Bornes sur le gradient pour une classe d'équations elliptiques singulières

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Outlines of the talk

- A class of elliptic problems singular at the boundary
- Motivation: a state constraint problem for the Brownian motion
- The PDE's results: Lipschitz solutions of singular equations
- Application to the control problem: qualitative behaviour of the optimal feedback

Let Ω be a C^2 bounded domain in $\mathbf{R}^N$, $N \geq 2$. $d(x) \equiv \mathrm{dist}(x, \partial\Omega)$

We consider a class of Hamilton–Jacobi equations

$$-\alpha\, \Delta v + v + H(x, \nabla v) = 0 \quad \text{in } \Omega\,,$$

where $H(x,p)$ is singular at the boundary. The typical structure:

$$-\alpha\, \Delta v + v + F(x) \cdot \nabla v + g(x, \nabla v) = f(x)$$

- $F(x) \cdot \nabla v$ is a singular transport term, ex. $|F(x)| \sim \dfrac{\sigma}{\mathrm{dist}(x, \partial\Omega)}$ with $F(x)$ "directed outward"

  ex: $F(x) = \frac{1}{d(x)} B(x)$, where $B(x) \cdot \nu(x) \geq \alpha$
- $g(x, \nabla v)$ is a nonlinear term with natural growth
- $f(x)$ is locally Lipschitz but possibly singular at $\partial\Omega$.

Model example:

$$-\alpha\,\Delta v + v + \frac{B(x)\cdot\nabla v}{d(x)} + c(x)|\nabla v|^2 = f(x) \quad \text{in } \Omega,$$

where $\alpha > 0$ and $d \equiv \mathrm{dist}(x,\partial\Omega)$ near $\partial\Omega$.

Main features:

- $B(x)$ "pushing towards outside": $B(x)\cdot\nu(x) \geq \alpha$
- *no sign condition* on $c(x)$
  (the Hamiltonian H is not coercive w.r.t. $|\nabla v|$)
- $f(x) \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and can be singular at $\partial\Omega$.

Rmk.: No boundary condition is prescribed

Main goals:

- Construction of solutions which are Lipschitz up to the boundary (despite the singularity in the equation)
- Uniqueness of bounded solutions
  (which would imply a regularity result: bounded solutions are globally Lipschitz)
- Informations on the boundary behaviour
  (typically, some Neumann condition induced by the singular transport term)
- Eventually, stability in the vanishing viscosity limit $\alpha \to 0$

Motivation: Constrain a Brownian motion by controlling the drift

In 1989 J.M. Lasry and P.L. Lions introduced a simple model to describe the possibility to confine a Brownian motion inside a domain Ω by controlling the drift.

Precisely:

- B_t is a standard Brownian motion (on a probability space $(\Theta, \mathcal{F}, \mathcal{F}_t, P)$) with values in R^N.
- $a : \Omega \to R^N$ is a vector field

Goal: control the dynamics of the process X_t given by

$$\begin{cases} dX_t = a(X_t)dt + \sqrt{2}\, dB_t \,, \\ X_0 = x \in \Omega \,. \end{cases}$$

in order that $X_t \in \Omega$ for every $t > 0$ (a.s.).

The drift $a_t = a(X_t)$ is interpreted as feedback control.

$$\begin{cases} dX_t = a(X_t)dt + \sqrt{2}\, dB_t\,, \\ X_0 = x \in \Omega\,. \end{cases} \tag{1}$$

Precisely: if $\tau_x = \inf\{t \geq 0\ :\ X_t \notin \Omega\}$, is the exit time variable, the admissible controls are:

$$a \in \mathcal{A} = \{a \in C(\Omega)^N : \quad P(\tau_x < \infty) = 0 \quad \forall x \in \Omega\}$$

Rmk: *It is necessary to use controls which are singular (at the boundary)* (a bounded drift can not constrain a Brownian motion inside Ω)

