Identifiability of point-wise Organic pollution in stream-waters

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Séminaire LJLL, UPMC, PARIS VI, 08/04/2011

Identifiabilité de sources ponctuelles de pollution organique dans les cours d'eau

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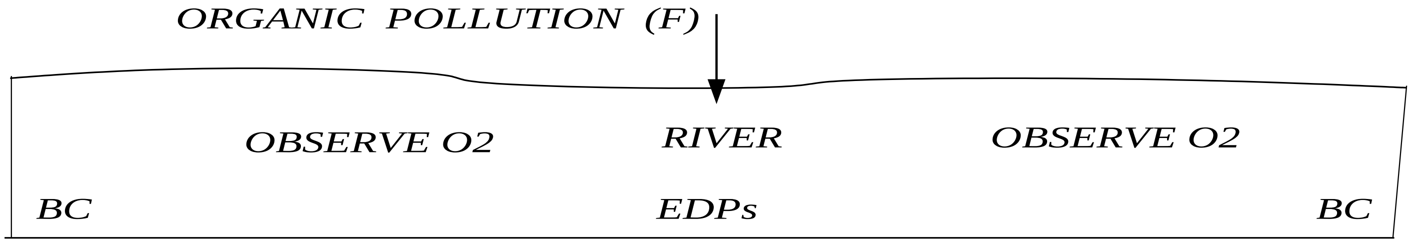
Croquis de l'exposé : Identifiabilité?



Objectif : Etudier l'opérateur

$$B : F \mapsto OBS(O2)$$

Est ce que B est

1. Injectif? C'est l'Identifiabilité!
2. Surjectif? Non!
3. Bicontinue? Non+(+)! ,

Dernier point = L'essentiel de la question numérique!

Pollution Organique

Bassin Versant Loire-Bretagne

Origine de la Pollution Organique

- Agriculture et Elevage : 60%
- Domestique : 30%
- Industrielle : 10%

Source : Une Etude relativement récente qui traîne sur l'un des sites WEB avec la terminaison .gouv.fr

Eutrophisation

Excès de matière organique $\Longrightarrow$ Excès de nutriments $\Longrightarrow$ Eutrophisation

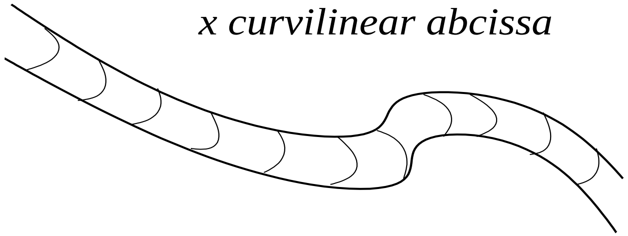
Photo Gauche : Rivière Thouet : les deux sèvres (Source : WEB!).
Elle se serait mise dans cet état en deux années (colonisée par l'élodée dense).

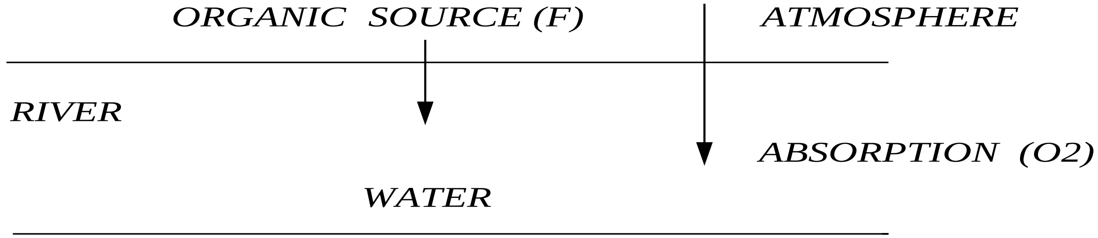
Outline

1. Biochemical Oxygen Demand-Dissolved Oxygen Model (BOD-DO Model)
2. Source Identifiability for a Scalar BOD Equation
   (M. Andrle, FBB, A. El Badia, 2009, —IP 2011—)
3. Source Identifiability for a 1D-Coupled BOD-DO System
   (M. Andrle, FBB, 2010)
4. Extension to mD.

BOD Model in a Channel

b : Biochemical Oxygen Demand (BOD) —DBO en français—.





Reaction (Streeter & Phelps, 1925). (Notation : $I = (0, L)$, $\partial_x b = b'$)

$$\partial_t b = -Rb + F \qquad \text{in } I \times (0, T).$$

Transport in the River

$$\partial_t b + Vb' + Rb = F \qquad \text{in } I \times (0, T),$$

Diffusion of Oxygen in water

$$\partial_t b - (Db')' + Vb' + Rb = F \qquad \text{in } I \times (0, T).$$

DO Model in a Channel

c_O : Dissolved Oxygen concentration

c_S : Dissolved Oxygen concentration at saturation (constant)

$c = c_S - c_O$: Dissolved Oxygen Deficit (to Saturation)

Pump out Oxygen ($\Longrightarrow$) Oxygen Deficit
($\Longrightarrow$) Oxygen Absorption from Atmosphere

(Diffusion, Transport, Reaction)

$$\partial_t c - (Dc')' + Vc' + R_* c = G \qquad \text{in } I \times (0, T).$$

BOD-DO Model in a Channel

Organic Pollution Source (F), Oxygen Source (G)

