Stability of Stationary Solution for the Lugiato-Lefever Equation

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1 Introduction and main theorem

We consider the stability of stationary solution for the nonlinear Schrödinger equation with damping and spatially homogeneous forcing terms:

$$\frac{\partial}{\partial t}A = -(1+i\theta)A + ib^2\frac{\partial^2}{\partial x^2}A + i|A|^2A + F, \qquad t>0, \quad x\in \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}. \tag{1}$$

Here, A denotes the slowly varying envelope of electric field, $\theta > 0$ denotes the detuning parameter, and $b > 0$ denotes the diffraction parameter. Let $F > 0$ be the spatially homogeneous input field. In [12], Lugiato and Lefever present equation (1) to model the so-called cavity soliton in the ring or the Fabry-Pérot cavity oscillator (see also [1, 2, 3, 9]). The existence and the stability of spatially nonhomogeneous stationary solutions for (1) have been studied by the authors [13]. In this paper, we prove the Strichartz estimate for the linear Schrödinger equation with potential and external forcing and investigate the stability of certain stationary solutions for (1) given in [13] under the L^2 perturbation. We decompose the solution $A(t)$ of (1) into effective dynamical components, following Buslaev and Perel'man [6] (see also Soffer and Weinstein [15]) and show the a priori estimates of those effective dynamical components, which ensure the asymptotic stability of stationary solution for (1). There are many papers on the asymptotic stability of a family of equilibria in the setting of nonlinear parabolic equation (see, e.g., Exercise 6 in section 5.1 on page 108 of Henry [11]). In the case of nonlinear parabolic equation, the smoothing property and the fractional power

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of infinitesimal generator for the holomorphic semigroup are useful. But, in the case of nonlinear Schrödinger equation, the Strichartz estimate plays a crucial role, which enables us to treat the rougher perturbation than the H^1 perturbation in the previous papers (see, e.g., Ghidaglia [10], X.-M. Wang [16], and Miyaji, Ohnishi and Tsutsumi [13]).

Theorem 1.1 *Assume D is a stationary solution of (1) such that the spectrum of the linearized operator around D for the stationary equation associated with (1) lies in $\{z \in \mathbf{C};\ \mathrm{Re} z \leq -\alpha\} \cup \{0\}$ for some $\alpha > 0$ and the eigenspace corresponding to zero eigenvalue is a one dimensional subspace in L^2 spanned by $\partial_x D$. Let $A_0 \in L^2$ and let $\varepsilon > 0$. For $c \in \mathbf{R}$, we put*

$$D_c(x) = D(x + c).$$

Then, there exist $\delta > 0$ and $0 \leq c_0 < 2\pi$ such that if initial data A_0 satisfies

$$\|A_0 - D\|_{L^2} < \delta,$$

we have

$$\begin{aligned} \sup_{t \geq 0} \Big[\inf_{0 \leq c < 2\pi} \|A(t) - D_c\|_{L^2} \Big] &< \varepsilon, \\ \|A(t) - D_{c_0}\|_{L^2} &\to 0 \quad (t \to \infty), \end{aligned}$$

where A is a solution of (1) with $A(0) = A_0$.

Remark 1.1 (i) It is known that there exist spatially nonhomogeneous stationary solution of (1) satisfying all the assumptions in Theorem 1.1 (see Theorems 2.1 and 2.2 in section 2).

(ii) If we consider equation (1) and the initial perturbation within the framework of H^1, it would be slightly easier to prove Theorem 1.1. Because $H^1 \subset L^\infty$ in the one dimensional case and the a priori estimates in H^1 needed for the proof of Theorem 1.1 follow from the energy estimates only. We note that Theorem 1.1 can cover the L^2 perturbation, which belongs to a bigger class than H^1.

The present paper is organized as follows. In section 2, we summarize the existence of stationary solution and the spectral analysis of linearized operator, which are mainly proved by the authors [13]. In section 3, we show the Strichartz estimate of the linear Schrödinger equation with complex potential and shift terms, which plays a crucial role in the proof of Theorem 1.1. Finally, in section 4, we give a sketch of the proof of Theorem 1.1.

2 Existence of Stationary Solution and Linear Stability

In this section, we summarize the results on the existence of stationary solution and the spectrum of the linearized operator, which have been obtained by the authors [13]. Let the spatially homogeneous stationary solutions A_S be defined as follows.

$$A_S = \frac{F}{1+i(\theta-\alpha)}, \qquad \alpha = |A_S|^2, \tag{2}$$

where α is uniquely determined by

$$F^2 = \alpha\{1+(\alpha-\theta)^2\}, \quad \theta < \sqrt{3}. \tag{3}$$

We now consider the stationary Lugiato-Lefever Equation and change the unknown function A to B as follows. Set $A = A_S(1+B)$. Then, B satisfies

$$\begin{aligned} 0 &= -(1+i\theta)B + ib^2\partial_x^2 B \\ &+ i\alpha(2B + \bar{B} + B^2 + 2|B|^2 + |B|^2 B), \quad x \in \mathbf{T}. \end{aligned} \tag{4}$$

Remark 2.1 Instead of F, we regard α as a bifurcation parameter. In that case, A_S and F are determined for given α through (2), (3).