If $a \in L^\infty$ then $\tau_x < \infty$. Indeed

$$\mathbb{E}\,(\tau_x) = V(x)$$

where V is the solution of

$$\begin{cases} -\Delta V - a \cdot \nabla V = 1 \quad \text{in } \Omega\,, \\ V = 0 \quad \text{on } \partial\Omega\,. \end{cases}$$

If $a \in L^\infty$ there exists a unique sol. $V \in L^\infty$ and then

$$\mathbb{E}\,(\tau_x) \leq \|V\|_\infty \qquad \forall x$$

Therefore:

- The set of admissible controls

$$\mathcal{A} = \{a \in C(\Omega)^N : \quad P(\tau_x < \infty) = 0 \quad \forall x \in \Omega\}$$

does not contain bounded functions.

- However $\mathcal{A} \neq \emptyset$. Admissible controls can be easily constructed as functions of $d(x)$ (distance from $\partial\Omega$) and push towards inside

Es: $a = \frac{\mu}{d(x)} \nabla d(x)$ is admissible for every $\mu \geq 1$.

(related to singular solutions of the Kolmogorov operator $-\Delta v - a(x) \cdot \nabla v$, e.g. $v(x) = \log(d(x))$)

[Lasry-Lions] introduce an optimality criterion over the set of admissible controls.

Easiest example: (minimization of "discounted" L^2– norm)

$$\inf_{a\in\mathcal{A}} J(a,x) := \mathbb{E}\left[\int_0^\infty \frac{1}{2}\,|a(X_t)|^2\, e^{-t} dt\right]$$

Then, [LL] prove that the minimum exists and there is a unique optimal (feedback) control.

The strategy is typical: use the dynamic programming principle and the PDE satisfied by the value function

$$u(x) = \inf_{a\in\mathcal{A}} J(a,x)$$

Given

$$u(x) = \inf_{a\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty \frac{1}{2}\,|a(X_t)|^2\, e^{-t} dt\right]$$

they prove

- u solves the Bellmann equation

$$-\Delta u + u + \frac{1}{2}|\nabla u|^2 = 0 \tag{2}$$

- u is the maximal solution of (2) and satisfies

$$u(x) \to +\infty \qquad \text{as } x \to \partial\Omega. \tag{3}$$

  Actually $u \in C^2(\Omega)$ is the unique solution of (2)–(3) and satisfies

$$u(x) \sim -2\log(d(x)) \qquad \text{as } x \to \partial\Omega$$

- the unique optimal control $a_0(x)$ is given by

$$a_o(x) = -\nabla u(x)$$

After [LL], we know that the optimal "constrained dynamics"

$$\begin{cases} dX_t = a(X_t)dt + \sqrt{2}\, dB_t \,, \\ X_0 = x \in \Omega \,, \end{cases}$$

is governed by the optimal control $a(X_t) = -\nabla u(X_t)$ where u solves

$$-\Delta u + u + \frac{1}{2} |\nabla u|^2 = 0$$

$$u \sim -2 \log d(x) \qquad \text{as } x \to \partial\Omega$$

Set

$$v = u + 2 \log d(x)$$

Then v solves

$$-\Delta v + v - 2\frac{\nabla d(x) \nabla v}{d(x)} + \frac{1}{2} |\nabla v|^2 = 2 \log d(x) + 2\frac{\Delta d}{d(x)}$$

Lipschitz bound for $v \Rightarrow \quad \nabla u = -2\frac{\nabla d}{d} + O(1)$
(description of the optimal drift)

This is our model example of singular equations:

$$(E) \qquad -\alpha\,\Delta v + v + \frac{B(x)\cdot\nabla v}{d(x)} + c(x)|\nabla v|^2 = f(x) \quad \text{in } \Omega$$

Model result:

Theorem (L-P)

Let $c(x),\, B(x) \in W^{1,\infty}(\Omega)$. Assume that

$$B(x)\cdot\nu \geq \sigma, \quad B(x)\cdot\tau = 0 \qquad \text{at } \partial\Omega$$

where $\sigma > \alpha$.
Assume that $f(x) \in W^{1,\infty}_{loc}(\Omega)$ satisfies (near the boundary)

$$|f| \leq \frac{\rho(d)}{d}, \quad |\nabla f| \leq \frac{\rho(d)}{d^2} \qquad \text{where } \int_0^1 \frac{\rho(s)}{s}\,ds < \infty\,.$$

Then there exists a solution v of (E) such that $v \in C^2(\Omega)\cap W^{1,\infty}(\Omega)$.
In addition, $\frac{\partial v(x)}{\partial \nu} \to 0$ as $x \to \partial\Omega$.

