OPTIMAL DESIGN IN
SMALL AMPLITUDE HOMOGENIZATION

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CONTENTS


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We consider two-phase optimal design problems:

\[ A_\chi(x) = (1 - \chi(x))A^0 + \chi(x)A^1, \quad \rho_\chi(x) = (1 - \chi(x))\rho^0 + \chi(x)\rho^1 \]

where \( \chi(x) = 0 \) or \( 1 \) is a characteristic function.
State equation \( \equiv \) wave equation (work with A. Kelly):

\[
\begin{cases}
\rho \chi \frac{\partial^2 u}{\partial t^2} - \text{div} (A \chi \nabla u) = f \quad \text{in } \Omega \times (0, T) \\
u = 0 \quad \text{on } \Gamma_D \times (0, T) \\
A \chi \nabla u \cdot n = 0 \quad \text{on } \Gamma_N \times (0, T) \\
u(x, 0) = u_{\text{init}}(x) \quad \text{in } \Omega \\
\frac{\partial u}{\partial t}(x, 0) = v_{\text{init}}(x) \quad \text{in } \Omega
\end{cases}
\]

\[
\inf_{\chi \in \mathcal{U}_{ad}} J(\chi) = \int_0^T \int_{\Omega} j(u, \nabla u) \, dx \, dt
\]

\[\mathcal{U}_{ad} = \left\{ \chi \in L^\infty(\Omega; \{0, 1\}), \text{ such that } \int_{\Omega} \chi(x) \, dx = \Theta |\Omega| \right\} .\]
Linear elastodynamics and multiple loads optimization are also possible.

Previous work with S. Gutierrez: state equation $\equiv$ steady-state elasticity.
Optimal design is ill-posed! Usually, no minimizer (counter-examples by F. Murat 72').

Minimizing sequences oscillate: the mixture of the two phases want to create a composite material.

Typical behavior for maximizing the vertical conductivity
The problem must be *relaxed* by introducing composite designs \(\equiv\) **homogenization method**.

Homogenization has been applied in the following cases:

- **Conductivity setting**: any objective function of the type
  \[
  J(\chi) = \int_0^T \int_\Omega j(\chi, u) \, dx \, dt
  \]

- **Elasticity setting**: only the stationary compliance
  \[
  J(\chi) = \int_\Omega f \cdot u \, dx
  \]

- In all known cases, the mere knowledge of **effective properties** (homogenized tensors) is enough.
What about elastodynamics or objective functions depending on the gradient?

\[ J(\chi) = \int_0^T \int_\Omega j(\nabla u) \, dx \, dt \]

- Almost nothing is known! Noticeable and limited exceptions (in the stationary case): Lipton, Bellido-Pedregal, Tartar, Casado-Diaz et al.

- It seems that the relaxation does not depend only on homogenized tensors but also on correctors.

- This problem is too difficult for us: so we simplify it!
Simplifying assumption: following the lead of Tartar, we assume a low contrast between the two phases

\[ A^1 = A^0(1+\eta) \Rightarrow A_\chi(x) = (1-\chi(x))A^0 + \chi(x)A^1 = A^0(1+\eta\chi(x)). \]

\[ \rho^1 = \rho^0(1+\eta) \Rightarrow \rho_\chi(x) = (1-\chi(x))\rho^0 + \chi(x)\rho^1 = \rho^0(1+\eta\chi(x)). \]

Simplifying assumption: the two phases have a low contrast.

\[ A^1 = A^0(1 + \eta) \quad \rho^1 = \rho^0(1 + \eta). \]

**Second order expansion for small \( \eta \):**

\[ u = u^0 + \eta u^1 + \eta^2 u^2 + \mathcal{O}(\eta^3). \]

- Plug this ansatz in the state equation.
- Plug this ansatz in the objective function.
- Drop all terms of order \( \mathcal{O}(\eta^3) \) or higher.

