Optimal polynomial approximation of PDEs with stochastic coefficients by Galerkin and collocation methods

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Outline

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Problem setting

Consider a differential problem

\[
\text{find } u : \quad \mathcal{L}(\mathbf{y})(u) = \mathcal{F}
\]

where the operator \( \mathcal{L}(\mathbf{y}) \) depends on a vector of \( N \) parameters:
\[
\mathbf{y} = (y_1, \ldots, y_N).
\]

The parameters are not perfectly known, or they have a lot of variability in repeated experiments.

Treat them as random variables with a given probability density function (estimated from experiments or from prior knowledge / expert opinion):
\[
\rho(\mathbf{y}) : \Gamma \to \mathbb{R}^+, \quad \int_\Gamma \rho(\mathbf{y}) d\mathbf{y} = 1
\]

We assume that for each \( \mathbf{y} \in \Gamma \) the corresponding solution \( u = u(\mathbf{y}) \) belongs to a given functional space \( V \)

\[
\Longrightarrow \quad u = u(\mathbf{y}) : \Gamma \to V
\]
Problem setting

- **Goal**: compute statistics of the solution $u$ or some quantity of interest $J(u)$. E.g.

$$\bar{J} = \mathbb{E}[J(u)] = \int_{\Gamma} J(u(y)) \rho(y) dy,$$

mean value

$$\text{Var}[J] = \mathbb{E}[J(u)^2] - \mathbb{E}[J(u)]^2$$

variance

$$P[J(u) > J_{cr}] = \int_{\Gamma} 1_{\{J(u(y)) > J_{cr}\}} \rho(y) dy$$

exceedance prob.
Problem setting: PDEs with random coefficients

Multivariate polynomial approximation

- Computations of statistical quantities typically imply lots of evaluations of $u(y) \leadsto$ lots of problems to solve
- **Idea**: build a reduced model $u_\Lambda(y) \approx u(y)$ that is cheap to evaluate and use it to compute statistical moments. E.g. 

$$\bar{J} \approx \bar{J}_\Lambda = \mathbb{E}[J(u_\Lambda)]$$

In this talk we consider multivariate polynomial reduced models.

- Let $\Lambda \subset \mathbb{N}^N$ be an index set of cardinality $|\Lambda| = M$, and consider the multivariate polynomial space

$$\mathbb{P}_\Lambda(\Gamma^N) = \text{span} \left\{ \prod_{n=1}^{N} y_n^{p_n}, \text{ with } p = (p_1, \ldots, p_N) \in \Lambda \right\}$$

- **find $M$ particular solutions** $u_p \in V, \forall p \in \Lambda$ and build 

$$u_\Lambda(y) = \sum_{p \in \Lambda} u_p y_1^{p_1} y_2^{p_2} \cdots y_N^{p_N}$$
Examples of pol. spaces: \( N = 2, \quad p = 16 \)

Tensor product: 
\[ p_n \leq w \]

Total degree: 
\[ \sum_n p_n \leq w \]

Hyperbolic cross: 
\[ \prod_n (p_n + 1) \leq w + 1 \]

Smolyak: 
\[ \sum_n f(p_n) \leq f(w) \]
\[ f(p) = \begin{cases} 
0, & p = 0 \\
1, & p = 1 \\
\lceil \log_2(p) \rceil, & p \geq 2 
\end{cases} \]
Anisotropic spaces: \( N = 2, \quad p_{\text{max}} = 16 \)

Tensor product:
\[ \alpha_n p_n \leq w \]

Total degree:
\[ \sum_n \alpha_n p_n \leq w \]

Hyperbolic cross:
\[ \prod_n (p_n + 1)^{\alpha_r} \leq w + 1 \]

Smolyak:
\[ \sum_n \alpha_n f(p_n) \leq f(w) \]
\[ f(p) = \begin{cases} 
0, & p = 0 \\
1, & p = 1 \\
\lfloor \log_2(p) \rfloor, & p \geq 2 
\end{cases} \]
Accuracy requirements

- We look at mean square error control (strong convergence)

\[ err_{2,\rho} = \int_\Gamma \| u(y) - u_\Lambda(y) \|_V^2 \rho(y) dy \text{ small} \]

An easy implication: [N.-Tempone, IJNME 09]

- Assume \( \| u(y) \|_V \) and \( \| u_\Lambda(y) \|_V \) uniformly bounded in \( \Gamma \)

- Let \( J : V \to \mathbb{R} \) a locally Lipschitz functional, with \( J(0) = 0 \)

Then

\[ \mathbb{E}[J(u)^q - J(u_\Lambda)^q] \leq C(q) \mathbb{E}[\| u - u_\Lambda \|_V^2]^{\frac{1}{2}} \]

i.e. convergence in \( \mathbb{E}[\| u - u_\Lambda \|_V^2]^{\frac{1}{2}} \) implies convergence of all moments of the functional \( J \).