For $\sigma = \alpha$, same result if ρ satisfies $\displaystyle\int_0^1 \frac{1}{s}\left(\int_0^s \frac{\rho(\tau)}{\tau}\,d\tau\right) ds < \infty\,.$

This Theorem is a simplified version of more general results.
Main extensions are:

- $B(x) \cdot \tau(x) = 0$ can be replaced by

$$\exists \quad \text{a symmetric positive matrix} \quad N(x) \in C^2 \ :$$

$$B(x) \cdot N(x)\tau(x) = 0 \qquad\qquad (\text{i.e. } N(x)B(x) = \nu(x)) \,.$$

  ($B(x)$ can be an oblique field, not necessarily orthogonal)

- operator $-\Delta v$ replaced by $-m_{ij}(x)\partial^2_{ij} v$, where $m_{ij} \in W^{1,\infty}(\Omega)$. In this case the main assumption is that

$$B(x) \cdot \nu(x) \geq \alpha \,, \qquad \text{where } \alpha = \sup_{x\in\partial\Omega} M(x)\nabla d \nabla d$$

- $c(x)$ and $B(x)$ do not need to be Lipschitz up to the boundary (but anyway bounded, with some continuity modulus..)

General version for the equation

$$-\alpha\,\Delta v + v + H(x,\nabla v) = 0 \quad \text{in } \Omega\,.$$

Structure assumptions:

$$\forall\, K \subset\subset \Omega \quad \exists C_K > 0\ :\quad |H(x,p)| \leq C_K\left(1+|p|^2\right) \qquad \text{(H0)}$$

($H(x,p)$ has natural growth in p, locally bdd. in x)

Near the boundary, H satisfies

$$|H(x,p) - p\cdot H_p(x,p)| \leq C_0|p|^2 + \frac{\rho(d(x))}{d(x)} \qquad \text{(H1)}$$

$$H_p(x,p)\cdot\nu(x) \geq \frac{\sigma}{d(x)} - C_1|p| \qquad \text{(H2)}$$

$$H_x(x,p)\cdot\frac{p}{|p|} \geq -\frac{\rho(d)}{d^2}|p| - \frac{\rho(d)}{d}|p|^2 - \frac{\rho(d)}{d^2} \qquad \text{(H4)}$$

Theorem

Assume that $H(x,p)$ *satisfies (H0)–(H4) and either*

$$\sigma > \alpha\,, \qquad \text{and} \quad \int_0^1 \frac{\rho(t)}{t}\,dt < \infty\,,$$

or

$$\sigma = \alpha\,, \qquad \text{and} \quad \int_0^1 \frac{1}{t}\left(\int_0^t \frac{\rho(\tau)}{\tau}\,d\tau\right) dt < \infty\,.$$

Then

(i) There exists a solution $v \in C^2(\Omega) \cap W^{1,\infty}(\Omega)$ *of*

$$-\alpha\,\Delta v + v + H(x,\nabla v) = 0 \quad \text{in}\ \Omega\,.$$

(ii) If $H(x,\cdot)$ *is convex,* v *is the unique bounded solution*

- When $H(x,\cdot)$ is convex, uniqueness holds in the class of bounded solutions

  Ex: this applies to linear equations, or to nonlinear H-J equations arising from dynamic programming principle (control pbs)

  Since *uniqueness holds in the class of bounded solutions*, in this case we deduce a regularity result:

  u bounded sol. $\Rightarrow$ u Lipschitz

- This kind of uniqueness/regularity holds *without any boundary information*

- The condition $\sigma \geq \alpha$ is optimal for this kind of result.