We first state the existence theorem of spatially nonhomogeneous stationary solutions which bifurcate from the spatially homogeneous stationary solution A_S.

Theorem 2.1 *There exist $b > 0$, $\sqrt{3} > \theta > 0$, $\eta > 0$, $n \in \mathbf{N}$, $B_0 \in \mathbf{C}$ such that equation (4) has a family of solutions $\{(\alpha(s), B(s)) \in \mathbf{R} \times H^2;\ -\eta < s < \eta\}$ satisfying*

$$\begin{aligned} &B(s) = sB_0\cos(2\pi nx) + r(s), \quad s \in (-\eta, \eta), \\ &\|r(s)\|_{H^2} = o(s) \quad (s \to 0), \\ &r(s) \perp \operatorname{span}\{\cos(2\pi nx)\}, \\ &r(\cdot, x) = r(\cdot, -x), \quad x \in \mathbf{T}, \\ &\alpha(s) = 1 - \frac{30\theta - 41}{9(2-\theta)^2}s^2 + o(s^2) \quad (s \to 0). \end{aligned}$$

For Theorem 2.1, see Theorem 3.1 on pages 2071 and 2072 in [13].

We next consider the spectrum of linearized operator and the linear stability of stationary solutions given by Theorem 2.1. We set

$$A_S\big(1+B(s)\big) = w(s) + iz(s), \quad s \in (-\eta, \eta),$$

where w and z are real-valued functions.

Let L be the linearized operator around (w, z) for (1):

$$L = \begin{pmatrix} -1 - 2wz & -\Delta_{b,\theta} - 2V_+ + V_- \\ \Delta_{b,\theta} + 2V_+ + V_- & -1 + 2wz \end{pmatrix}, \tag{5}$$

where

$$\begin{aligned} \Delta_{b,\theta} &= b^2\partial_x^2 - \theta, \\ V_\pm &= w^2 \pm z^2. \end{aligned}$$

Theorem 2.2 *Assume* $0 < \theta < 41/30$. *Then,* $\exists\eta' > 0$, $\exists\gamma \in C((-\eta', \eta'); \mathbf{R})$ *such that* $\gamma(s) > 0$ $(0 < |s| < \eta')$, $\gamma(0) = 0$ *and*

$$\sigma(L) \subset \{z \in \mathbf{C};\ \mathrm{Re}\ z\ \leq -\gamma\} \cup \{0\} \qquad (0 < |s| < \eta').$$

In addition, when $\eta' > |s| > 0$, *the eigenspace belonging to the zero eigenvalue of* L *consists of the derivative of stationary solution* (w, z), *which is an odd function.*

For the proof of Theorem 2.2, see the proof of Theorem 3.1 in [13].

Remark 2.2 We can completely analyze the spectrum of the linearized operator near the bifurcation point. Because at the bifurcation point, the linearized operator around the homogeneous stationary solution is reduced to the Sturm-Liouville operator with constant coefficients.

Remark 2.3 Theorem 2.2 implies the linear stability of stationary solutions given by Theorem 2.1 for $\theta < 41/30$ within the framework of even functions (in fact, the nonlinear stability is proved in [13]). In the ODE case, it is well known that when all eigenvalues of the linearized operator have negative real part except for zero eigenvalue and every orbit starting from a neighborhood of stationary solution has a non-empty ω-limit set, the stationary solution is nonlinearly stable (see, e.g., Proposition 1.1 on pages 2 and 3 in [4]). In the nonlinear PDE case, the situations are more complicated, but suitable a priori estimates often ensure the similar conclusion to the ODE case.

Remark 2.4 It can be proved that when $\theta > 41/30$, the stationary solution given by Theorem 2.1 is nonlinearly unstable (see [13]).

3 Strichartz Estimate

In this section, we prove the global Strichartz estimate in time for the Schrödinger equation on one-dimensional torus with linear time-independent potential and external forcing.

$$i\partial_t u = -\partial_x^2 u + Vu + f, \quad t > 0, \ x \in \mathbf{T} \tag{6}$$
$$u(0,x) = u_0(x). \tag{7}$$

We assume the following two hypotheses:
(A1) V is a complex-valued function in $L^\infty(\mathbf{T})$,
(A2) $\exists \gamma > 0$ such that

$$\sigma\big(i(\partial_x^2 - V)\big) \subset \{z \in \mathbf{C} \mid \operatorname{Re} z \le -\gamma\}.$$

Remark 3.1 (i) We note that (A2) is equivalent to the following:

$$\exists \gamma > 0;\ \operatorname{Re}\big(-i(\partial_x^2 - V)v, v\big) \ge \gamma \|v\|_{L^2}^2, \quad v \in H^1.$$

This implies that potential V has a damping effect, which yields the global Strichartz estimate in time for (6).