- Weak convergence could (should) be considered as well.
Example 1: Thermal conduction with inclusions of random conductivity (baked cookies problem)

\[ \begin{cases} \quad \text{div}(a \nabla u) = f, & \text{in } D \\ \quad u = 0 & \text{on } \partial D \end{cases} \]

- Each circular inclusion \( C_i, \ i = 1, \ldots, 8 \) has a random conductivity coefficient \( y_i \).

\[ a(y_1, \ldots, y_N, x) = a_0 + \sum_{n=1}^{N} (y_n - a_0) \mathbb{1}_{C_n}(x), \quad x \in D, \ y \in \Gamma \]

- 8-dimensional parametric problem
Example II: Darcy flow in a medium with random permeability (groundwater flow problem)

\begin{align*}
    \mathbf{u} &= -a \nabla p \\
    \text{div} \mathbf{u} &= f
\end{align*}

- $a = a(\omega, x)$: random permeability field (with $a > 0$ a.s.)
- Each realization of the stochastic process gives a spatially varying permeability field.
- The random field can be conditioned to available measurements (e.g. by Kriging techniques)
- Infinite-dimensional parametric problem!
Problem setting: PDEs with random coefficients

Approximation of an $\infty$-dimensional random field

Let $\{b_n(x)\}$ be a complete orthonormal basis in $L^2(D)$ (trigon., wavelet, Karhunen-Loève, ...) and $a(\omega, x)$ a $\infty$-dimensional random field with finite second moments. Then, $a$ can be expanded as

$$a(\omega, x) = \mathbb{E}[a](x) + \sum_{n=1}^{\infty} y_n(\omega) b_n(x)$$

with $y_n(\omega) = \int_D (a(\omega, x) - \mathbb{E}[a](x)) b_n(x) \, dx$

If the basis $\{b_n\}$ has spectral approx. properties and the realizations of $a$ are smooth, then $\text{Var}[y_n] \xrightarrow{n \to \infty} 0$ suff. fast and we can truncate the series

$$a(\omega, x) \approx a_N(\omega, x) = \mathbb{E}[a](x) + \sum_{n=1}^{N} y_n(\omega) b_n(x)$$

**WARNING**: the truncated expansion might not be positive almost surely!

Possible remedy: $a(\omega, x) \approx a_N(\omega, x) = a_{\text{min}} + e^{b_0(x)} + \sum_{n=1}^{N} y_n(\omega) b_n(x)$
**Galerkin projection**

[Ghanem-Spanos, Karniadakis et al, Matthies-Keese, Schwab-Todor et al., Knio-Le Maître et al, Babuska et al., . . . ]

- Project the equation onto the subspace $\mathbb{P}_\Lambda(\Gamma)$
- Suitable for stochastic problems

Let $\{\psi_j\}_{j=1}^M$ be an orthonormal basis w.r.t. the probability density $\rho(y)$. Expand $u_\Lambda(y)$ on the basis: $u_\Lambda(y) = \sum_{j=1}^M u_j \psi_j(y)$

**Galerkin formulation**

Find $u_j \in V, j = 1, \ldots, M$ s.t.

$$\mathbb{E}\left[ L(y) \left( \sum_{j=1}^M u_j \psi_j \right) \psi_i \right] = \mathbb{E}[F \psi_i], \quad i = 1, \ldots, M$$

- This approach leads to solving $M$ coupled deterministic problems; difficult to assemble and need good preconditioners.
Collocation on sparse grids

[Smolyak ’63, Griebel et al ’98-’03-’04, Barthelmann-Novak-Ritter ’00, Hesthaven-Xiu ’05, N.-Tempone-Webster ’08, Zabaras et al ’07]

1. Choose a set of points \( y^{(j)} \in \Gamma, j = 1, \ldots, \tilde{M} \)
2. Compute the solutions \( u_j \in V : \mathcal{L}(y^{(j)})(u_j) = F \)
3. Interpolate the obtained values: \( u_\Lambda(y) = \sum_{j=1}^{\tilde{M}} u_j \phi_j(y) \).
\( \phi_j \in \mathbb{P}_\Lambda(\Gamma) \): suitable combinations of Lagrange polynomials

