Long time behavior of solutions of Vlasov-like Equations

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- Equations
  - Vlasov Poisson Equation and related models
  - 2D Euler

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- Other possible limiting behaviors
- What can we say in general
- Conclusions

The Vlasov Equation

$$\partial_t f + v\partial_x f + F_f \partial_v f = 0,$$

$f(x,v) : \mathbf{S}^1 \times \mathbf{R}$ is the phase space density

$$\rho(x) = \int dv f(x,v)$$

is the space density

$$F_f(x) = \int dy \mathcal{F}(x-y)$$

is the force.

Vlasov-Poisson Equation - VPE

Vlasov-Poisson Equation

$\mathcal{F} = \partial_x \mathcal{V}$
$\mathcal{V} = (\partial_{xx})^- 1 \delta$
for $x \in [0, 2\pi)$: $\mathcal{F}(x) = \frac{1}{2} - \frac{x}{2\pi}$

Hamiltonian Mean Field Model - HMF model

$\mathcal{V} = \cos(x - y)$
In this case
$\mathcal{F} = C(t)\cos x + S(t)\sin x$

2D Euler Equation

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

$$\omega : D \to \mathbf{R}$$

is the vorticity
$u(x, t)$, the velocity field, is given by

$$u = \nabla^{\perp}(-\Delta)^{-1}\omega$$

Some cases:
$D = \mathbf{R}^2$, $D = \mathbf{T}^2$, $D = \mathbf{R} \times [0, h]$

For VPE the density $f(x, v, t)$ is transported along the trajectories of an Hamiltonian system:

$$\dot{x} = v, \dot{v} = F_f(x)$$

The hamiltonian is a functional of the density f itself: self-consistent force field.
Therefore the area of the level stes of f is conseved.
Same for 2D Euler: the vorticity is transported along the trajectories of an Hamiltonian system.

Stable Stationary Solutions o VPE

$$f = g(v)$$

is a stationary solution of VPE

If g is not increasing and $\int f(1+v^2)dxdv < \infty$
then f is a stationary stable solution of VPE
Marchioro and Pulvirenti (1986).

BGK waves

Bernstein, Greene and Kruskal (1957) discovered the existence of inhomogeneous traveling wave solution of VPE

$$f = f_0(x - u_0 t, v - u_0)$$

BGK waves are time-periodic solutions of VPE.
They solve

$$f(x, v) = G(H(x, v)) = G(\frac{1}{2}v^2 + V(x))$$

where G is a generic $\mathbf{C}^2$ function, $H(x, v)$ is the 2D Hamiltonian, and $V(x)$ is the potential energy.
V must be compatible with its operative definition:

$$V_{xx} = \int_{-\infty}^{\infty} dvG(\frac{1}{2}v^2 + V(x)) - 1$$

Stable Stationary Solutions of 2D Euler

Any radial vorticity

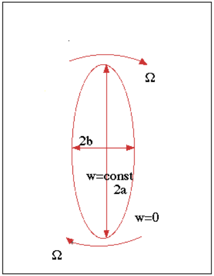
$$\omega = g(\rho), \quad \rho = \sqrt{x^2 + y^2}$$

is a stationary solution of 2D Euler

If g is not increasing and $\int \omega(1 + \rho^2)dxdv < \infty$
then f is a stationary stable solution of VPE
Marchioro and Pulvirenti.
In particular the circular vortex patch is stable (Pulvirenti and Wan).

Kirchoff Ellipse

Kirchhoff (1876) showed that elliptical patches are rotating solutions of 2D Euler.



The patch rotates with angular velocity $w\frac{ab}{a^2+b^2}$
The patch is stable in shape for a < 3 b.

Landau Damping

Landau, on the basis of the analysis of the VPE linearized around an equilibrium conjectured that for initial data close to euilibrium

$$f_0 = f(v) + \epsilon g(x, v)$$

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asymptotically the electric field will vanish and the phase space density will become homogeneous. The linear case has been fully characterized (see for instance Maslov and Fedoryuk) Existence of a class of damped analytic solutions has been proved with a scattering approach by C. and Maffei (1998) . Also non small solutions allowed. The result has been extended to close to equilibrium solutions by Hwang and Velasquez (2009). Initial data cannot be characterized.

Landau Damping

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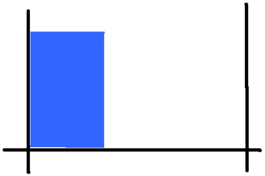
Recently Lin and Zeng have shown that close to equilibrium BGK waves exist for small regularity: $\mathbf{W}^{s,p}: \quad s < 1 + \frac{1}{p}$
Therefore No Landau Damping for small regularity.