(ii) The linearized equation of (1) has the form : $i\partial_t u = -\partial_x^2 u + V_1 u + V_2 \bar{u}$. This includes the conjugate linear term $V_2\bar{u}$. But the Strichartz estimate of this equation can be proved in the same way, if we replace (A2) by

$$\exists \gamma > 0;\ \operatorname{Re}\big(-i(\partial_x^2 - V_1)v + iV_2\bar{v}, v\big) \ge \gamma \|v\|_{L^2}^2, \quad v \in H^1.$$

We set

$$U_0(t) = e^{it\partial_x^2}, \quad U(t) = e^{it(\partial_x^2 - V)}, \quad \mathbf{R}_+ = (0, \infty).$$

We first give two lemmas, which are useful for the proof of the Strichartz estimate of (6)-(7). We begin with the Christ-Kiselev lemma (see [8]).

Lemma 3.1 *Let X, Y be Banach spaces and let $T > 0$. Assume $K(t,s)$ is continuous from $[0,T] \times [0,T]$ to $B(X,Y)$ and that $1 \le p < q \le \infty$. We put*

$$Sf(t) = \int_0^T K(t,s) f(s)\, ds,$$
$$\tilde{S}f(t) = \int_0^t K(t,s) f(s)\, ds.$$

$$\|Sf\|_{L^q\left((0,T);Y\right)} \leq C\|f\|_{L^p\left((0,T);X\right)}$$

Then, we have

$$\|\tilde{S}f\|_{L^q\left((0,T);Y\right)} \leq \tilde{C}\|f\|_{L^p\left((0,T);X\right)}$$

For Lemma 3.1, see Lemma 3.1 on page 2179 in [14].

The following lemma is concerned with the Strichartz estimate of (6)-(7) without potential (see Propositions 2.1 on page 112 and Proposition 2.33 on page 116 in [5]), which is traced back to Zygmund [17].

Lemma 3.2

$$\|U_0(\cdot)u_0\|_{L^4\left((0,2\pi)\times\mathbf{T}\right)} \leq C\|u_0\|_{L^2(\mathbf{T})},$$
$$\left\|\int_{t_0}^t U_0(t-s)f(s)\ ds\right\|_{L^4((t_0,t_0+2\pi)\times\mathbf{T})} \leq C\|f\|_{L^{4/3}((t_0,t_0+2\pi)\times\mathbf{T})},$$

where t_0 is an arbitrary real number.

For the convenience of the reader, we present the proof of Lemma 3.2.

Proof. We show the first inequality. By the Fourier expansion, we have

$$\begin{aligned} u_0 &= \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}\hat{u}_0(m)e^{imx}, \\ \hat{u}_0(m) &= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} u_0(x)e^{-imx}\ dx. \end{aligned}$$

Then, since we have

$$U_0(t)u_0 = \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}\hat{u}_0(m)e^{-i(m^2t-mx)},$$

a simple computation yields

$$\begin{aligned} &\int_{\mathbf{T}}\int_0^{2\pi}\left|U_0(t)u_0\right|^4\ dtdx = \frac{1}{(2\pi)^2}\times \\ &\int_{\mathbf{T}}\int_0^{2\pi}\sum_{\substack{m_1,m_2,\\ m_3,m_4}}\hat{u}_0(m_1)\hat{u}_0(m_2)\overline{\hat{u}_0(m_3)\hat{u}_0(m_4)} \\ &\times e^{-i(m_1^2+m_2^2-m_3^2-m_4^2)t}e^{i(m_1+m_2-m_3-m_4)x}\ dtdx. \end{aligned} \tag{8}$$

On the right hand side of (8), the integrals remain if

$$m_1^2+m_2^2-m_3^2-m_4^2=0, \quad m_1+m_2-m_3-m_4=0$$

$$\Longleftrightarrow \left\{ \begin{array}{l} m_1=m_3, \\ m_2=m_4, \end{array} \right. \text{ or } \left\{ \begin{array}{l} m_1=m_4, \\ m_2=m_3. \end{array} \right.$$

Thus, the right hand side of (8) is equale to:

$$\begin{aligned} & 2\Big(\sum_{m_1} \big|\hat{u}_0(m_1)\big|^2 \ \Big) \ \Big(\sum_{m_2} \big|\hat{u}_0(m_2)\big|^2\Big) \\ & \qquad = \ 2\|u_0\|_{L^2(\mathbf{T})}^4. \end{aligned}$$

We next show the second inequality. We put

$$G=\int_{t_0}^{t_0+2\pi} U_0(t-s)f(s)\ ds.$$

The first inequality proved above yields

$$\|G\|_{L^4((t_0,t_0+2\pi)\times\mathbf{T})} \le C\bigg\| \int_{t_0}^{t_0+2\pi} U_0(-s)f(s)\ ds\bigg\|_{L^2(\mathbf{T})}. \tag{9}$$