- Always leads to solving \( \tilde{M} \) uncoupled deterministic problems
- The number \( \tilde{M} \) of points needed is larger than the dimension \( M \) of the polynomial space (Except for tensor product spaces).
(Generalized) Sparse Grid approximation

1. Choose 1D abscissae. E.g.
   - Clenshaw-Curtis (extrema on Chebyshev polynomials)
   - Gauss points w.r.t. the weight $\rho_n$, assuming that the probability density factorizes as $\rho(y) = \prod_{n=1}^N \rho_n(y_n)$

2. Take a sequence of 1D polynomial interpolant operators $\mathcal{U}_n^{m(i)} : C^0(\Gamma_n) \to \mathbb{P}_{m(i)-1}(\Gamma_n)$ with increasing number of points.

   The $i$-th interpolant uses $m(i)$ abscissae $\vartheta^i_n = \{y_{n,1}, \ldots, y_{n,m_i}\}$.

3. Take differences of consecutive operators:
   $$\Delta_n^{m(i)} = \mathcal{U}_n^{m(i)} - \mathcal{U}_n^{m(i-1)}, \quad \mathcal{U}_n^{m(0)} = 0.$$ 

4. Multidimensional Smolyak approx.: let $i = [i_1, \ldots, i_N] \in \mathbb{N}_+^N$, and $\Lambda \subset \mathbb{N}_+^N$ an index set
   $$u^{SC}_\Lambda = \sum_{i \in \Lambda} \left( \Delta_1^{m(i_1)} \otimes \cdots \otimes \Delta_N^{m(i_N)} \right) (u)$$
By choosing properly the function $m$ and the set $\Lambda$ one can obtain a polynomial approximation in any given multivariate polynomial space ([Back-N.-Tamellini-Tempone LNCSE vol. 76, 2010])

Examples of sparse grids: $N = 2$, max. polynomial degree $p = 16$
Thermal conduction with random inclusions

- Conductivity coefficient: matrix $k=1$
- Circular inclusions: $k|_{\Omega_i} \sim \mathcal{U}(0.01, 0.8)$
- $\rightarrow$ 8 iid uniform random variables
- Forcing term $f = 100\mathbf{1}_F$
- Zero boundary conditions
- Quantity of interest $\psi(u) = \int_F u$

![Image showing thermal conduction with random inclusions](image_url)
Convergence plot for $\mathbb{E}[\psi(u)]$

![Galerkin](image1)

![Collocation](image2)

error versus estimated cost

(see [Back-N.-Tamellini-Tempone, LNCSE ’10])
Thermal conduction with random inclusions – anisotropic version

- Conductivity coefficient: matrix $k=1$
- Circular inclusions: $k_{\Omega_i} \sim \gamma_i U(0.01, 0.8) \
\rightarrow 4$ indep. uniform random variables
- Forcing term $f = 100$
- Zero boundary conditions
- Quantity of interest $\psi(u) = \int_F u$

![Diagram showing thermal conduction with random inclusions]
Convergence plot for $\mathbb{E}[\psi(u)]$

Anisotropic Total degree spaces with different anisotropy weights $\alpha_n$

Error versus estimated cost

(see [Back-N.-Tamellini-Tempone, LNCSE '10])
Optimization of polynomial spaces

- Consider the diffusion problem

\[
\begin{align*}
- \text{div}(a(x, y_1, \ldots, y_N) \nabla u) &= f, \quad \text{in } D \\
u &= 0, \quad \text{on } \partial D
\end{align*}
\]

- Assume \( a(y) \geq \alpha > 0, \forall y \in \Gamma \) (uniform coerciveness)

Analyticity result \([\text{Back-N.-Tamellini-Tempone '11, Babuska-N.-Tempone '05, Cohen-DeVore-Schwab '09/'10}]\)

- Let \( i = (i_1, \ldots, i_N) \in \mathbb{N}^N \) and \( r = (r_1, \ldots, r_N) > 0 \). Set \( r^i = \prod_n r_n^{i_n} \).