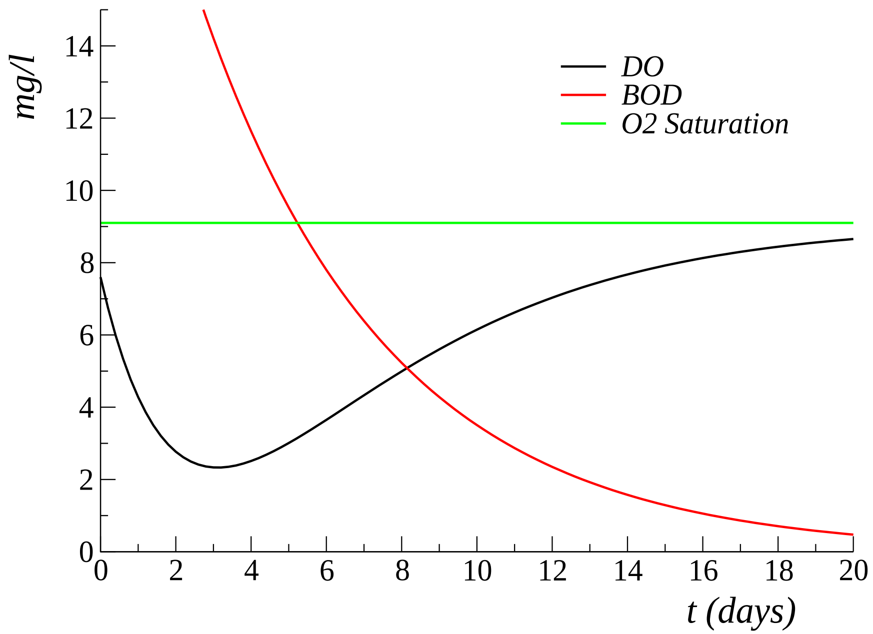
$$\begin{aligned} \partial_t b - (Db')' + Vb' + Rb &= F && \text{in } I \times (0,T), \\ \partial_t c - (Dc')' + Vc' + R_* c - Rb &= G && \text{in } I \times (0,T), \\ b(0,t) = c(0,t) &= 0, && \text{in } (0,T), \\ Db'(L,t) = Dc'(L,t) &= 0, && \text{in } (0,T), \\ b(0,\cdot) = c(0,\cdot) &= 0, && \text{in } I. \end{aligned}$$

Wide Bibliography in this model

Brown (EPA, 1987), Okubo (Springer-Verlag, 1980),

Bermúdez, (Dunod, 1993)

Streeter-Phelps Model (1925)



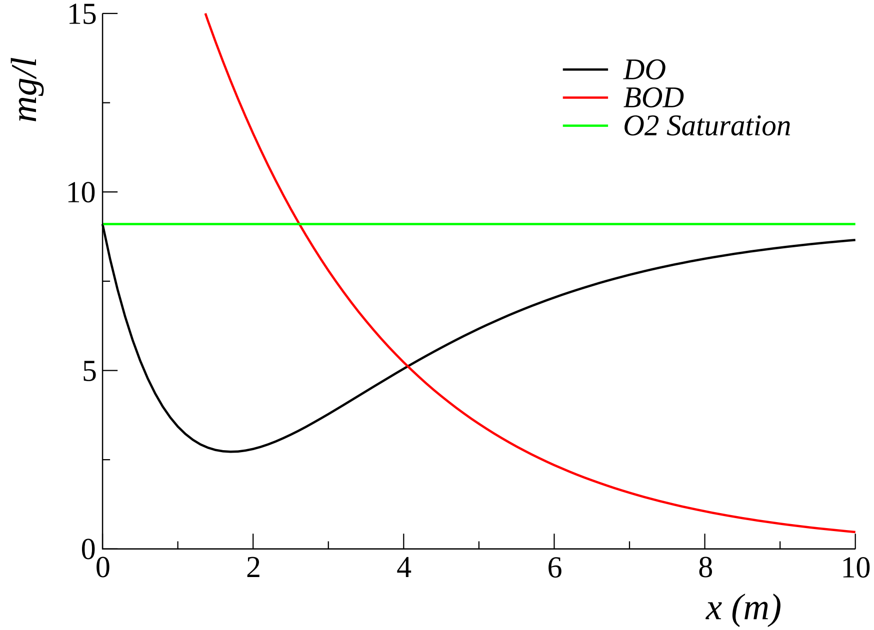


Diagram to the left : Initial pollution, (No Diffusion nor Advection)
Diagram to the right : Steady pollution, (No diffusion).

Point-wise Pollution Detection

Point-wise Sources to identify

(Organic Pollution Source , Oxygen Sink)

$$(F(x,t)\ ,\ G(x,t)) = (f(t)\delta_{x-r(t)}\ ,\ g(t)\delta_{x-s(t)}), \qquad \text{in } I\times(0,T).$$

Observations

Available Observations, only on c the Dissolved O_2

$$B\,[F,G] = \Big\{ (c, Dc')\,(\xi_1,\cdot)\ ,\ (c, Dc')\,(\xi_2,\cdot) \Big\}, \qquad \text{in } (0,T).$$

Assumption : Position of observations (upstream, downstream)

$$\xi_1 < r(t), s(t) < \xi_2, \qquad \text{in } (0,T).$$

How to use these Observations?

Direct use of the Observations on the DO (c)

Use the observations $(c, Dc')(\xi_1, \cdot), (c, Dc')(\xi_2, \cdot)$ available in real-time and directly cope with the detection for the Coupled BOD-DO system.

Obtain Experimentally Observations on the BOD (b)

Long Chemical Protocol ($\Longrightarrow$) Recovery of $(b(\xi_1, \cdot), b(\xi_2, \cdot))$ on BOD

($\Longrightarrow$) Source Detection for the BOD scalar equation

($\Longrightarrow$) Lasts five Days! Too Long!

I. Scalar BOD Equation

Inverse Problems, Volume 27 , 2011

Source Detection in the BOD Equation

$$\partial_t b - (Db')' + [Vb'] + Rb = F \qquad \text{in } I \times (0,T),$$
$$b(0,t) = 0, \qquad \text{in } (0,T),$$
$$Db'(L,t) = 0, \qquad \text{in } (0,T),$$
$$b(0,\cdot) = 0, \qquad \text{in } I.$$

Steady or Moving Source

$$F(x,t) = f(t)\delta_{x-r(t)} \ , \qquad \text{in } I \times (0,T).$$

Observations on b

$$B\,[F] = (b\,(\xi_1,\cdot) \ , \ b\,(\xi_2,\cdot)), \qquad \text{in } (0,T).$$

Observed Data $\alpha = (\alpha_1 \ , \ \alpha_2)$

$$\text{Find } F; \qquad B\,[F] = \alpha = \Big(\alpha_1\,(\cdot) \ , \ \alpha_2\,(\cdot)\Big), \qquad \text{in } (0,T).$$

This problem is severely ill-posed.