On the other hand,

$$\begin{aligned} & \bigg\| \int_{t_0}^{t_0+2\pi} U_0(-s)f(s)\ ds\bigg\|_{L^2(\mathbf{T})}^2 \\ & = \Big\langle f(s), \int_{t_0}^{t_0+2\pi} U_0(s-s')f(s')\ ds' \Big\rangle \\ & \le C\|f\|_{L^{4/3}((t_0,t_0+2\pi)\times\mathbf{T})} \bigg\| \int_{t_0}^{t_0+2\pi} U_0(s-s')\ f(s')ds' \bigg\|_{L^4((t_0,t_0+2\pi)\times\mathbf{T})}, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product of $L^2\big((t_0,t_0+2\pi)\times\mathbf{T}\big)$. Combining this inequality and (9), we have

$$\big\|G\big\|_{L^4((t_0,t_0+2\pi)\times\mathbf{T})} \le C\|f\|_{L^{4/3}((t_0,t_0+2\pi)\times\mathbf{T})}.$$

This and Lemma 3.1 implies the second inequality. $\square$

We first consider the Strichartz estimate for (6)-(7) without external forcing term.

Theorem 3.3 *Assume (A1) and (A2). Let* $0 < \gamma' < \gamma$*. Then, for* $s \geq 0$*, we have*

$$\|e^{\gamma' t}U(t)u_0\|_{L^2(\mathbf{R}_+;L^2(\mathbf{T}))} \leq C\|u_0\|_{L^2(\mathbf{T})},$$
$$\|e^{\gamma' t}U(t)u_0\|_{L^4(\mathbf{R}_+\times\mathbf{T})} \leq C\|u_0\|_{L^2(\mathbf{T})}.$$

Proof of Theorem 3.3 For the sake of simplicity, we set $s = 0$. The first inequality follows immediately from (A2) and the standard L^2 inequality. For the proof of the second inequality, we consider the Cauchy problem of (6) with initial data prescribed at $t = t_0$, where $t_0 \geq 0$. By Duhamel's principle,

$$\begin{aligned} u(t) &= U_0(t-t_0)u(t_0) - iF(t), \quad t \geq t_0, \\ F(t) &= \int_{t_0}^{t} U_0(t-s)Vu(s)ds. \end{aligned}$$

If we can prove

$$\begin{aligned} &\|F\|_{L^4\left((t_0,t_0+2\pi)\times\mathbf{T}\right)} \\ &\leq C\|u\|_{L^\infty\left((t_0,t_0+2\pi);L^2(\mathbf{T})\right)}, \end{aligned} \tag{10}$$

then, for any $n \in \mathbf{N}\cup\{0\}$, we have by Lemma 3.2

$$\begin{aligned} \|u\|_{L^4\left((2\pi n,2\pi(n+1))\times\mathbf{T}\right)} &\leq C\|u(2\pi n)\|_{L^2(\mathbf{T})} \\ &+C\|u\|_{L^\infty\left((2\pi n,2\pi(n+1));L^2(\mathbf{T})\right)}. \end{aligned} \tag{11}$$

On the other hand, for $0 < \gamma' < \gamma$, (A2) yields

$$\begin{aligned} &\|U(t)u_0\|_{L^2(\mathbf{T})} \\ &\leq C\exp\Big(-\frac{\gamma+\gamma'}{2}t\Big)\|u_0\|_{L^2(\mathbf{T})}, \quad t \geq 0. \end{aligned}$$

We use this inequality to bound the two terms on the right hand side of (11) by

$$C\exp\Big(-\pi(\gamma+\gamma')n\Big)\|u_0\|_{L^2(\mathbf{T})}.$$

Accordingly, we conclude that for $0 < \gamma' < \gamma$,

$$\begin{aligned} &\|e^{\gamma' t}u\|_{L^4(\mathbf{R}_+\times\mathbf{T})} \\ &\leq \sum_{n=0}^{\infty} e^{2\pi\gamma' n}\|u\|_{L^4\left((2\pi n,2\pi(n+1))\times\mathbf{T}\right)} \\ &\leq C\sum_{n=0}^{\infty} e^{-\pi(\gamma-\gamma')n}\|u_0\|_{L^2(\mathbf{T})} \\ &\leq C\|u_0\|_{L^2(\mathbf{T})}, \end{aligned}$$

which implies Theorem 3.3.

It remains only to prove estimate (10). We easily see by Lemma 3.2 that

$$\begin{aligned}
&\left\| \int_{t_0}^{t_0+2\pi} U_0(t-s)f(s)\; ds \right\|_{L^4\left((t_0,t_0+2\pi)\times\mathbf{T}\right)} \\
&= \left\| U_0(t)\int_{t_0}^{t_0+2\pi} U_0(-s)f(s)\; ds \right\|_{L^4\left((t_0,t_0+2\pi)\times\mathbf{T}\right)} \\
&\le C\left\| \int_{t_0}^{t_0+2\pi} U_0(-s)f(s)\; ds \right\|_{L^2(\mathbf{T})} \\
&\le C\|f\|_{L^1\left((t_0,t_0+2\pi);L^2(\mathbf{T})\right)}.
\end{aligned}$$

Lemma 3.1 ensures that the integral operator

$$\int_{t_0}^{t} U_0(t-s)f(s)\; ds$$

has the same estimate as above. Therefore, we obtain

$$\begin{aligned}
&\|F\|_{L^4\left((t_0,t_0+2\pi)\times\mathbf{T}\right)} \\
&\le C\|Vu\|_{L^1\left((t_0,t_0+2\pi);L^2(\mathbf{T})\right)} \\
&\le C\|u\|_{L^\infty\left((t_0,t_0+2\pi);L^2(\mathbf{T})\right)},
\end{aligned}$$

which yields inequality (10). $\square$

We next consider the Strichartz estimate for (6)-(7) with $u_0 = 0$.