- Assume \( \| \frac{1}{a} \frac{\partial^i a}{\partial y^i} \|_{L^\infty(D)} \leq r^i \) uniformly in \( y \)

Then \( \| \frac{\partial^i u}{\partial y^i} \|_V \leq C |i|! (\log_2 r)^i \) uniformly in \( y \)

\( u : \Gamma \to V \) is analytic and can be extended analytically to \( \Sigma = \{ z \in \mathbb{C}^N : \sum_{n=1}^N r_n |z_n - \tilde{y}_n| < \log 2 \text{ for some } \tilde{y} \in \Gamma \} \)
Remark:
The assumption is satisfied both for *linear* and *exponential* expansions of a random field:

- **linear expansion:** \( a(y, x) = a_0 + \sum_{n=1}^{N} b_n(x) y_n \),
  with \( a_{\min} = a_0 - \sum_{n=1}^{N} \| b_n \|_{L^\infty(D)} > 0 \).
  In this case \( r_n = \| b_n \|_{L^\infty(D)}/a_{\min} \)

- **exponential expansion:** \( a(y, x) = a_0 + \exp \left( \sum_{n=1}^{N} b_n(x) y_n \right) \)
  with \( a_0 > 0 \). In this case: \( r_n = \| b_n \|_{L^\infty(D)} \).

Better estimates on analyticity region can be obtained by complex analysis.
Optimization of polynomial spaces

**Galerkin**

\[ u_{\Lambda}^{SG} = \sum_{p \in \Lambda} u_p(x) \psi_p(y) \]

find \( u_{\Lambda}^{SG} \) by Galerkin projection of the equation on

\[ P_{\Lambda} = \text{span}\{\psi_p, \ p \in \Lambda\}. \]

**Collocation**

\[ u_{\Lambda}^{SC} = \sum_{i \in \Lambda, n=1,\ldots,N} \bigotimes \Delta_{n}^{m(i_n)}[u]. \]

Compute \( u_{\Lambda}^{SC} \) by collocation on the corresponding sparse grid

**Question:** What is the best index set \( \Lambda \) in both cases?
Galerkin projection – best $M$ term approximation

- **Galerkin optimality:**
  \[
  \|u - u^{SG}_\Lambda\|_{V \otimes L^2_\rho(\Gamma)} \leq C \inf_{v_\Lambda \in V \otimes P_\Lambda} \|u - v_\Lambda\|_{V \otimes L^2_\rho(\Gamma)}
  \]

- Let \(\{\psi_p, \ p \in \mathbb{N}^N\}\) be the orthonormal basis of Legendre multivariate polynomials and \(v_\Lambda\) the truncated Legendre expansion of \(u\)
  \[
  v_\Lambda = \sum_{p \in \Lambda} \mathbb{E}[u \psi_p] \psi_p
  \]

- Parseval’s identity:
  \[
  \|u - v_\Lambda\|^2_{V \otimes L^2_\rho(\Gamma)} = \|u - \sum_{p \in \Lambda} \mathbb{E}[u \psi_p] \psi_p\|^2_{V \otimes L^2_\rho(\Gamma)} = \sum_{p \notin \Lambda} \|\mathbb{E}[u \psi_p]\|^2_V
  \]

**Best $M$ terms approximation**

The optimal index set \(\Lambda\) of cardinality \(M\) is the one that contains the \(M\) largest Legendre coefficients \(\|\mathbb{E}[u \psi_p]\|_V\).
Abstract construction of quasi optimal spaces

- Suppose we have an *a priori* estimate of the form

\[ \| \mathbb{E}[u \psi_p] \|_V \leq G(p) \] (1)

- Fix a threshold \( \epsilon \in \mathbb{R}_+ \), and define the index set \( \Lambda \) as

\[ \Lambda(\epsilon) = \{ p \in \mathbb{N}^N : G(p) \geq \epsilon \} \]

or equivalently

\[ \Lambda(w) = \{ p \in \mathbb{N}^N : -\log G(p) \leq w, \ w = \lceil -\log \epsilon \rceil \} \]

- The sharper the estimate (1), the better \( \Lambda \) approximates the “best M terms” index set.
Estimate of Legendre coefficients

For the diffusion problem with random coefficients, the solution $u(y)$ is analytic in $\Gamma$ and the following estimate of the Legendre coefficients holds [Cohen-DeVore-Schwab '10, Back-N.-Tamellini-Tempone '11]

$$
\|E[u_{\psi p}]\|_V \leq C_0 e^{-\sum_n g_n p_n} \frac{|p|!}{p!}
$$

for some $g_n > 0$, with $|p| = \sum_n p_n$, $p! = \prod_n p_n!$. Then the induced optimal index set is (TD-FC)

$$
\Lambda(w) = \left\{ p \in \mathbb{N}^N : \sum_n g_n p_n - \log \frac{|p|!}{p!} \leq w \right\}
$$