Possible limiting behaviors: figures

$$\partial_t f + v\partial_x f = 0$$



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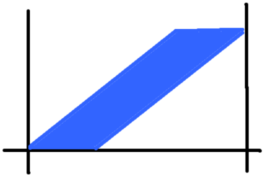

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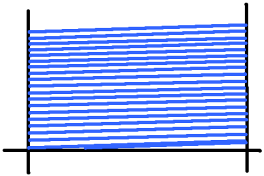

In the case of Landau Damping, weakly,

$$f(x,v,t) \xrightarrow[t\to\infty]{w} f^+(v)$$

and more precisely: it exists f^+ such that, strongly,

$$||f(x,v,t) - f^{+\infty}(x-vt,v)|| \to$$

where

$$< f^{+\infty} >_x = f^+(v)$$

For any concave functional $S(f)$, for instance the entropy, unless $f_0 = f^+$

$$S(f) = -\int f \log f$$

$$S(f^+) > S(f_0)$$

So entropy increases.

Possible limiting behaviors: simulations

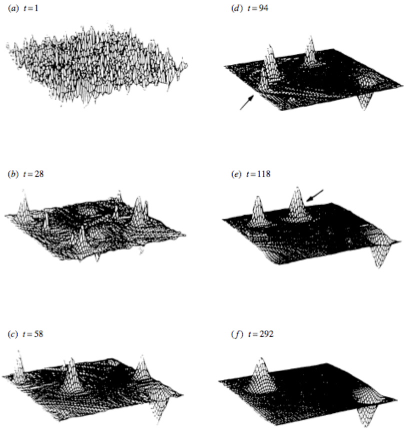

Figure 2. 3D perspective plots of the vorticity versus x and y for six successive times from a 2D Navier–Stokes simulation. (From Matthaeus *et al.* (1991).)

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- Is it possible to prove damping to BGK solutions ?

What we can say in general: Shnirelman construction

Mixing operators in $\mathbf{L}^2$

$$K * \phi(x) \equiv \int_M dy k(x,y)\phi(y)$$

where $\forall x : \quad \int_M k(x,y)dy = 1$ and $\forall y : \quad \int_M k(x,y)dx = 1$
bistochastic operators
Examples:

- $k(x,y) = \delta(x-y)$
- k is the heat kernel

$\mathcal{K} = \{K\}$ defines a partial order between vorticity fields.

Definition. Let $\mathbf{V}^s = \{u \in \mathbf{H}^s : \div u = 0, u \cdot n_{\partial_M} = 0\}$ the set of admissible velocity field

Definition. If $u_1, u_2 \in \mathbf{V}^1$
then $u_1 \prec u_2$ iff $curl \;\; u_1 \prec curl \;\; u_2$

Definition. For any $u_0 \in \mathbf{V}^1$ define

$$\Omega_{u_0} = \{u \in H^1 : u \prec u_0, \;\; ||u||_{L^2} = ||u_0||_{L^2}\}$$

If $u(t)$ is the solution of the Euler Equation with initial data u_0, and $\mathcal{O}(u_0) = \{u(t) : t \in \mathbf{R}\}$is the orbit of u_0 then $\bar{\mathcal{O}}(u_0) \in \Omega_{u_0}$.

Definition. Minimal Elements of Ω_{u_0} with respect to the order relation $\prec$ are called minimal flows..

It is possible to prove that minimal flows are stationary stable solutions of 2D Euler.

A first conjecture.
The set of minimal flows is an attractor for the 2D Euler flow.

Motivation: if the fluid does not go to stationary solutions level lines of vorticity are stretched and stretched and therefore the solution reacesh a minimal element.

The conjecture is probably wrong because more complicate behaviors are expected form simulations

Generalized minimal flows (Shnirelman).

A vector field is called a GMF if for any $v \in \bar{\mathcal{O}}(u)$:

$$||curl\ \ v||_{L^2} = ||curl\ \ u||_{L^2}$$

Now the conjecture is that (for the Navier Stokes Equation with Random Forcing in the null viscosity limit) the attractor is concentrated on Generalized Minimal Flows

In the case of Landau Damping, weakly,

$$f(x,v,t) \xrightarrow[t\to\infty]{w} f^{+}(v)$$

and more precisely: it exists f^{+} such that, strongly,

$$||f(x,v,t) - f^{+\infty}(x - vt, v)|| \to$$

where

$$< f^{+\infty} >_x = f^{+}(v)$$

For any concave functional $S(f)$, for instance the entropy

$$S(f) = -\int f \log f$$

$$S(f^{+}) \geq S(f_0)$$

Equality holds only if $f^{+} = f_0$

A strictly related conjecture

essentially rephrasing it in the Vlasov and deterministic case

Given f_0 define

$$\Omega(f_0) = \{weak \;\; limit \;\; points \;\; of \;\; f(x, v, t) : \;\; t \to +\infty\}$$