Identifiability

Théo. **1** *Assume that* $B[F_1] = B[F_2]$, *then* $F_1 = F_2$ *that is*

$$(r_1(\cdot), f_1(\cdot)) = (r_2(\cdot), f_2(\cdot)), \qquad \textit{in } (0,T).$$

Proof Simplification : fixed sources (r_1, r_2). Now, define

$$b_1 \ (\Longleftrightarrow) \ F_1, \qquad b_2 \ (\Longleftrightarrow) \ F_2$$

Set $\eta = (b_1 - b_2)$. Assume that $B[F_1] = B[F_2]$ (with $r_1 < r_2$). Then

$$\eta(\xi_1, \cdot) = \eta(\xi_2, \cdot) = 0, \qquad \text{in } (0,T).$$

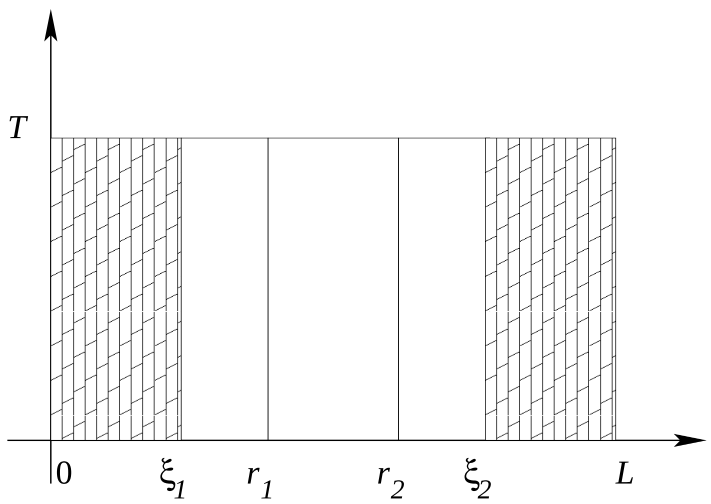
Aim $(\Longrightarrow)$ Show in three steps that

$$\Big(\eta \equiv 0, \qquad \text{in } I \times (0,T) \ \Big)(\Longrightarrow) \ \Big((r_1, f_1(\cdot)) = (r_2, f_2(\cdot)), \qquad \text{in } (0,T)\Big).$$

First Step

Solve heat sub-problems in external strips $(0, \xi_1) \times (0, T)$ (in $(0, \xi_2) \times (0, T)$)

$$\begin{aligned} \partial_t \eta - (D\eta')' + R\eta = 0 & \qquad \text{in } (0, \xi_1) \times (0, T), \\ \eta(0, t) = \eta(\xi_1, t) = 0 & \qquad \text{in } (0, T), \\ \eta(0, \cdot) = 0 & \qquad \text{in } (0, \xi_1). \end{aligned}$$



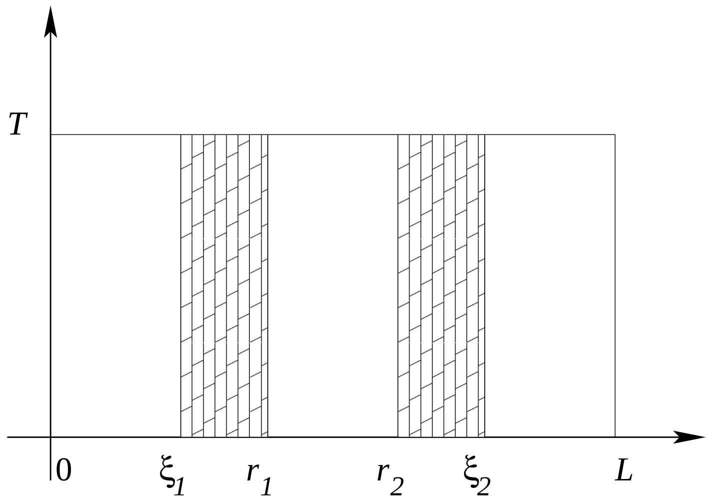
Then, we obtain that

$$\begin{aligned} \eta \equiv 0, & \qquad \text{in } (0, \xi_1) \times (0, T), \\ \eta \equiv 0, & \qquad \text{in } (\xi_2, L) \times (0, T). \end{aligned}$$

Second Step

Solve sideways sub-problems in (ξ_1, r_1) (in (r_2, ξ_2)). Cauchy's conditions at $x = \xi_1$

$$\begin{aligned} \partial_t \eta - (D\eta')' + R\eta &= 0 && \text{in } (\xi_1, r_1) \times (0,T), \\ \eta(\xi_1, t) = D\eta'(\xi_1, t) &= 0 && \text{in } (0,T), \\ \eta(0, \cdot) &= 0 && \text{in } (\xi_1, r_1). \end{aligned}$$



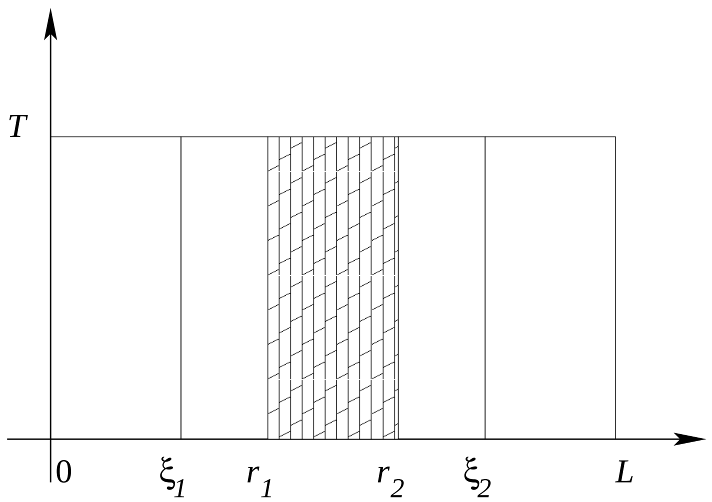
Unique continuation theorem (Saut et Sheurer, 1987),
Carleman estimate (Klibanov, 2006, A. Ben Abdallah et al, 2008-10),
We derive that

$$\eta \equiv 0, \qquad \text{in } (\xi_1, r_1) \cup (r_2, \xi_2) \times (0,T).$$