Theorem 3.4 *Assume (A1) and (A2). Let u be the solution of (6)-(7) with $u_0 = 0$. Then, for any γ' with $0 < \gamma' < \gamma$, we have*

$$\begin{aligned}
\|e^{\gamma' t}u\|_{L^\infty(\mathbf{R}_+;L^2(\mathbf{T}))} &\le C\|e^{\gamma t}f\|_{L^{4/3}(\mathbf{R}_+\times\mathbf{T})}, \\
\|e^{\gamma' t}u\|_{L^4(\mathbf{R}_+\times\mathbf{T})} &\le C\|e^{\gamma t}f\|_{L^{4/3}(\mathbf{R}_+\times\mathbf{T})}.
\end{aligned}$$

Remark 3.2 It seems to be likely that $e^{\gamma t}$ can be replaced by $e^{\gamma' t}$ on the right hand sides of the two inequalities in Theorem 3.4.

Proof of Theorem 3.4 We begin with the proof of the second inequality. We follow the same strategy as in the case of the homogeneous Schrödinger equation. We take the L^2 scalar product of (6) and $e^{(\gamma'+\gamma)t}u$ and integrate the resulting equation in t to have by (A2)

$$\begin{aligned}
\left\|e^{(\gamma'+\gamma)t/2}u(t)\right\|^2_{L^2(\mathbf{T})} &\le C\|e^{\gamma\cdot}f\|_{L^{4/3}((0,t)\times\mathbf{T})} \\
&\quad\times\|e^{\gamma'\cdot}u\|_{L^4((0,t)\times\mathbf{T})}, \quad t>0.
\end{aligned} \tag{12}$$

We note that Duhamel's principle yields

$$
\begin{aligned}
u(t) &= U(t-t_0)u(t_0) + G(t), \\
G(t) &= -i\int_{t_0}^{t} U(t-s)f(s)\ ds
\end{aligned}
\tag{13}
$$

for any $t_0 \geq 0$. If we can prove

$$
\begin{aligned}
&\|G\|_{L^4((t_0,t_0+2\pi)\times\mathbf{T})} \\
&\leq C\|f\|_{L^{4/3}((t_0,t_0+2\pi)\times\mathbf{T})},
\end{aligned}
\tag{14}
$$

then, for any $n \in \mathbf{N}\cup\{0\}$, we have by Theorem 3.3

$$
\begin{aligned}
\|u\|_{L^4\big((2\pi n,2\pi(n+1))\times\mathbf{T}\big)} &\leq C\|u(2\pi n)\|_{L^2(\mathbf{T})} \\
&\quad + C\|f\|_{L^{4/3}\big((2\pi n,2\pi(n+1))\times\mathbf{T}\big)}.
\end{aligned}
\tag{15}
$$

Inequalities (12) and (15) yield

$$
\begin{aligned}
&\|u\|_{L^4\big((2\pi n,2\pi(n+1))\times\mathbf{T}\big)} \\
&\leq Ce^{-\pi(\gamma'+\gamma)n}\|e^{\gamma t}f\|^{1/2}_{L^{4/3}((0,2\pi n)\times\mathbf{T})}\|e^{\gamma' t}u\|^{1/2}_{L^4((0,2\pi n)\times\mathbf{T})} \\
&\quad + C\|f\|_{L^{4/3}\big((2\pi n,2\pi(n+1))\times\mathbf{T}\big)}.
\end{aligned}
$$

Accordingly, we conclude that for $0 < \gamma' < \gamma$,

$$
\begin{aligned}
\|e^{\gamma' t}u\|_{L^4(\mathbf{R}_+\times\mathbf{T})} &\leq \textstyle\sum_{n=0}^{\infty} e^{2\pi\gamma' n}\|u\|_{L^4\big((2\pi n,2\pi(n+1))\times\mathbf{T}\big)} \\
&\leq C\textstyle\sum_{n=0}^{\infty} e^{-\pi(\gamma-\gamma')n}\Big[\|e^{\gamma t}f\|^{1/2}_{L^{4/3}((0,2\pi n)\times\mathbf{T})}\|e^{\gamma' t}u\|^{1/2}_{L^4((0,2\pi n)\times\mathbf{T})} \\
&\quad + \|e^{(\gamma'+\gamma)t/2}f\|_{L^{4/3}\big((2\pi n,2\pi(n+1))\times\mathbf{T}\big)}\Big] \\
&\leq C\|e^{\gamma t}f\|_{L^{4/3}(\mathbf{R}_+\times\mathbf{T})} + \tfrac{1}{2}\|e^{\gamma' t}u\|_{L^4(\mathbf{R}_+\times\mathbf{T})}.
\end{aligned}
$$

This shows Theorem 3.4.