The result is called “small amplitude” optimal design problem.
State equation

\[
\begin{align*}
\rho \chi \frac{\partial^2 u}{\partial t^2} - \text{div} (A \chi \nabla u) &= f & \text{in } \Omega \times (0, T) \\
u &= 0 & \text{on } \Gamma_D \times (0, T) \\
A \chi \nabla u \cdot n &= 0 & \text{on } \Gamma_N \times (0, T) \\
u(x, 0) &= u_{\text{init}}(x) , \quad \frac{\partial u}{\partial t}(x, 0) &= v_{\text{init}}(x) & \text{in } \Omega
\end{align*}
\]

0-th order equation (without $\chi$ !)

\[
\begin{align*}
\rho^0 \frac{\partial^2 u^0}{\partial t^2} - \text{div} (A^0 \nabla u^0) &= f & \text{in } \Omega \times (0, T) \\
u^0 &= 0 & \text{on } \Gamma_D \times (0, T) \\
A^0 \nabla u^0 \cdot n &= 0 & \text{on } \Gamma_N \times (0, T) \\
u^0(x, 0) &= u_{\text{init}}(x) , \quad \frac{\partial u^0}{\partial t}(x, 0) &= v_{\text{init}}(x) & \text{in } \Omega
\end{align*}
\]
1-st order equation  (linear in $\chi$)

\[ \left\{ \begin{align*}
\rho^0 \frac{\partial^2 u^1}{\partial t^2} - \text{div} \left( A^0 \nabla u^1 \right) &= -\rho^0 \chi \frac{\partial^2 u^0}{\partial t^2} + \text{div} \left( \chi A^0 \nabla u^0 \right) \quad \text{in } \Omega \times (0, T) \\
 u^1 &= 0 \quad \text{on } \Gamma_D \times (0, T) \\
 A^0 \nabla u^1 \cdot n &= -\chi A^0 \nabla u^0 \cdot n \quad \text{on } \Gamma_N \times (0, T) \\
 u^1(x, 0) &= 0, \quad \frac{\partial u^1}{\partial t}(x, 0) = 0 \quad \text{in } \Omega
\end{align*} \right. \]

2-nd order equation  (quadratic in $\chi$)

\[ \left\{ \begin{align*}
\rho^0 \frac{\partial^2 u^2}{\partial t^2} - \text{div} \left( A^0 \nabla u^2 \right) &= -\rho^0 \chi \frac{\partial^2 u^1}{\partial t^2} + \text{div} \left( \chi A^0 \nabla u^1 \right) \quad \text{in } \Omega \times (0, T) \\
 u^2 &= 0 \quad \text{on } \Gamma_D \times (0, T) \\
 A^0 \nabla u^2 \cdot n &= -\chi A^0 \nabla u^1 \cdot n \quad \text{on } \Gamma_N \times (0, T) \\
 u^2(x, 0) &= 0, \quad \frac{\partial u^2}{\partial t}(x, 0) = 0 \quad \text{in } \Omega
\end{align*} \right. \]
Objective function

\[ J(\chi) = \int_0^T \int_\Omega j(\nabla u) \, dx \, dt \quad \Rightarrow J(\chi) = J_{sa}(\chi) + O(\eta^3) \]

with

\[ J_{sa}(\chi) = \int_0^T \int_\Omega j(\nabla u^0) \, dx \, dt + \eta \int_0^T \int_\Omega j'(\nabla u^0) \cdot \nabla u^1 \, dx \, dt \]
\[ + \eta^2 \int_0^T \int_\Omega \left( j'(\nabla u^0) \cdot \nabla u^2 + \frac{1}{2} j''(\nabla u^0) \nabla u^1 \cdot \nabla u^1 \right) \, dx \, dt. \]

Technical assumption: for any \( \lambda \in \mathbb{R}^N \),

\[ |j(\lambda)| \leq C(|\lambda|^2 + 1), \quad |j'(\lambda)| \leq C(|\lambda| + 1), \quad |j''(\lambda)| \leq C. \]
Small amplitude optimal design

\[ \inf_{\chi \in \mathcal{U}_{ad}} J_{sa}(\chi) \]

with \( \mathcal{U}_{ad} = \left\{ \chi \in L^\infty(\Omega; \{0, 1\}), \text{ such that } \int_{\Omega} \chi(x) \, dx = \Theta |\Omega| \right\} \).

Equations for \( u^0, u^1, u^2 \) have the same constant coefficients.

\( u^0 \) does not depend on \( \chi \), \( u^1 \) depends linearly on \( \chi \), \( u^2 \) depends quadratically on \( \chi \).

\( J_{sa}(\chi) \) depends quadratically on \( \chi \).
With some smoothness assumptions on the data, one can get a uniform error estimate of order $\eta^3$.

Still an ill-posed problem! Non existence of solutions.

Relaxation by means of $H$-measures.
**H-measures (Gérard, Tartar)**

It is a default measure which quantifies the lack of compactness of weakly converging sequences in $L^2(\mathbb{R}^N)$.