Third and Final Step

Solve internal sub-problem in (r_1, r_2).

$$
\begin{aligned}
\partial_t \eta - (D\eta')' + R\eta = \; 0 \qquad & \text{in } (r_1, r_2) \times (0, T), \\
\eta(r_1, t) = \eta(r_2, t) = 0, \qquad & \text{in } (0, T), \\
\eta(0, \cdot) = 0, \qquad & \text{in } (r_1, r_2).
\end{aligned}
$$



Then, we have to

$$\eta \equiv 0, \qquad \text{in } (r_1, r_2) \times (0, T).$$

We conclude therefore to

$$\eta \equiv 0, \qquad \text{in } I \times (0, T).$$

The final result is then obtained which is $F_1 = F_2$, or $(f_1(\cdot), r_1(\cdot)) = (f_2(\cdot), r_2(\cdot))$.

Numerical Examples (I)

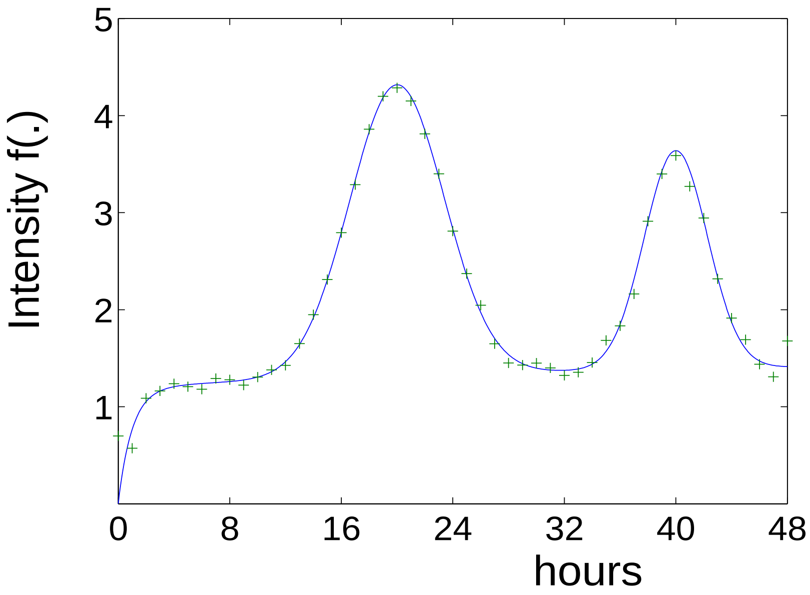
$$L = 1000\ m, \qquad T = 48 \times 3600\ s, \qquad (\xi_1, \xi_2) = (200, 800)\ m$$

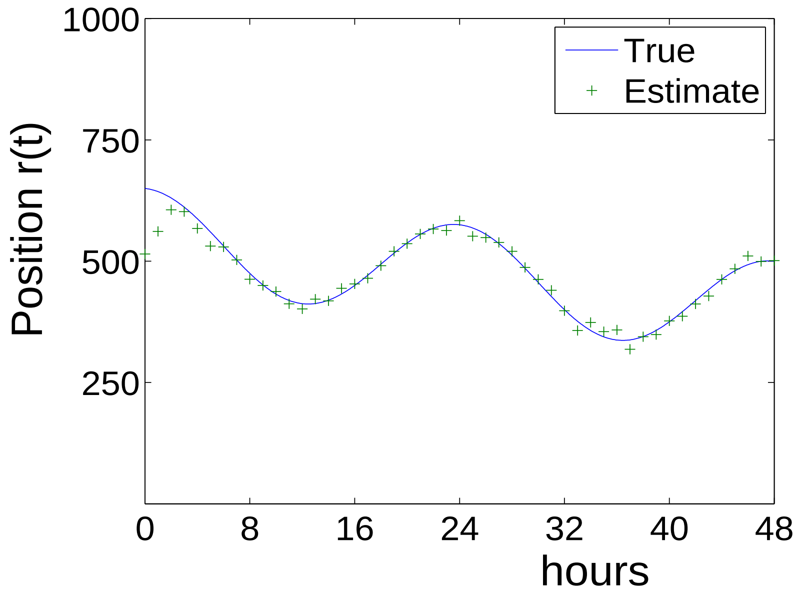
$$D = 40\,\mathrm{m}^2\mathrm{s}^{-1}, \qquad R_* = R = 1 \times 10^{-5}\,\mathrm{s}^{-1}$$

$$r\,(t) = L\Big\{\frac{13}{20} + \frac{1}{10}\left(\cos\left(4\pi\frac{t}{T}\right) - 1\right) - \frac{3t}{20T}\Big\}$$

$$f\,(t) = 3\Big\{\exp -\left(\frac{10}{T}\left(t - 7.2 \times 10^4\right)\right)^2 + \frac{3}{4}\exp -\left(\frac{15}{T}\left(t - 1.44 \times 10^5\right)\right)^2 + \frac{2}{5}\left(1 - \exp\frac{-t}{3.6 \times 10^3}\right) + \frac{1}{10}\left(1 - \exp\frac{-t}{1.44 \times 10^5}\right)\Big\}$$

Numerical Examples (II)