It remains only to prove (14). We first note that G satisfies

$$
\begin{aligned}
& i\partial_t G = -\partial_x^2 G + VG + f, \quad t > t_0,\ x \in \mathbf{T} \\
& G(t_0,x) = 0, \quad x \in \mathbf{T}.
\end{aligned}
\tag{16}
$$

Therefore, Duhamel's principle yields

$$
\begin{aligned}
& G(t) = -iG_1(t) - iG_2(t), \quad t \geq t_0, \\
& G_1(t) = \int_{t_0}^{t} U_0(t-s)VG(s)\ ds, \quad G_2(t) = \int_{t_0}^{t} U_0(t-s)f(s)\ ds.
\end{aligned}
$$

Furthermore, by (16) and (A2), we have

$$\|G(t)\|_{L^2(\mathbf{T})}^2 \leq C\|f\|_{L^{4/3}((t_0,t)\times\mathbf{T})}\|G\|_{L^4((t_0,t)\times\mathbf{T})}, \quad t > t_0. \tag{17}$$

On the other hand, by Lemma 3.2, we have

$$\begin{aligned}\|G\|_{L^4((t_0,t_0+2\pi)\times\mathbf{T})} \leq C\big[&\|G\|_{L^\infty\left((t_0,t_0+2\pi);L^2(\mathbf{T})\right)} \\ &+\|f\|_{L^{4/3}\left((t_0,t_0+2\pi);L^2(\mathbf{T})\right)}\big].\end{aligned}$$

By combining this inequality and (17), we obtain (14).

The first inequality follows from (17) and the second inequality proved above. □

Remark 3.3 We note that Theorems 3.3 and 3.4 also hold for the linear Schrödinger equation with shift term:

$$\begin{aligned}&i\partial_t u = -\partial_x^2 u - i\dot{c}(t)\partial_x u + Vu + f, \quad t > 0,\ x \in \mathbf{T} \\ &u(0,x) = u_0(x),\end{aligned} \tag{18}$$

where $c(t)$ is a continuously differentiable real-valued function. In fact, if we put

$$U_1(t,s) = e^{i[(t-s)\partial_x^2 - (c(t)-c(s))\partial_x]} \qquad (t \geq s \geq 0),$$

then we have Lemma 3.2 without any change for $U_1(t,0)$. Because the shift term $-i\dot{c}(t)\partial_x u$ only leads to the spatial translation of solution. The norms appearing in Lemma 3.2 are invariant under the spatial translation and so the spatial translation caused by the shift term has no influence on the Strichartz estimate for the Schrödinger equation :

$$i\partial_t u = -\partial_x^2 u - i\dot{c}(t)\partial_x u + f, \qquad t > 0, \quad x \in \mathbf{T}.$$

Furthermore, the shift term $-i\dot{c}(t)\partial_x u$ has no influence on the exponential decay in $L^2(\mathbf{T})$ of solution, either. This enables us to prove Theorems 3.3 and 3.4.

4 Proof of Theorem 1.1

In this section, we describe the proof of Theorem 1.1. Let L be the linearized operator around $D(x)$ defined as in (5). We denote the subspace $\mathrm{span}\{\partial_x D(x))\}$ and its complementary subspace in $L^2(\mathbf{T})$ by L_0^2 and L_-^2 , respectively. We choose L^2_- such that L^2_- is an invariant subspace of L. Let $\widetilde{\partial_x D}$ denote the normalization in L^2 of $\partial_x D$. Let Q and P be the projections from $L^2(\mathbf{T})$ to L^2_- and from $L^2(\mathbf{T})$ to L^2_0, respectively. The projections Q and P are explicitly expressed as follows.

$$Qf = f - (f, E)\widetilde{\partial_x D}, \qquad Pf = (f, E)\widetilde{\partial_x D},$$

where E is the normalized eigenfunction belonging to the zero eigenvalue of the adjoint operator of L. We choose $2\pi > c_1 \geq 0$ such that

$$\|A_0(\cdot + c_1) - D(\cdot)\|_{L^2} = \min_{2\pi > c \geq 0} \|A_0(\cdot + c) - D(\cdot)\|_{L^2}.$$