- The factorial term $\frac{|p|!}{p!}$ accounts for the interaction between the random variables and is purely isotropic
- Estimate (2) is meaningful only if $\sum_n e^{-g_n} < 1$!
Numerical tests

We consider the 1D problem

\[
\begin{aligned}
-(a(x, y)u(x, y)')' &= 1 \quad x \in D = (0, 1), y \in \Gamma \\
u(0, y) = u(1, y) &= 0, \quad y \in \Gamma
\end{aligned}
\]

with several choices of \(a(x, y)\) and compute \(\Theta(u) = u(\frac{1}{2})\).

We compare:

- **(Aniso) TD space:**

\[
\Lambda(w) = \left\{ p \in \mathbb{N}^N : \sum_n g_n p_n \leq w \right\}.
\]

- **(Aniso) TD-FC space:**

\[
\Lambda(w) = \left\{ p \in \mathbb{N}^N : \sum_{n=1}^N g_n p_n - \log \frac{|p|!}{p!} \leq w \right\}.
\]
Test 1: \( a(\mathbf{x}, \mathbf{y}) = 1 + 0.1y_1 + 0.5y_2 \)

**Figure:** Legendre coeffs of \( \Theta(u) \) in lexicographic order, with TD and TD-FC estimates

The Legendre coefficients have been computed with a sufficiently high level sparse grids.
Optimization of polynomial spaces

“true” Legendre coeffs.

iso-TD estimate.

aniso TD estimate

TD-FC estimate.
Test 2

\[ \log a(x, y) = y_1 + 0.2 \sin(\pi x) y_2 + 0.04 \sin(2\pi x) y_3 + 0.008 \sin(3\pi x) y_4 \]
Optimization of sparse grids

\[ u_M = S^m_{\Lambda}[u] = \sum_{i \in \Lambda} \bigotimes_{n=1}^{N} \Delta^{m(i_n)}_n[u]. \]

We use a knapsack problem-approach [Griebel-Knapek '09, Gerstner-Griebel '03, Bungartz-Griebel '04]: for each multiindex \( i \) estimate

- \( \Delta E(i) \): how much error decreases if \( i \) is added to \( \Lambda \) (error contribution)
- \( \Delta W(i) \): how much work, i.e. number of evaluations, increases if \( i \) is added to \( \Lambda \) (work contribution)

Then estimate the **profit** of each \( i \) as

\[ P(i) = \frac{\Delta E(i)}{\Delta W(i)} \]

and build the sparse grid using the set \( \Lambda \) of the \( M \) indices with the largest profit.
Estimate for $\Delta W$

Suppose we use nested abscissae, e.g. Clenshaw Curtis. The number of points added to the grid by $i$ is then

$$\Delta W(i) = \text{nb. new pts. in } \bigotimes_{n=1}^{N} \Delta^{m(i_n)} = \prod_{n=1}^{N} \left( m(i_n) - m(i_n - 1) \right)$$

Recall that for Clenshaw-Curtis

$$m(i) = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i = 1 \\ 2^{i-1} + 1 & \text{if } i > 1, \end{cases}$$

and the Lebesgue constant is

$$\mathbb{L}(m(i)) = \frac{2}{\pi} \log(m(i_n) + 1) + 1$$
Estimate for $\Delta E$

1. rewrite $\Delta E(i)$ as

$$
\Delta E(i) = \| (u - S_{\Lambda} [u]) - (u - S_{\{\Lambda \cup i\}} [u]) \|_{V \otimes L^2_\rho(\Gamma)} = \\
\| \sum_{j \in \{\Lambda \cup i\}} \Delta^m(j) [u] - \sum_{j \in \{\Lambda\}} \Delta^m(j) [u] \|_{V \otimes L^2_\rho(\Gamma)} = \| \Delta^m(i) [u] \|_{V \otimes L^2_\rho(\Gamma)}.
$$

2. use the following estimate (numerically validated)

$$
\Delta E(i)[u] \approx \| u_{m(i-1)} \|_V \prod_{n=1}^{N} (1 + \| m(i_{n-1}) \|)
$$

where $u_{m(i-1)}$ is the corresponding Legendre coefficient.