  Fichera condition: in the linear case, you can prescribe Dirichlet boundary data in

$$\{x \in \partial\Omega \,:\, B(x)\cdot\nu(x) < \alpha\}$$

  So, any global gradient bound in that case depends on boundary data.

Role of boundary conditions: actually, some Neumann condition is hidden in the equation. Ex:

$$(E) \qquad -\alpha\Delta v + v + \frac{B(x)\cdot\nabla v}{d(x)} + c(x)|\nabla v|^2 = f(x) \quad \text{in } \Omega,$$

assuming $B \in W^{1,\infty}(\Omega)$:

$$B(x)\cdot\nu \geq \alpha, \quad B(x)\cdot\tau = 0 \qquad \text{at } \partial\Omega$$

Easy heuristics: if v is not singular, we expect from the equation that

$$B(x)\cdot\nabla v \;\Big|_{\partial\Omega} = \lim_{d(x)\to 0} d(x)f(x)$$

and if $f = o\left(\frac{1}{d(x)}\right)$ then $B(x)\cdot\nabla v = 0$, hence $\frac{\partial v}{\partial\nu} = 0$.

In a weak sense, this is true even without knowing that v is Lipschitz.

Theorem

Let $c(x) \in L^\infty$, $f(x) = o(\frac{1}{d})$ as $d(x) \to 0$. Assuming $B \in W^{1,\infty}(\Omega)$:

$$B(x) \cdot \nu \geq \alpha, \quad B(x) \cdot \tau = 0 \qquad \text{at } \partial\Omega$$

then any $v \in C(\overline{\Omega})$ (viscosity) solution of

$$(E) \qquad -\alpha \Delta v + v + \frac{B(x) \cdot \nabla v}{d(x)} + c(x)|\nabla v|^2 = f(x) \quad \text{in } \Omega,$$

satisfies $\frac{\partial v}{\partial \nu} = 0$ (in viscosity sense).
If $v \in W^{1,\infty}(\Omega)$, then $\frac{\partial v}{\partial \nu} \to 0$ as $x \to \partial\Omega$.

Rmk: (work in progress) Using the weaker framework of viscosity solutions we can prove directly that any $v \in C_b(\Omega)$, which is a viscosity solution inside Ω, is Lipschitz up to $\partial\Omega$
(bounded solutions are Lipschitz, independently of the uniqueness result)

Method used to construct Lipschitz solutions:

(i) Approximate with (regular) Neumann problems

$$\begin{cases} -\alpha\Delta v_n + v_n + H(x, \nabla v_n) = f(x) & \text{in } \Omega_n\,, \\ \frac{\partial v_n}{\partial \nu} = 0 & \text{on } \partial\Omega_n\,, \end{cases}$$

where $\Omega_n = \{x \in \Omega \,:\, d(x) > \frac{1}{n}\}$.

(ii) Use the Bernstein's method (Serrin, P.L. Lions) to obtain

$$|\nabla v_n| \leq C\left(1 + \sup_{d(x)=\delta} |\nabla v_n|\right) \qquad \text{for all } x \,:\, \frac{1}{n} < d(x) < \delta$$

for some $\delta > 0$.

(iii) Interior elliptic estimates give

$$\|\nabla v_n\|_{L^\infty(K)} \leq C(\mathrm{dist}(K, \partial\Omega)) \qquad \forall K \subset\subset \Omega$$

(ii)+ (iii) imply

$$|\nabla v_n| \leq C \qquad \forall x \in \Omega$$

How does the direction of the transport term play a coercive role ?

Main technical tool: use a weighted version of Bernstein's technique.