Without loss of generality, we may change the initial data $A_0(x)$ to $A_0(x+c_1)$, since equation (1) is invariant under the spatial translation. We denote $A_0(x + c_1)$ by $A_0(x)$ again and we have

$$\mathrm{Re}(A_0, \partial_x D) = 0. \tag{19}$$

It is expedient to work with the real and the imaginary parts of complex-valued function and to regard the space $L^2(\mathbf{T})$ as a real Hilbert space with scalar product $\mathrm{Re}(\cdot, \cdot)$. In that case, (19) implies that A_0 is orthogonal to the subspace spanned by $\partial_x D$. Let w and z denote the real and the imaginary parts of the stationary solution $D(x)$, respectively. We now decompose the solution A into effective dynamical components, following Buslaev and Perel'man [6] and Soffer and Weinstein [15] (for the case of the nonlinear parabolic equation, see Exercise 6 in section 5.1 on page 108 of Henry [11]). For the solution A of (1), we make an ansatz as follows.

$$A(t, x) = D(x + c(t)) + u(t, x + c(t)) + iv(t, x + c(t)), \qquad u, v \in L^2_-, \tag{20}$$

where $u(t, x)$ and $v(t, x)$ are real-valued functions, and $c(t)$ is continuously differentiable function with $c(0) = 0$ to be determined later. If we insert the ansatz (20) into (1) and remove the spatial translation $c(t)$ by the change of variables, then we rewrite equation (1) as in the follwoing form.

$$\partial_t T - LT + \dot{c}(t)Q\partial_x T = Q\mathcal{F}(x, T), \quad t > 0,\ x \in \mathbf{T}, \tag{21}$$

$$\dot{c}(t) = \frac{(\mathcal{F}(x, T), E)}{a - (T, \partial_x E)}, \quad t > 0, \qquad a = (\partial_x D, E), \tag{22}$$

$$T(0, x) = T_0(x) \quad (x \in \mathbf{T}), \qquad c(0) = 0, \tag{23}$$

where

$$T(t,x)=\begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix}\in L^2_-,\quad T_0(x)=\begin{pmatrix}\mathrm{Re}\big(A_0(x)-D(x)\big)\\ \mathrm{Im}\big(A_0(x)-D(x)\big)\end{pmatrix},$$
$$\mathcal{F}\in C^2(\mathbf{T}\times\mathbf{R}^2;\mathbf{R}^4),$$
$$|\mathcal{F}(x,T_1)-\mathcal{F}(x,T_2)|\le C\big(|T_1|+|T_2|+|T_1|^2+|T_2|^2\big)|T_1-T_2|$$
$$(x\in\mathbf{T},\quad T_1,T_2\in\mathbf{R}^2),$$
$$|\partial_T\mathcal{F}(x,T_1)-\partial_T\mathcal{F}(x,T_2)|\le C\big(1+|T_1|+|T_2|\big)|T_1-T_2|$$
$$(x\in\mathbf{T},\quad T_1,T_2\in\mathbf{R}^2),$$

Here, $|T|=\sqrt{u^2+v^2}$ for $T=(u,v)\in\mathbf{R}^2$. Let $U_c(t,s)$ $(t\ge s\ge 0)$ denote the evolution operator associated with the infinitesimal generator $L-\dot c(t)\partial_x$ for each $c(t)$. We note that the Strichartz estimates such as Theorems 3.3 and 3.4 are applicable to the first and the second components of the solution T of (21) (see Remark 3.1 (ii) and Remark 3.3). In fact, we have

$$\dot c(t)Q\partial_x T=\dot c(t)\partial_x T-\dot c(t)P\partial_x T,$$

and the term $\dot c(t)P\partial_x T$ can be regarded as a small regular perturbation as long as $\dot c(t)$ is small. Furthermore, we note that the unique global existence of solution $A(t)$ on the time interval $(-\eta,\infty)$ for the Cauchy problem of (1) with initial data in $L^2(\mathbf{T})$ follows from the result by Bourgain [5], where η is a small positive constant depending only on the initial data. For a given solution $A(t)$, by (20), we set

$$T(t,x)=A(t,x-c(t))-D(x).$$

We insert T into (22) to have by (23)

$$c(t)=\int_0^t\frac{(\mathcal{F}(x,T),E)}{a-(T,\partial_x E)}\,ds,\quad t>-\eta. \tag{24}$$

For a short time, we have the solution $c(t)$ of (24) by the implicit function theorem as long as T is small. Because a direct computation yields

$$\begin{aligned}&\frac{\partial}{\partial c}\left(c-\int_0^t\frac{(\mathcal{F}(x,T),E)}{a-(T,\partial_x E)}\,ds\right)\\ &=1-\int_0^t\frac{\big((\partial_T\mathcal{F}(x,T)\partial_x A(s,x-c),E\big)}{a-(T,\partial_x E)}\,ds\\ &\quad+\int_0^t\frac{(\partial_x A(s,x-c),\partial_x E)(\mathcal{F}(x,T),E)}{\big(a-(T,\partial_x E)\big)^2}\,ds\end{aligned}$$

$$
\begin{aligned}
&= 1 + \int_0^t \frac{\big((\mathcal{F}(x,T), \partial_x E\big) + \big(\partial_x \mathcal{F}(x,T), E\big) - \big(\partial_x D, E\big)}{a - (T, \partial_x E)}\, ds \\
&\quad - \int_0^t \frac{(T, \partial_x^2 E)(\mathcal{F}(x,T), E) - \big(\partial_x D, \partial_x E\big)}{\big(a - (T, \partial_x E)\big)^2}\, ds.
\end{aligned}
$$