3. estimate $u_{m(i-1)}$ as in the Galerkin case

$$
\| u_{m(i-1)} \|_V \leq C_0 e^{-\sum_n g_n m(i_{n-1})} \frac{|m(i - 1)|!}{m(i - 1)!}
$$
Example: Comparison $\Delta E$ vs. estimate:

- Let $y_1, y_2 \sim \mathcal{U}(-1, 1)$.

\[
\begin{align*}
-\nabla \cdot [(1 + c_1 y_1 + c_2 y_2) \nabla u(x, y_1, y_2)] &= f(x) \quad x \in D \\
u(x) &= 0 \quad x \in \partial D
\end{align*}
\]

- $u(x, y_1, y_2) = \frac{\Delta^{-1} f(x)}{1 + c_1 y_1 + c_2 y_2}$ admits a Legendre expansion.

- Nested knots: Clenshaw-Curtis: $m(i) = 2^{i+1} - 1$, $y^k = \cos\left(\frac{k\pi}{m(i)}\right)$
All the pieces together

The set \( \{ \mathbf{i} \in \mathbb{N}^N : P(\mathbf{i}) \geq \epsilon \} \) is then equivalent to

\[
\left\{ \mathbf{i} \in \mathbb{N}_+^N : \sum_{i=n}^{N} m(i_n - 1)g_n - \log \frac{|m(\mathbf{i} - 1)|!}{m(\mathbf{i} - 1)!} - \sum_{n=1}^{N} \log \frac{2}{\pi} \log \frac{m(i_n - 1) + 1}{m(i_n) - m(i_n - 1)} \leq \mathcal{W} \right\}
\]

(EW - Error Work grids)

where

- Legendre coeff + Lebesgue constant = error estimate
- work estimate
Numerical test 1 - Uniform case

\[
\begin{aligned}
-(a(x,y)u(x,y))' &= 1 & x \in D = (0,1), \\
u(0,y) = u(1,y) &= 0
\end{aligned}
\]

- \(y \in \Gamma = [-1,1]^N, \ N = 2, 4\)
- different choices of diffusion coefficient \(a(x,y)\).
- We focus on a linear functional \(\psi : V \rightarrow \mathbb{R}, \ \psi(v) = v(\frac{1}{2})\);
- Convergence: \(\|\psi(u_{SG}) - \psi(u)\|_{L^2(\Gamma)} \) vs. nb of points in sparse grid
- We compare
  - standard isotropic Smolyak Sp. Grid, \(I = \{i \in \mathbb{N}^N : \sum_{n=1}^{N} (i_n - 1) \leq w\}\)
  - the Knapsack grid derived
  - “best \(M\) terms”: knapsack grid, with \textit{computed profits} \(P(i)\)
  - dimension adaptive algorithm [Gerstner-Griebel ’03, Klimke, PhD ’06],
    “www.ians.uni-stuttgart.de/spinterp”
Numerical examples

\[ a = 1 + 0.3y_1 + 0.3y_2 \]

\[ a = 1 + 0.1y_1 + 0.5y_2 \]
Numerical examples

\[ a(x, y) = 4 + y_1 + 0.2 \sin(\pi x) y_2 + 0.04 \sin(2\pi x) y_3 + 0.008 \sin(3\pi x) y_4 \]

\[ \log a(x, y) = y_1 + 0.2 \sin(\pi x) y_2 + 0.04 \sin(2\pi x) y_3 + 0.008 \sin(3\pi x) y_4 \]
Numerical examples

Numerical test 2 - 1D lognormal field

\[ L = 1, \ D = [0, L]^2. \]

\[
\begin{cases}
-\nabla \cdot a(y, x) \nabla u(y, x) = 0 \\
u = 1 \text{ on } x = 0, \ h = 0 \text{ on } x = 1 \\
\text{no flux otherwise}
\end{cases}
\]

\[ a(x, y) = e^{\gamma(x, y)} \]

\[ \mu_\gamma(x) = 0 \]

\[ \text{Cov}_\gamma(x, x') = \sigma^2 e^{-\frac{|x_1 - x'_1|^2}{LC^2}} \]

We approximate \( \gamma \) as

\[
\gamma(y, x) \approx \mu(x) + \sigma a_0 Y_0 + \sigma \sum_{k=1}^{K} a_k \left[ Y_{2k-1} \cos \left( \frac{\pi}{L} kx_1 \right) + Y_{2k} \sin \left( \frac{\pi}{L} kx_1 \right) \right]
\]

with \( Y_i \sim \mathcal{N}(0, 1) \), i.i.d.