**Theorem.** Let $u_\epsilon = (u^i_\epsilon)_{1 \leq i \leq p} \to 0$ weakly in $L^2(\mathbb{R}^N)^p$. There exist a subsequence and complex-valued Radon measures $(\mu_{ij}(x, \xi))_{1 \leq i, j \leq p}$ on $\mathbb{R}^N \times S_{N-1}$ such that, $\forall \phi_1(x), \phi_2(x) \in C_0(\mathbb{R}^N)$ and $\psi(\xi) \in C(S_{N-1})$, it satisfies

$$
\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \mathcal{F} \left( \phi_1 u^i_\epsilon \right)(\xi) \overline{\mathcal{F} \left( \phi_2 u^j_\epsilon \right)(\xi)} \psi \left( \frac{\xi}{|\xi|} \right) d\xi = \int_{\mathbb{R}^N} \int_{S_{N-1}} \phi_1(x) \overline{\phi_2(x)} \psi(\xi) \mu_{ij}(dx, d\xi)
$$

where $(\mathcal{F} \phi)(\xi) = \int_{\mathbb{R}^N} \phi(x) e^{-2i\pi x \cdot \xi} dx$ is the Fourier transform.

The matrix $\mu = (\mu_{ij})_{1 \leq i, j \leq p}$ is called the $H$-measure of $u_\epsilon$.

It is hermitian and non-negative, $\mu_{ij} = \overline{\mu_{ji}}$, $\mu \lambda \cdot \overline{\lambda} \geq 0 \ \forall \lambda \in \mathbb{C}^p$. 
An example just to understand... (Periodic oscillations)

\[ u_\varepsilon(x) \equiv u_0 \left( x, \frac{x}{\varepsilon} \right) \]

with smooth \( u_0(x, y) \) defined on \( \mathbb{R}^N \times \mathbb{T}^N \) (i.e. periodic in \( y \)) and

\[ \int_{\mathbb{T}^N} u_0(x, y) \, dy = 0 \quad \forall x \in \mathbb{R}^N. \]

It satisfies \( u_\varepsilon \to 0 \). Writing \( u_0 \) as a Fourier series in \( y \), i.e.

\[ u_0(x, y) = \sum_{m \in \mathbb{Z}^N, m \neq 0} u^m(x) e^{2i\pi m \cdot y}, \]

the \( H \)-measure of the entire sequence \( u_\varepsilon(x) \) is

\[ \mu(x, \xi) = \sum_{m \in \mathbb{Z}^N, m \neq 0} |u^m(x)|^2 \delta_{m/|m|}(\xi), \]

where \( \delta_{m/|m|} \) is the Dirac mass at \( \frac{m}{|m|} \).
A pseudo-differential operator \( q \) is defined through its symbol \( (q_{ij}(x, \xi))_{1 \leq i, j \leq p} \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N) \) by

\[
(q(u))_i(x) = \sum_{j=1}^{p} \mathcal{F}^{-1}(q_{ij}(x, \cdot) \mathcal{F}u_j(\cdot))(x)
\]

We consider only pseudo-differential operators with symbol, homogeneous of degree 0 in \( \xi \).

**Theorem (Gérard, Tartar).** Let \( u_\epsilon \to 0 \) weakly in \( L^2(\mathbb{R}^N)^p \).

There exist a subsequence and an \( H \)-measure \( \mu \) such that, for any polyhomogeneous pseudo-differential operator \( q \) of degree 0 with symbol \( (q_{ij}(x, \xi))_{1 \leq i, j \leq p} \),

\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^N} q(u_\epsilon) \cdot \bar{u}_\epsilon \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \sum_{i,j=1}^{p} q_{ij}(x, \xi) \mu_{ij}(dx, d\xi).
\]
The case of characteristic functions

Lemma (Kohn, Tartar). Let $\chi_\varepsilon(x)$ be a sequence of characteristic functions converging weakly-* to a limit $\theta$ in $L^\infty(\Omega; [0, 1])$. The $H$-measure of $(\chi_\varepsilon - \theta)$ is

$$\mu(x, \xi) = \theta(x)(1 - \theta(x))\nu(x, \xi)$$

where $\xi \rightarrow \nu$ is a probability measure on the unit sphere $S_{N-1}$, i.e.

$$\nu(\xi) \geq 0 \ \forall \xi \in S_{N-1}, \text{ and } \int_{S_{N-1}} d\nu(\xi) = 1.$$

Furthermore, any such measure can be attained as the $H$-measure of a sequence $\chi_\varepsilon$ of characteristic functions.
III- APPLICATION

Computation of the relaxation of small-amplitude optimal design.