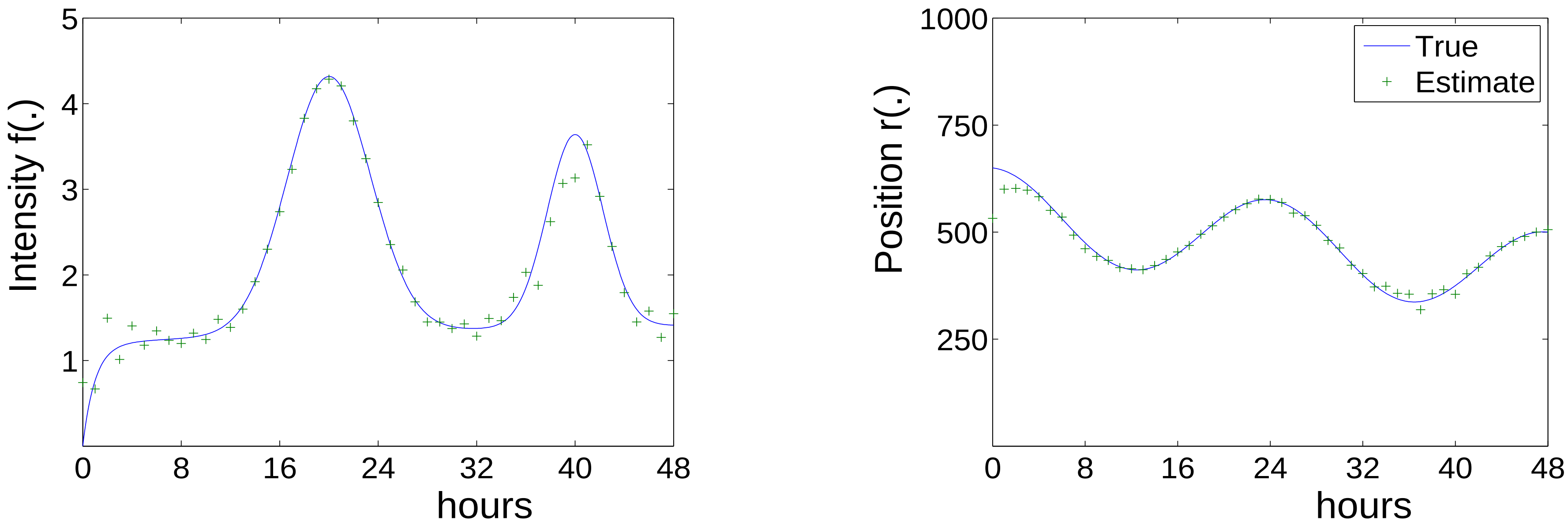




$$V = 0.2 \text{ ms}^{-1}$$

Noise on Observations 0.05, Errors : Intensity 0.05, Location 0.042

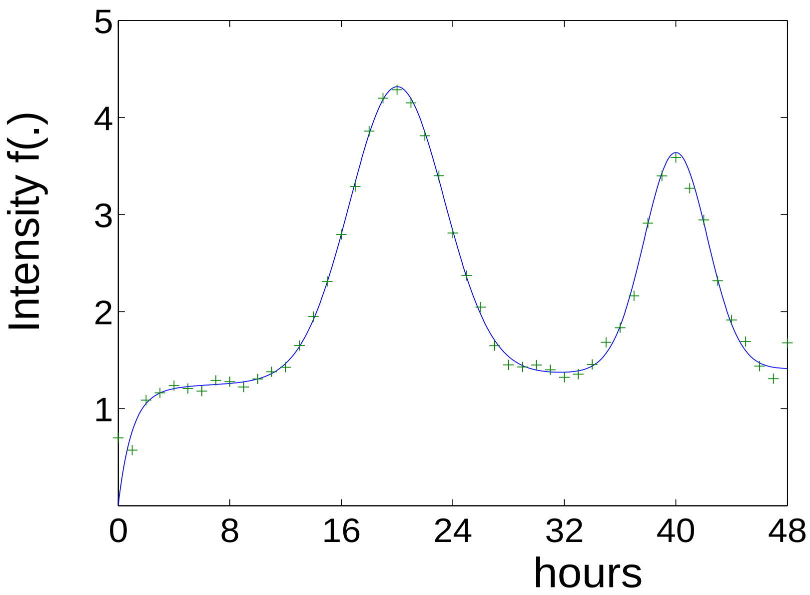
Numerical Examples (III)

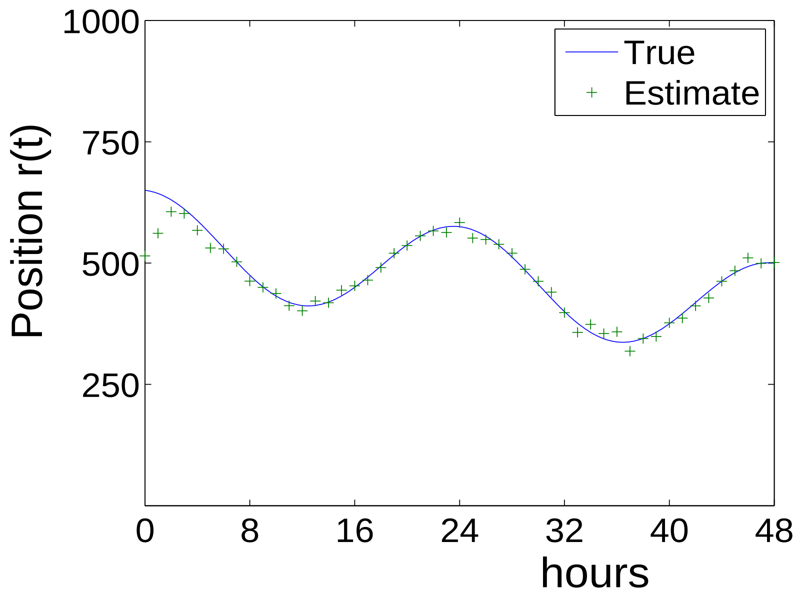


$$V = 0.5 \ \ \mathrm{ms}^{-1}$$

Noise on Observations 0.05, Errors : Intensity 0.079, Location 0.032

Numerical Examples (IV)





$$V = 0.2 \ \ \mathrm{ms}^{-1}$$

Noise on Observations 0.125, Errors : Intensity 0.062, Location 0.08

II. BOD-DO Model

No Direct Observations on BOD
Cauchy Observations on DO

Sources Detection

(Pollution Source, Oxygen Sink) $\qquad (F, G) = (f(t)\delta_{x-r(t)}, g(t)\delta_{x-s(t)})$

$$
\begin{aligned}
\partial_t b - (Db')' + Rb &= F && \text{in } I \times (0,T), \\
\partial_t c - (Dc')' + R_* c - Rb &= G && \text{in } I \times (0,T), \\
b(0,t) = c(0,t) &= 0, && \text{in } (0,T), \\
Db'(L,t) = Dc'(L,t) &= 0, && \text{in } (0,T), \\
b(0,\cdot) = c(0,\cdot) &= 0, && \text{in } I.
\end{aligned}
$$