The right hand side of this formula does not vanish at $(t, c) = (0, 0)$. We note that the right hand side of this formula makes sense for $|t| < \tau$, provided that for some $\tau > 0$, T is in $L^\infty((-\tau, \tau); L^2(\mathbf{T})) \cap L^4((-\tau, \tau) \times \mathbf{T})$ and T is small in $L^\infty((-\tau, \tau); L^2(\mathbf{T}))$. These facts and the implicit function theorem imply the existence of $c(t)$ for a short time. From the above construction of the function $c(t)$, it automatically follows that $T(t)$ must satisfy (21). Because the L^2_0-component of T satisfies

$$
\frac{d}{dt}(PT) = 0, \quad t > 0, \qquad (PT)(0) = 0,
$$

which implies that $T(t) \in L^2_-$ for each $t > 0$. Therefore, if we have proved the a priori estimates of $(T(t), c(t))$, then we obtain Theorem 1.1.

By the L^2 estimate, Theorem 3.3 and Remark 3.3, for some $0 < \gamma' < \gamma$, we have

$$
\begin{aligned}
\big\| e^{\gamma' t}\big(U_c(t,0)T_0\big)\big\|_{L^\infty(\mathbf{R}_+; L^2)} &\le C\|T_0\|_{L^2}, \\
\big\| e^{\gamma' t}\big(U_c(t,0)T_0\big)\big\|_{L^4(\mathbf{R}_+ \times \mathbf{T})} &\le C\|T_0\|_{L^2}.
\end{aligned}
$$

We put $\delta = C\|T_0\|_{L^2}$. Later, we choose T_0 so small in $L^2(\mathbf{T})$ that δ is sufficiently small. We define the space X_δ by

$$
\begin{aligned}
\big\{(T, c) \in \big(L^\infty(\mathbf{R}_+; L^2) \cap L^4(\mathbf{R}_+ \times \mathbf{T})\big)^2 \times C_b([0, \infty)\big); \\
c(0) = 0, \quad \|(T, c)\|_{X_\delta} \le 2\delta\big\},
\end{aligned}
$$

where $C_b([0, \infty))$ denotes the space of all bounded continuous functions defined on $[0, \infty)$ and

$$
\|(T, c)\|_{X_\delta} = \max\big\{\|e^{\gamma' t} T\|_{L^\infty(\mathbf{R}_+; L^2)}, \|e^{\gamma' t} T\|_{L^4(\mathbf{R}_+ \times \mathbf{T})}, \|e^{\gamma' t} \dot{c}\|_{L^2((0,\infty); \mathbf{R})}\big\}.
$$

Let $(T, c) \in X_\delta$ be the solution of (21)-(23). We now prove the a priori estimates for (T, c), which ensure the global existence of (T, c) and

$$
\|T(t)\|_{L^2(\mathbf{T})} \to 0, \tag{25}
$$

$$
\exists c_0 \in \mathbf{R}; \quad c(t) \to c_0 \quad (t \to \infty). \tag{26}
$$

We first show that if $\delta > 0$ is sufficiently small, (T, c) is small in X_δ. We apply Theorems 3.3 and 3.4, together with Remark 3.3, to (21) and have by the Hölder inequality and the assumption $\gamma' < \gamma$

$$\begin{aligned}
&\|e^{\gamma' t}T\|_{L^\infty(\mathbf{R}_+;L^2)} \le \delta \\
&\qquad +C\big(\|e^{\gamma' t}T\|^2_{L^4(\mathbf{R}_+\times\mathbf{T})} + \|e^{\gamma' t}T\|^3_{L^4(\mathbf{R}_+\times\mathbf{T})}\big) \\
&\le \delta + C(\delta+\delta^2)\delta, \\
&\|e^{\gamma' t}T\|_{L^4(\mathbf{R}_+\times\mathbf{T})} \le \delta \\
&\qquad +C\big(\|e^{\gamma' t}T\|^2_{L^4(\mathbf{R}_+\times\mathbf{T})} + \|e^{\gamma' t}T\|^3_{L^4(\mathbf{R}_+\times\mathbf{T})}\big) \\
&\le \delta + C(\delta+\delta^2)\delta
\end{aligned}$$

for $T \in X_\delta$. We easily see that

$$\|e^{\gamma' t}\dot{c}\|_{C_b([0,\infty))} \le C\frac{(\delta+\delta^2)\delta}{a - C\delta}.$$

Here, if we choose $\delta > 0$ such that

$$C(\delta+\delta^2) \le 1, \quad C\frac{(\delta+\delta^2)}{a-C\delta} < 1, \quad a - C\delta > 0,$$

then we can conclude that

$$\|(T,c)\|_{X_\delta} \le 2\delta.$$

This implies (25) and (26), that is, the asymptotic stability of D.

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