Take a minimizing sequence $\chi_n \rightarrow \theta$ in $L^\infty(\Omega; [0, 1])$ weakly-*.

While $u^0$ is does not depend on $\chi_n$, $u^1_n, u^2_n$ are bounded.

\[
\begin{cases}
\rho^0 \frac{\partial^2 u^1_n}{\partial t^2} - \text{div} (A^0 \nabla u^1_n) = -\rho^0 \chi_n \frac{\partial^2 u^0}{\partial t^2} + \text{div} (\chi_n A^0 \nabla u^0), \\
\rho^0 \frac{\partial^2 u^2_n}{\partial t^2} - \text{div} (A^0 \nabla u^2_n) = -\rho^0 \chi_n \frac{\partial^2 u^1_n}{\partial t^2} + \text{div} (\chi_n A^0 \nabla u^1_n).
\end{cases}
\]

We can pass to the limit for $u^1_n$ by weak convergence and for $u^2_n$ thanks to $H$-measures.
Relaxation of the objective function

\[ J_{sa}(\chi_n) = \int_0^T \int_{\Omega} j(\nabla u^0) \, dx \, dt + \eta \int_0^T \int_{\Omega} j'(\nabla u^0) \cdot \nabla u_1^n \, dx \, dt \]
\[ + \eta^2 \int_0^T \int_{\Omega} \left( j'(\nabla u^0) \cdot \nabla u_2^n + \frac{1}{2} j''(\nabla u^0) \nabla u_1^n \cdot \nabla u_1^n \right) \, dx \, dt. \]

We can pass to the limit for \( u_1^n \) by weak convergence.

We can pass to the limit for \( u_2^n \) and the quadratic term in \( u_1^n \) thanks to \( H \)-measures.

In the end, the limits will depend on the \( H \)-measure of the sequence \( \chi_n \) (which is thus a new variable for optimization).
Main idea for passing to the limit (steady-state case)

Recall that \(-\text{div} \left( A^0 \nabla u^1_n \right) = \text{div} \left( \chi_n A^0 \nabla u^0 \right)\).

If \(\Omega = \mathbb{R}^N\) and \(\nabla u^0\) is constant, the solution is explicitly given by

\[ F(\nabla u^1_n) = -F(\chi_n) \frac{A^0 \nabla u^0 \cdot \xi}{A^0 \xi \cdot \xi} \xi. \]

A “similar” computation shows that \(\nabla u^1_n\) depends linearly on \(\chi_n\) through the pseudo-differential operator

\[ q(x, \xi) = -\frac{A^0 \nabla u^0(x) \cdot \xi}{A^0 \xi \cdot \xi} \xi. \]
Therefore, we deduce that

\[
\lim_{n \to +\infty} \int_{\Omega} \chi_n A^0 \nabla u^1_n \cdot \nabla \phi \, dx = \int_{\Omega} \theta A^0 \nabla u^1 \cdot \nabla \phi \, dx \\
- \int_{\Omega} \int_{\mathbb{S}^{N-1}} \theta (1 - \theta) \left( \frac{A^0 \nabla u^0 \cdot \xi}{A^0 \xi \cdot \xi} \right) \nu(dx, d\xi)
\]

where \(\theta (1 - \theta)\nu\) is the \(H\)-measure of \((\chi_n - \theta)\).
What is different or not with the wave equation?

✔ The sequence $\chi_n(x)$ oscillates only in space, not in time.

✗ There is a serious difficulty with a priori estimates!
Lemma. Let \( u(\eta) \) be the unique solution in \( H^1_0(\Omega) \) of
\[
- \text{div} \left( A(\eta) \nabla u(\eta) \right) = f.
\]
If \( \eta \to A(\eta) \) is analytic, so is \( \eta \to u(\eta) \).

This result is false with the solution of
\[
\frac{\partial^2 u}{\partial t^2} - c^2(\eta) \Delta u = f.
\]
In 1-d with \( f = 0 \), the solution is \( u(t, x) = u^+(x - ct) + u^-(x + ct) \)
which, upon differentiation with respect to \( c \), looses one degree of regularity.
Consequence of the poor a priori estimates

Some smoothness assumptions on the data are necessary to establish the relaxed problem.

