{e}^{4\pi(1+\epsilon)\|u_n\|_{L^\infty}^2} \leq C\left(C_\lambda + \frac{\|u_n\|_{\mathcal{C}^{\frac{1}{4}}}}{\sqrt{(1-\rho)}}\right)^{4\lambda\pi(1+\epsilon)(1-\rho)},$$

$limsup_{n\to\infty}\ E^n < 1-\rho.$

• Hölder

$$\|f(u_n)\|^{1+\epsilon}_{L^{1+\epsilon}([0,T],L^{2+\epsilon}(\mathbb{R}^2))} \leq C(\eta, T)(T^{\frac{1}{4}} + \|u_n\|_{L^4([0,T]),\mathcal{C}^{1/4})})^{\theta}.$$

• Strichartz estimates : subcritical case ($E_0 < 1$)

$$\|u\|_{L^4([0,T];\mathcal{C}^{1/4})} \leq C(T, E_0).$$

- Strichartz estimates

- Estimate of the source term

- Absorption argument

Idea of the proof : critical case

$w_n = u_n - v_n$

$$\partial_t^2 w_n - \Delta w_n + w_n = -f(u_n)$$

$$f(u_n) = f(v_n + w_n) = f(v_n) + f'(v_n)\, w_n + \frac{1}{2}\, f''(v_n + \theta_n\, w_n)\, w_n^2,$$

- $\|f(v_n)\|_{L^1([0,T];L^2(\mathbb{R}^2))} \to 0$

- convergence in measure to 0

- $\|f(v_n)\|_{L^{1+\epsilon}([0,T],L^{2+\epsilon}(\mathbb{R}^2))}$

$$limsup_{n\to\infty}\, \|v_n\|_{L^\infty([0,T];\mathcal{L})} < \frac{1}{\sqrt{4\pi}}.$$

- $\|f'(v_n)\, w_n\|_{L^1([0,T];L^2(\mathbb{R}^2))} \leq \varepsilon_n \,\|w_n\|_{ST(I)}, \quad \varepsilon_n \to 0$

- Hölder

$$\|f'(v_n)\, w_n\|_{L^1([0,T];L^2)} \leq \|w_n\|_{L^{1+\frac{1}{\eta}}([0,T];L^{2+\frac{2}{\eta}})} \|f'(v_n)\|_{L^{1+\eta}([0,T];L^{2+2\eta})}$$

- Strichartz estimates for w_n

- $\|f'(v_n)\|_{L^{1+\eta}([0,T];L^{2+2\eta})} \to 0$, for η small enough

$limsup_{n\to\infty} \|v_n\|_{L^\infty([0,T];\mathcal{L})} < \frac{1}{\sqrt{4\pi}}$

- $\|f''(v_n + \theta_n w_n)\, w_n^2\|_{L^1([0,T];L^2(\mathbb{R}^2))} \leq \varepsilon_n \|w_n\|^2_{ST(I)}, \quad \varepsilon_n \to 0$

provided that

$$limsup_{n\to\infty} \|w_n\|_{L^\infty([0,T];H^1)} \leq \frac{1 - L\sqrt{4\pi}}{2}.$$

where $L := limsup_{n\to\infty} \|v_n\|_{L^\infty([0,T];\mathcal{L})} < \frac{1}{\sqrt{4\pi}}$.

- Hölder + Strichartz

$$K_n \leq \|w_n\|^2_{ST(I)} \left(\|f''(v_n)\|_{L^{1+\eta}([0,T];L^{2+2\eta})} + \|f''(u_n)\|_{L^{1+\eta}([0,T];L^{2+2\eta})}\right)$$

with $\quad K_n = \|f''(v_n + \theta_n w_n)\, w_n^2\|_{L^1([0,T];L^2)}$.

$$K_n \leq \varepsilon_n \|w_n\|^2_{ST(I)}, \quad \varepsilon_n \to 0.$$

- above arguments

$-limsup_{n\to\infty} \|u_n\|_{L^\infty([0,T];\mathcal{L})} < \frac{1}{\sqrt{4\pi}}$

- Consequence : Strichartz + absorption

$$\|w_n\|_{ST(I)} \leq C\varepsilon_n$$

- Conclusion : the theorem is established provided that

$$limsup_{n\to\infty} \|w_n\|_{L^\infty([0,T];H^1)} \leq \frac{1 - L\sqrt{4\pi}}{2}.$$

$$T^* = \sup\left\{0 \leq t \leq T; \quad limsup_{n\to\infty} \|w_n\|_{L^\infty([0,t];H^1)} \leq \frac{1 - L\sqrt{4\pi}}{2}\right\}$$

$T^* = T$ by classical arguments

- $T^* > 0$: $w_n(0) = 0$

- contradiction if $T^* < T$ ($\|w_n\|_{ST([0,T^*])} \to 0$).