Given the Fourier series \( \sigma^2 e^{-\frac{|z|^2}{LC^2}} = \sum_{k=0}^{\infty} c_k \cos \left( \frac{\pi}{L} k z \right), \ a_k = \sqrt{c_k}. \)
Numerical examples

Numerical test 2 - 1D lognormal field

- Quantity of interest:

\[ \mathbb{E}[\Phi(u)], \text{ with } \Phi = \left[ \int_0^L k(\cdot, x) \frac{\partial u(\cdot, x)}{\partial x} \, dx \right] \]

- Convergence:

\[ |\mathbb{E}[\Phi(u_{SG})] - \mathbb{E}[\Phi(u)]| \]

- We compare Monte Carlo estimate with Knapsack grids

- Gauss-Hermite-Patterson points (nested Gauss-Hermite)

- Estimate of Hermite coefficients decay (heuristic)

\[ \|u_i\|_V \leq \prod_{n=1}^N \frac{e^{-g_n i_n}}{\sqrt{i_n!}} \]

- Estimate of Lebesgue constant (heuristic) \( \mathbb{L}_n^{m(i_n)} \approx 1 \)
Numerical test 2 - 1D lognormal field

Here $LC = 0.2$, $\sigma = 0.3$.

$K = 6 \rightarrow N = 13$ r.v., and 99% of total variability of $e^\gamma$.

$K = 10 \rightarrow N = 21$ r.v., and 99.99% of total variability of $e^\gamma$.

![Graph showing numerical results](image-url)
Numerical test 3 - 2D lognormal field

$L = 1$, $D = [0, L]^2$.

\[\begin{align*}
-\nabla \cdot a(x, y) \nabla h(y, x) &= 0 \\
\text{B.C. : see figure}
\end{align*}\]

- $a(x, y) = e^{\gamma(y, x)}$
- $\mu_{\gamma}(x) = 0$
- $\text{Cov}_{\gamma}(x, x') = \sigma^2 e^{-\frac{|x-x'|^2}{Lc^2}}$

We approximate $\gamma$ as

\[\gamma(x, y) \approx \mu(x) + \sigma \sum_{k \in K} a_k [Y_{k,1} \cos(\pi k_1 x_1) \cos(\pi k_2 x_2) + Y_{k,2} \cos(\pi k_1 x_1) \sin(\pi k_2 x_2) + Y_{k,3} \sin(\pi k_1 x_1) \cos(\pi k_2 x_2) + Y_{k,4} \sin(\pi k_1 x_1) \sin(\pi k_2 x_2)]\]

with $Y_i \sim N(0, 1)$, i.i.d.

Given the Fourier series $\sigma^2 e^{-\frac{|z|^2}{Lc^2}} = \sum_{k=0}^{\infty} c_k \cos \left( \frac{\pi}{L} k z \right)$, $a_k = \sqrt{c_{k_1} c_{k_2}}$. 
Numerical examples

Numerical test 3 - 2D lognormal field

Here $LC = 0.4$, $\sigma = 0.3$.

$N = 21$ r.v., 92% of total variability of $e^\gamma$. 

![Graph showing lognormal field results]
Hyperbolic problems

\[
\begin{cases}
\frac{\partial^2 u}{\partial t^2} - \text{div}(a^2(x, y_1(\omega), \ldots, y_N(\omega)) \nabla u) = f, & \text{in } D, \ t > 0 \\
u = 0, & \text{on } \partial D, \ t > 0 \\
v|_{t=0} = u_0, \quad \frac{\partial u}{\partial t}|_{t=0} = v_0, & \text{in } D
\end{cases}
\]

assume \( a(x, y(\omega)) \leq a_{\text{max}} < \infty, \ \forall y \in \Gamma, \ \forall x \in D \) \hspace{2cm} (uniform boundedness)

- The solution is in general not smooth
Example: 1D problem

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - y^2 \frac{\partial^2 u}{\partial x^2} &= 0, & \text{in } \mathbb{R}, \ t > 0 \\
u(x, 0) &= u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = 0, & \text{in } \mathbb{R}
\end{align*}
\]