However, strong assumptions are required to prove that the small-amplitude problem is close to the original one, i.e., there exists $C > 0$ such that, for any $\chi$,

$$|J(\chi) - J_{sa}(\chi)| \leq C\eta^3.$$  

In particular we need $u_{\text{init}} = 0$ and $v_{\text{init}} = 0$ for the gradient-based objective function!
Relaxed state equations

For a limit density \( \theta(x) \in L^\infty(\Omega; [0, 1]) \) and probability measure \( \nu(x, \xi) \), the relaxed state equations are

\[
\begin{align*}
\rho^0 \frac{\partial^2 u^0}{\partial t^2} - \text{div} \left( A^0 \nabla u^0 \right) &= f, \\
\rho^0 \frac{\partial^2 u^1}{\partial t^2} - \text{div} \left( A^0 \nabla u^1 \right) &= \text{div} \left( \theta A^0 \nabla u^0 \right), \\
\rho^0 \frac{\partial^2 u^2}{\partial t^2} - \text{div} \left( A^0 \nabla u^2 \right) &= \text{div} \left( \theta A^0 \nabla u^1 \right) - \text{div} \left( \theta(1 - \theta) A^0 M A^0 \nabla u^0 \right),
\end{align*}
\]

with

\[
M(x) = \int_{S^{N-1}} \frac{\xi \otimes \xi}{A^0 \xi \cdot \xi} \nu(x, d\xi).
\]

This last term is the "trace" of homogenization!

\( \theta \) "replaces" \( \chi \) and \( \nu \) is a new variable of optimization.
Relaxed objective function

Passing to the limit in the objective function, using again $H$-measures, we obtain

$$J_{sa}^*(\theta, \nu) = \int_0^T \int_\Omega j(\nabla u^0) \, dx \, dt + \eta \int_0^T \int_\Omega j'(\nabla u^0) \cdot \nabla u^1 \, dx \, dt$$

$$+ \eta^2 \int_0^T \int_\Omega \left( j'(\nabla u^0) \cdot \nabla u^2 + \frac{1}{2} j''(\nabla u^0) \nabla u^1 \cdot \nabla u^1 \right) \, dx \, dt$$

$$+ \eta^2 \int_0^T \int_\Omega \theta(1 - \theta) A^0 N A^0 \nabla u^0 \cdot \nabla u^0 \, dx \, dt,$$

with

$$N(x) = \frac{1}{2} \int_{S^{N-1}} \frac{j''(\nabla u^0) \xi \cdot \xi}{(A^0 \xi \cdot \xi)^2} \xi \otimes \xi \nu(x, d\xi)$$

This last term is the "trace" of correctors in homogenization!
Relaxed small-amplitude optimal design

\[
\min_{(\theta, \nu) \in U_{ad}^*} J_{sa}^*(\theta, \nu)
\]

with

\[U_{ad}^* = \left\{ (\theta, \nu) \in L^\infty(\Omega; [0, 1]) \times \mathcal{P}(\Omega, \mathbb{S}_{N-1}) \text{ such that } \int_\Omega \theta(x) \, dx = \Theta |\Omega| \right\}\]

\[\mathcal{P}(\Omega, \mathbb{S}_{N-1}) = \left\{ \nu(x, \xi) \text{ Radon measure on } \Omega \times \mathbb{S}_{N-1} \text{ such that: } \begin{array}{l} 
\nu(x, \xi) \geq 0, \quad \int_{\mathbb{S}_{N-1}} \nu(x, \xi) \, d\xi = 1 \text{ a.e. } x \in \Omega
\end{array} \right\}.\]
**Theorem.** The relaxed problem is a true relaxation in the sense that

1. it admits at least one minimizer \((\theta, \nu)\),

2. any minimizer \((\theta, \nu)\) is the limit of a minimizing sequence \(\chi_n\) of the original problem, i.e.
   \[
   \chi_n \rightharpoonup \theta, \quad \nu = H\text{-measure of } (\chi_n - \theta), \quad \lim_{n \to +\infty} J_{sa}(\chi_n) = J_{sa}^*(\theta, \nu),
   \]

3. any minimizing sequence \(\chi_n\) of the original problem converges in the previous sense to a minimizer \((\theta, \nu)\).

Well-posed problem! But still complicated...
Theorem.
Rank-one laminates are optimal in the relaxed problem.

There exist minimizers such that the probability measure \( \nu \) is a Dirac mass (in \( \xi \)). Furthermore, the optimal Dirac mass \( H \)-measure does not depend on the density \( \theta \).