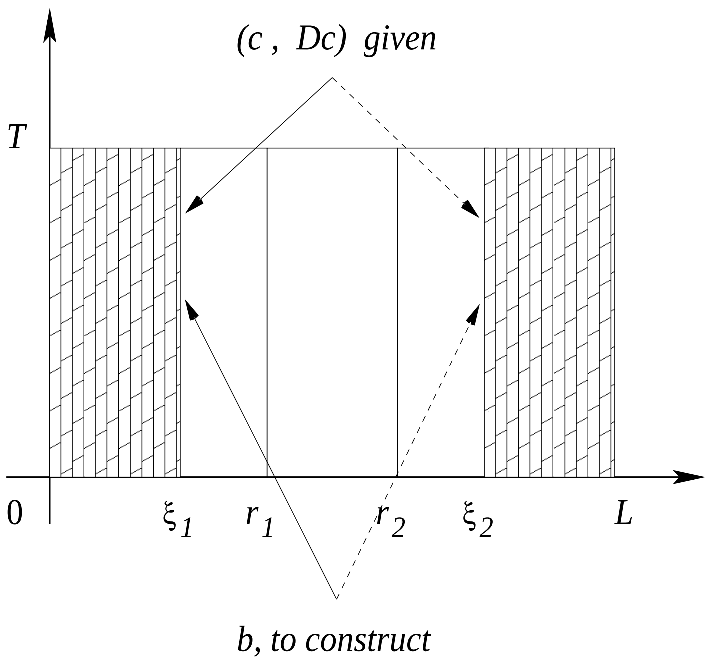
Observations available on the Dissolved Oxygen Concentration c

$$
B\,[F, G] = \Big\{ (c, Dc')\,(\xi_1, \cdot)\ ,\ (c, Dc')\,(\xi_2, \cdot) \Big\}, \qquad \text{in } (0,T).
$$

Observed Data (α, β)

$$
\text{Find } (F, G); \qquad B\,[F, G] = \Big\{ (\alpha, \beta)_1\,(\cdot)\ ,\ (\alpha, \beta)_2\,(\cdot) \Big\}, \qquad \text{in } (0,T).
$$

METHODOLOGY



Show that the operators are injective

$$(c(\xi_1,\cdot), Dc'(\xi_1,\cdot)) \mapsto b\,(\xi_1,\cdot)\,,$$
$$(c(\xi_2,\cdot), Dc'(\xi_2,\cdot)) \mapsto b\,(\xi_2,\cdot)\,.$$

Then, join (PART I) in Step 2 to obtain separately identifiability of F and of G.

Sideways BOD-DO Model

Two sub-problems in the time-space strips $(0,\xi) \times (0,T) = I_\xi \times (0,T)$.

$$
\begin{aligned}
\partial_t b - (Db')' + Rb = 0 \qquad & \text{in } I_\xi \times (0,T), \\
\partial_t c - (Dc')' + R_* c - Rb = 0 \qquad & \text{in } I_\xi \times (0,T), \\
b(0,t) = c(0,t) = 0, \qquad & \text{in } (0,T), \\
(b(0,\cdot), c(0,\cdot)) = (0,0), \qquad & \text{in } I_\xi. \\
c(\xi,t) = \alpha(t), \qquad & \text{in } (0,T), \\
Dc'(\xi,t) = \beta(t), \qquad & \text{in } (0,T).
\end{aligned}
$$

Uniqueness is the aim!

$$
(\alpha,\beta) = (0,0) \ (\Longrightarrow) \ (b,c) = (0,0).
$$

ILL-POSEDNESS (I)

Set $\beta(t) = 0$. Consider $\gamma(t) = Db'(\xi, t)$ as unknown. Solve sequentially

$$\begin{aligned} \partial_t b_\gamma - (Db'_\gamma)' + Rb_\gamma = 0 \qquad & \text{in } (0,\xi) \times (0,T), \\ b_\gamma(0,t) = 0, \qquad & \text{in } (0,T), \\ Db'_\gamma(\xi,t) = \gamma(t), \qquad & \text{in } (0,T), \\ b_\gamma(0,\cdot) = 0, \qquad & \text{in } (0,\xi). \end{aligned}$$

$$\begin{aligned} \partial_t c_\gamma - (Dc'_\gamma)' + R_* c_\gamma = Rb_\gamma \qquad & \text{in } (0,\xi) \times (0,T), \\ c_\gamma(0,t) = 0, \qquad & \text{in } (0,T), \\ Dc'_\gamma(\xi,t) = 0, \qquad & \text{in } (0,T), \\ c_\gamma(0,\cdot) = 0, \qquad & \text{in } (0,\xi). \end{aligned}$$

We have to solve the ill-posed equation on $\gamma \in L^2(0,T)$, that is

$$S\gamma(t) = c_\gamma(\xi,t) = \alpha(t), \qquad \text{in } (0,T).$$

ILL-POSEDNESS (II)

Fourier Computations ($D = R = R_* = 1$)

$$b_\gamma(x,t) = \frac{2}{\xi}\sum_{k\in\mathbb{N}}\Big((-1)^k\int_0^t \gamma(s)e^{-\lambda_k(t-s)}\,ds\Big)\sin\Big((k+\frac{1}{2})\frac{\pi}{\xi}x\Big),$$

$$c_\gamma(x,t) = \frac{2}{\xi}\sum_{k\in\mathbb{N}}\Big((-1)^k\int_0^t \gamma(s)(t-s)e^{-\lambda_k(t-s)}\,ds\Big)\sin\Big((k+\frac{1}{2})\frac{\pi}{\xi}x\Big),$$

$$(S\gamma)(t) = \int_0^t K(t-s)\gamma(s)\,ds, \qquad \forall t\in(0,T),$$

$$\lambda_k = \Big[(k+\frac{1}{2})\frac{\pi}{\xi}\Big]^2 + 1, \qquad K(s) = \frac{2}{\xi}\sum_{k\in\mathbb{N}} s e^{-\lambda_k s}, \qquad \forall s\in(0,T).$$

($S\gamma = \alpha$) is a Volterra equation.