D’Alambert formula: \( u(x, t) = \frac{1}{2} u_0(x - yt) + \frac{1}{2} u_0(x + yt) \)

\[
\Rightarrow \quad \frac{\partial u}{\partial y}(x, t) = -\frac{t}{2} \frac{\partial u_0}{\partial \xi} \bigg|_{\xi = x - yt} + \frac{t}{2} \frac{\partial u_0}{\partial \xi} \bigg|_{\xi = x + yt}
\]

Remarks:

- \( u_0(\cdot) \in C^k(\mathbb{R}) \implies u(y)|_{(x,t)} \in C^k(\Gamma) \)

  In particular, a discontinuous initial datum implies discontinuous dependence on the parameter (wave speed)

- However, consider a functional \( J(y) = \int_{\mathbb{R}} u(y; x, T)g(x) \, dx \) with \( g \in C^m(\mathbb{R}) \) with compact support. Then, \( J(y)|_{(x,t)} \in C^{k+m}(\Gamma) \).

- Linear functionals of the solution can be much smoother than the solution itself. It might still be good to approximate \( J(y) \) by polynomials (not \( u \) itself).
Layered random medium

Assume that the wave speed has the form

\[ a(x, y(\omega)) = \sum_{n=1}^{N} y_n(\omega) \rho_n(x) \mathbb{1}_{D_n}(x) \]

where \( \mathbb{1}_{D_n} \) are characteristic functions corresponding to a non-overlapping partition of the domain: \( D = \bigcup_{n=1}^{N} D_n \) with smooth interfaces, \( \rho_n \) are smooth functions and \( y_n \in \Gamma_n \) are random variables.

Result 1 [Motamed-N.-Tempone '11]

Given \( f \in L^2(0, T; H^1_0(D)) \), \( u_0 \in H^1_0(D) \), \( v_0 \in L^2(D) \),

- the solution has in general only one bounded derivative \( \partial_{y_n} u \in L^2(0, T; L^2(D)) \).
- The solution might be smoother for smoother data not intersecting any interface between the layers.
Layered random medium

Result 2 [Motamed-N.-Tempone ’11]

Consider a finite dimensional approximation of the equation in space with discretization parameter $h$.

- The discrete solution $u_h(y)$ is always analytic with respect to $y$ for all $y \in \mathbb{R}^N$. However, if we replace $y$ with $z \in \Sigma \equiv \{z \in \mathbb{C}^N : \text{dist}(\Gamma, z) \leq \tau\}$, and study the problem in the complex domain

$$\max_{z \in \mathbb{C}^N} \|u_h(z)\|_{L^2(0,T;H_0^1)} \leq C \frac{h}{\tau} \exp\{\gamma T \frac{\tau}{h}\}$$

- Consider a tensor product polynomial approximation if $y$ of degree $p$. We have two regimes
  - for $hp \gg CT \leadsto$ exponential convergence in $p$
  - for $hp \ll CT \leadsto$ algebraic slow convergence in $p$ due to small regularity of $u$ w.r.t. $y$. 
Wave equation in two layered random medium

- wave equation in two-layered medium
- random wave speed in each layer
- smooth initial deformation across the interface

Initial solution \( E[u](x, t = 1) \)

Standard deviation \( std[u](x, t = 1) \)
Isotropic Smolyak grid approximation on Gauss-Legendre abscissae

Finite difference approximation in space + leapfrog in time

Maximum error in the expected value at $t=1$ versus $\#$ of collocation points $\tilde{M}$.

The solution has low regularity $\sim$ slow convergence (the convergence rate in $\tilde{M}$ depends on the mesh size $\Delta x$).
Smooth case:

- The initial displacement is smooth and confined in the second layer
- Maximum error in the expected value at $t=1$ versus $\#$ of collocation points $\tilde{M}$. 

![Graph showing numerical results for smooth case with random coefficients.]
Conclusions

1. Solution may depend on a **high number of parameters / random variables**. An **accurate choice of the approximation space** is needed, to avoid unaffordable computational costs.

2. Under regularity assumptions it makes sense to look for global polynomial approximations, either modal (galerkin procedure) or nodal (collocation procedure).

3. We propose a **general procedure** based on **estimates of Legendre coefficients / profits of grids** to build optimal polynomial approximations.

4. There is **no “one for all” recipe**: the structure of the problem (hence of the solution) leads to the appropriate choice of approximation.
J. Bäck and F. Nobile and L. Tamellini and R. Tempone

J. Bäck and F. Nobile and L. Tamellini and R. Tempone

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