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- If g_1 and g_2 are in $\Omega(f_0)$ then

$$S(g_1) = S(g_2)$$

  The first is quasi obvious, the second it is not

Counterexample

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- then evolve for a time n with T
- ......
  There are two limit points of f, one is f_0 and the other is $\lim_{n\to\infty} T^n f_0 =< f_0 >_x$ and their entropy is not the same.

Limiting behavior: Lagrangian-chaotic motion

In this picture nothing hosts the fact that the asymptotic motion is Lagrangian-chaotic if, for instance, it is Eulerian-periodic. That is, for some $T > 0$,

$$\bar{f}(x, v, t) = \bar{f}(x, vt + T),$$

and therefore

$$\bar{H}(x, v, t) = \bar{H}(x, v, t + T),$$

Limiting behavior: Lagrangian-chaotic motion

where

$$\bar{f}(x,v,t) = \lim_{n\to\infty} f(x,v,t+nT),$$

and where

$$\bar{H} = \frac{v^2}{2} + V_{\bar{f}}$$

In this case it is reasonable that
$\bar{f}$ is constant on the invariant sets of the flow Φ^T
induced by $\bar{H}$ in the time interval $[0, T]$.

Limiting behavior: Lagrangian-chaotic motion

A new interesting behavior has been discovered numerically by H. Morita and K. Kaneco, PRL (2006)

(see also A. Antoniazzi, D. Fanelli, J. Barrè, P. H. Chavanis, T. Dauxois, S. Ruffo)

They consider the HMF model

$$V_f(x) \equiv \int dydv\ f(x,v)\cos(x-y)$$

with a far from equilibrium initial condition:

$$f_0 = e^{-\frac{v^2}{2T_0}} e^{-\frac{M_0 \cos x}{Teq(M_0)}}$$

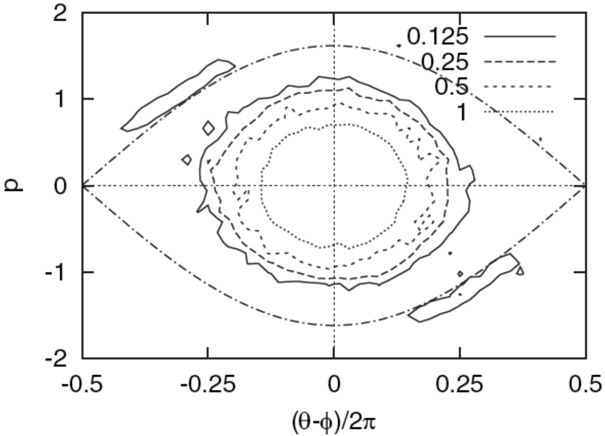
where M_0 is the magnetization, $T_eq(M_0)$ is the corresponding equilibrium temperature, and where $T_0 \neq T_{eq}$ is a (temperature) parameter.

Limiting behavior: Lagrangian-chaotic motion

The asymptotic state, depending on the parameter T_0,seems to be

stationary solution $\rightarrow$ periodic solution $\rightarrow$ quasi-periodic solution

Explanation - Poincaré Section



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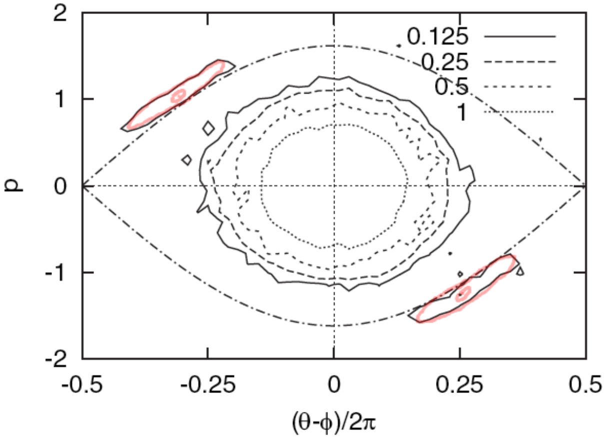

Conclusions

It seems reasonable that the VPE flow (and the 2D Euler flow) cannot be to complicated asymptotically. Because, if too complicate, mixing would lead to simple behavior.
Reasonable to expect asymptotically to reach a simple motion: may be even chaotic but with a few degrees of freedom involved.