S is a convolution operator ($\Longrightarrow$) S^{-1} is unbounded.

Uniqueness : First Result ($T = \infty$)

Solve equation

$$(S\gamma)(t) = K \star \gamma(t) = 0 \qquad \forall t \in (0, \infty).$$

Use Laplace transform yields that

$$\widehat{K}(p)\hat{\gamma}(p) = 0, \qquad \forall p \in (0, \infty),$$

The Laplace Transform of $K(\cdot)$ is

$$\widehat{K}(p) = \frac{2}{\xi} \sum_{k \in \mathbb{N}} \frac{1}{(p + \lambda_k)^2} > 0, \qquad \forall p \in (0, \infty).$$

We obtain then that

$$(\hat{\gamma}(p) = 0, \ \ \forall p \in (0, \infty)) \implies (\gamma(t) = 0, \ \ \forall t \in (0, \infty)).$$

Then

$$b(t, x) = c(t, x) = 0, \qquad \forall t \in (0, \infty), \ \forall x \in (0, L).$$

General Case : Abstract EDO Formulation

Set $(\alpha, \beta) = (0,0)$. Consider the system

$$\partial_t \begin{pmatrix} b \\ c \end{pmatrix} + A \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} b \\ c \end{pmatrix}(0,\cdot) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The unbounded operator A is defined as follows

$$A \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} -(Db')' + Rb \\ -(Dc')' + R_* c - Rc \end{pmatrix}$$

Its domain is given by

$$D(A) = \Big\{ (b,c) \in \boldsymbol{H}^1(I_\xi), \qquad ((Db')', (Dc')') \in \boldsymbol{L}^2(I_\xi) \\ (b(0), c(0)) = (0,0),\ \ c(\xi) = (Dc')(\xi) = 0 \Big\}.$$

Analysis of A

Assume $(f, g) \in L^2(I_\xi)$. Consider the stationary system

$$A \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

Hence

$$\begin{aligned} -(Db')' + Rb &= f \qquad \text{in } I_\xi, \\ -(Dc')' + R_* c - Rb &= g \qquad \text{in } I_\xi, \\ b(0) = c(0) &= 0, \\ c(\xi) = 0, \qquad Dc'(\xi) &= 0, \end{aligned}$$

Functional framework

$$b \in V = \left\{ \varphi \in H^1(I_\xi) \qquad \varphi(0) = 0 \right\}, \qquad c \in Q = H_0^1(I_\xi).$$

Saddle-Point Problem

Find $(b, c) \in V \times Q$ such that

$$\int_{I_\xi} Db'\psi'\,dx + \int_{I_\xi} Rb\psi\,dx = \int_{I_\xi} f\psi\,dx, \qquad \forall\psi \in Q,$$

$$\int_{I_\xi} Dc'\varphi'\,dx + \int_{I_\xi} R_* c\varphi\,dx - \int_{I_\xi} Rb\varphi\,dx = \int_{I_\xi} g\varphi\,dx, \qquad \forall\varphi \in V.$$

This is a saddle point problem: $(b, c) \in V \times Q$

$$m(b,\psi) = \int_{I_\xi} f\psi\,dx, \qquad \forall\psi \in Q,$$

$$-a(b,\varphi) + m_*(\varphi, c) = \int_{I_\xi} g\varphi\,dx, \qquad \forall\varphi \in V.$$

Saddle Point Theory (Brezzi, 1974), (Bernardi, Canuto, Maday, 1988) .

Well Posedness of the Steady Problem

Define the Kernel of $m_{(*)}(\cdot,\cdot)$

$$\mathcal{N}\left(m_{(*)}\right) = \left\{\varphi \in V, \qquad m_{(*)}\left(\varphi,\psi\right) = 0 \quad \forall\psi \in Q\right\}.$$

The bilinear form $a(\cdot,\cdot)$ satisfies a two inf-sup conditions on $\mathcal{N}(m) \times \mathcal{N}(m_*)$,

$$\inf_{\psi\in\mathcal{N}(m)} \sup_{\varphi\in\mathcal{N}(m_*)} \frac{a(\varphi,\psi)}{\|\varphi\|_{H^1}\|\psi\|_{H^1}} \geq \eta, \qquad \inf_{\psi\in\mathcal{N}(m_*)} \sup_{\varphi\in\mathcal{N}(m)} \frac{a(\varphi,\psi)}{\|\varphi\|_{H^1}\|\psi\|_{H^1}} \geq \eta_*.$$

The bilinear form $m_{(*)}(\cdot,\cdot)$ satisfies the inf-sup condition in $V \times Q$,

$$\inf_{\psi\in Q} \sup_{\varphi\in V} \frac{m_{(*)}(\varphi,\psi)}{\|\varphi\|_{H^1}\|\psi\|_{H^1}} \geq 1.$$

Prop. **2** *The mixed problem has a unique solution* $(b,c) \in V \times Q$ *such that*

$$\|b\|_{H^1} + \|c\|_{H^1} \leq C(\|f\|_{L^2} + \|g\|_{L^2}).$$

The Resolvent of A

Prop. 3 *Let $(f,g) \in \boldsymbol{L}^2(I_\xi)$. Set*

$$(b_\lambda, c_\lambda) = \mathcal{R}(\lambda, A)(f,g) = (\lambda + A)^{-1}(f,g).$$

We have that

$$\|b_\lambda\|_{L^2} + \lambda \|c_\lambda\|_{L^2} \leq C(\|f\|_{L^2} + \|g\|_{L^2}),$$

The resolvent $\mathcal{R}(\lambda)$ is then a bounded in $\boldsymbol{L}^2(I_\xi)$

$$\|\mathcal{R}(\lambda)\|_{(\boldsymbol{L}^2(I) \to \boldsymbol{L}^2(I_\xi))} \leq C'$$

Rem. 1 *We have not*

$$\|\mathcal{R}(\lambda)\|_{(\boldsymbol{L}^2(I_\xi) \to \boldsymbol{L}^2(I_\xi))} \leq \frac{C}{\lambda}.$$

Hille-Yosida Theory fails (Expected!).