Remarks.

\[ \blacktriangleright \text{Design parameters: density } \theta, \text{ single lamination direction } \xi^*. \]

\[ \blacktriangleright \text{Rank-one laminates are always optimal! Same result for elasticity, multiple loads, etc.} \]
Proof

\[ J_{sa}^*(\theta, \nu) = \int_0^T \int_{\Omega} j(\nabla u^0) \, dx \, dt + \eta \int_0^T \int_{\Omega} j'(\nabla u^0) \cdot \nabla u^1 \, dx \, dt \]
\[ + \eta^2 \int_0^T \int_{\Omega} \left( j'(\nabla u^0) \cdot \nabla u^2 + \frac{1}{2} j''(\nabla u^0) \nabla u^1 \cdot \nabla u^1 \right) \, dx \, dt \]
\[ + \eta^2 \int_0^T \int_{\Omega} \theta(1 - \theta) A^0 N A^0 \nabla u^0 \cdot \nabla u^0 \, dx \, dt, \]

\( J_{sa}^* \) depends linearly on the \( H \)-measure \( \nu \) because \( u^2 \) and \( N \) depend linearly on \( \nu \in \mathcal{P}(\Omega, S_{N-1}) \).

A linear function on the convex set \( \mathcal{P} \) is minimal on the extremal points of \( \mathcal{P} \) which are Dirac masses.

\[ \Rightarrow \quad \nu = \delta_{\xi^*} \]
-V- NUMERICAL RESULTS

Numerical algorithm (steepest descent):

- **Initialization:** We introduce an adjoint to compute the optimal direction of lamination $\xi^*$ which is independent of $\theta$. The microstructure (lamination) is computed once.

- **Iterations for $k \geq 1$:** We introduce another adjoint to compute $\nabla_{\theta} J^*_{sa}(\theta_k)$ and update $\theta_{k+1}$. The two adjoints are integrated backwards with a stored state.
Tricks

- **Finite element solver:** FreeFem++, $P_2/P_0$ FEM.

- In the end we **penalize the composite zones**, e.g., by adding to the objective function a term of the type
  \[
  \int_{\Omega} \theta(1 - \theta) \, dx
  \]
  or by changing the density at each step
  \[
  \theta_{pen} = \frac{1 - \cos(\pi \theta_{opt})}{2}
  \]

- **Black** = strong phase on the pictures.

- All computations are done for the **elasticity system**.
Compliance minimization for the long cantilever with $\eta = 0.99$ and volume=60%: relaxed (left), penalized (right).
Cantilever: dynamic case

\[ T = 10, \ \Delta t = 0.25 \]

\[ f(t, x) = (\sin \frac{\pi t}{20}) f(x) \]

\[ u_{\text{init}} = v_{\text{init}} = 0 \]

\[ J(\chi) = \int_0^T \int_{\Omega} \partial_t u \cdot f \, dx \, dt = \frac{1}{2} \int_{\Omega} \left( \rho \chi \left( \frac{\partial u}{\partial t} \right)^2 + A \chi e(u) \cdot e(u) \right) (T) \, dx \]

\[ \eta = 0.90 \text{ and volume}=50\%: \]
Maximization of the dissipation for the long cantilever: relaxed (left), penalized (right).
Same case with $f(t,x) = (\sin \pi t/5) f(x)$
Strain minimization: steady-state case

Square fixed at the bottom, vertically loaded at the top with $\eta = 0.1$ and volume=50%.
Strain minimization: dynamic case

\[ T = 10, \quad f(t, x) = f(x), \quad u_{\text{init}} = v_{\text{init}} = 0, \quad J(\chi) = \int_0^T \int_{\Omega} |e(u)|^2 \, dx \, dt \]
Stress minimization: steady-state case

Square fixed at the bottom, vertically loaded at the top with $\eta = 0.1$ and volume=50%.
Stress minimization: dynamic case

\[ T = 10, \quad f(t, x) = f(x), \quad u_{\text{init}} = v_{\text{init}} = 0, \quad J(\chi) = \int_0^T \int_{\Omega} |Ae(u)|^2 \, dx \, dt \]
Minimal dissipation of a wheel

\[ T = 2\pi, \quad f(t, x) = (\sin t, \cos t), \quad u_{\text{init}} = v_{\text{init}} = \text{solution after 100 rotations}, \quad J(\chi) = \int_0^T \int_\Omega f \cdot \partial_t u \, dt \, dx \]