Uniqueness (I) : Pazy's Theorem

Théo. **4 (A. Pazy)** *If* $\mathcal{R}(\lambda)$ *exists for large real-numbers* $\lambda(\geq 0)$ *and*

$$\limsup_{\lambda \to +\infty} \frac{1}{\lambda} \log \|\mathcal{R}(\lambda)\|_{(\boldsymbol{L}^2(I_\xi) \to \boldsymbol{L}^2(I_\xi))} = 0,$$

then the ODE problem has at most one solution.

Rem. 2 *The norm of the resolvent may behave like*

$$\|\mathcal{R}(\lambda)\|_{(\boldsymbol{L}^2(I_\xi) \to \boldsymbol{L}^2(I_\xi))} = \exp(\lambda \epsilon(\lambda)).$$

A Uniqueness Result for the Sideways Problem

Théo. **5** *The Sideways Problem has at most one solution.*

PROOF We have that $\mathcal{R}(\lambda)$ is well defined for all $\lambda \geq 0$. The uniform resolvent estimate yields that

$$\limsup_{\lambda \to \infty} \frac{1}{\lambda} \log \|\mathcal{R}(\lambda)\|_{(\boldsymbol{L}^2(I_\xi) \to \boldsymbol{L}^2(I_\xi))} = 0.$$

Applying Pazy's Theorem completes the proof.

IDENTIFIABLITY FOR BOD-DO

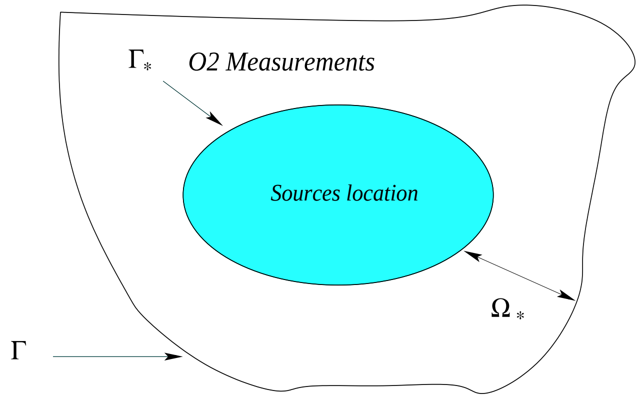
THÉO. **6** *Assume that* $B\left[F_1, G_1\right] = B\left[F_2, G_2\right]$, *then*

$$(F_1, G_1) = (F_2, G_2),$$

or equivalently

$$(r_1(\cdot), f_1(\cdot)) = (r_2(\cdot), f_2(\cdot)), \qquad \textit{in } (0, T)$$
$$(s_1(\cdot), g_1(\cdot)) = (s_2(\cdot), g_2(\cdot)), \qquad \textit{in } (0, T).$$

mD Extension



$$\partial_t b - \text{ div } (D\nabla b) + Rb = \sum_{m\le m_*} f_m(t)\delta_{\boldsymbol{x}-\boldsymbol{r}_m} \qquad \text{in } \Omega\times(0,T)$$

$$\partial_t c - \text{ div } (D\nabla c) + R_* c - Rb = \sum_{n\le n_*} g_n(t)\delta_{\boldsymbol{x}-\boldsymbol{s}_n} \qquad \text{in } \Omega\times(0,T)$$

$$(D\partial_{\boldsymbol{n}} b, D\partial_{\boldsymbol{n}} c) = (0,0) \qquad \text{in } \Gamma\times(0,T)$$

$$(b(\cdot,0), c(\cdot,0)) = (0,0) \qquad \text{in } \Omega.$$

Cauchy Observations on c along Γ_*

$$(c, D\partial_{\boldsymbol{n}} c) = (\alpha,\beta) \qquad \text{in } \Gamma_*\times(0,T).$$

Sideways problem mD

$$\begin{aligned} \partial_t b - \text{ div } (D\nabla b) + Rb = 0 \qquad & \text{in } \Omega_* \times (0,T) \\ \partial_t c - \text{ div } (D\nabla c) + R_* c - Rb = 0 \qquad & \text{in } \Omega_* \times (0,T) \\ (D\partial_{\boldsymbol{n}} b, D\partial_{\boldsymbol{n}} c) = (0,0) \qquad & \text{in } \Gamma \times (0,T) \\ (c, D\partial_{\boldsymbol{n}} c) = (\alpha, \beta) \qquad & \text{in } \Gamma_* \times (0,T) \\ (b(\cdot,0), c(\cdot,0)) = (0,0) \qquad & \text{in } \Omega. \end{aligned}$$

The framework $(b,c) \in H^1 \times H^1$ fails!

(Remember the Stream function-Vorticity Formulation of the Stokes problem!).

Pick-up the non-standard framework proposed by Y. Maday [BGM, 1992]

+ Further technical Work.

Possible Applications of Identifiability

1. Discriminate responsabilities in the pollution.

$$\text{Moving Source } (\Longrightarrow) \text{ Ship responsability}$$
$$\text{Fixed Source } (\Longrightarrow) \text{ Factory responsability}$$

2. Provide Authorities with Valuable information on the location and the release rate of the Organic pollution ($\Longrightarrow$) (Hope!) they'll make Right Decisions.

Perspectives

1. Extension to two- and three-dimensions ($\Longrightarrow$) Estuaries and Lakes (Under Investigation, Finished!).

2. Regularization algorithms and Realistic computational simulations (Andrle PhD. Thesis, December 2011)!

3. Non-linear Model for poorly Oxygenized (or Highly Polluted) Waters. The reaction coefficients (R, R_*) depend on the Dissolved Oxygen (c). How? (Logistic (?) or Michaelis-Menten ($-$) term, or something else?)