Classical Bernstein's method: $w := |\nabla u|^2$ satisfies

$$\begin{aligned} -\Delta w &\leq -2\nabla(\Delta u)\nabla u \\ &= -2\nabla(u + H(x, \nabla u))\nabla u \\ &= -2w - H_p \cdot \nabla w - 2H_x \cdot \nabla u \end{aligned}$$

Multiplying by a weight $\phi(d(x))$ one gets

$$\begin{aligned} -\phi(d)\Delta w &\leq -2\phi(d)w - \phi(d)H_p \cdot \nabla w + \ldots \\ &= -2\phi(d)w - H_p \cdot \nabla(\phi(d)w) + \phi'(d)w\, H_p \cdot \nabla d + \ldots \end{aligned}$$

Now replace w with $z = \phi(d)w$, to get

$$-\Delta z + 2z + H_p \cdot \nabla z + \frac{\phi'(d)}{\phi(d)} z \underbrace{H_p \cdot \nu}_{\geq \frac{\sigma}{d}} \leq \ldots..$$

$(z \leq C \iff \phi(d)|\nabla u|^2 \leq C$. If $\phi(d)$ is bdd. then u is Lipschitz)

A similar weighted version of Bernstein's method was introduced by [P.L. Lions, 85] to deal with Neumann problems:

$$\nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega \quad \Rightarrow \nabla[|\nabla u|^2] \cdot \nu \leq C|\nabla u|^2$$

hence, for any function $\phi(d(x))$ such that $C\phi - \phi' \leq 0$,

$$\nabla \left[\phi(d)|\nabla u|^2\right] \cdot \nu \leq [C\phi(d) - \phi'(d)] \, |\nabla u|^2 \leq 0$$

hence $z := \phi(d)|\nabla u|^2$ cannot have maximum on $\partial\Omega$.

If you are able to handle the inequality

$$-\Delta z + 2z + H_p \cdot \nabla z + \frac{\phi'(d)}{\phi(d)} z \underbrace{H_p \cdot \nu}_{\geq \frac{\sigma}{d}} \leq .....$$

by maximum principle it follows a bound on $\max z = \max [\phi(d)|\nabla u|^2]$.

Vanishing viscosity limit: the boundary estimates can be uniform with respect to the ellipticity constant α. However

- It is important to combine the vanishing viscosity with a vanishing singular transport, ex. approximate

$$v + \frac{B(x) \cdot \nabla v}{d(x)} + c(x)|\nabla v|^2 = f(x)$$

  with

$$v - \alpha \Delta v - \alpha \frac{\nabla d(x) \nabla v}{d(x)} + \frac{B(x) \cdot \nabla v}{d(x)} + c(x)|\nabla v|^2 = f(x)$$

  (note: here $\alpha \to 0$. Now one only needs $B(x) \cdot \nu(x) > 0$)

- We need to assume $c(x) > 0$ in Ω in order to have the interior Lipschitz estimates (no more ellipticity here !)
  However, $c(x)$ may vanish at $\partial\Omega$ (there is no coercivity w.r.t. $|\nabla v|$ as $x \to \partial\Omega$)

Theorem (L-P)

Assume that $B(x) \in W^{1,\infty}(\Omega)^N$ *satisfies*

$$B(x)\cdot\nu > 0, \quad B(x)\cdot\tau = 0 \qquad at \;\; \partial\Omega$$

and $f(x) \in W^{1,\infty}_{loc}(\Omega)$ *satisfies*

$$|f| \leq \frac{\rho(d)}{d}, \quad |\nabla f| \leq \frac{\rho(d)}{d^2} \qquad where \;\; \int_0^1 \frac{\rho(s)}{s}\, ds < \infty\,.$$

Moreover suppose that $c(x) \in W^{1,\infty}_{loc}(\Omega)$, $c(x) > 0$ *and (near* $\partial\Omega$*):*

$$|\nabla c(x)|^2 \leq \frac{\rho(d)}{d^2}\, c(x)\,.$$

Then there exists $v \in W^{1,\infty}(\Omega)$ *which is a (viscosity) solution of*

$$v + \frac{B(x)\cdot\nabla v}{d(x)} + c(x)|\nabla v|^2 = f(x) \qquad in \; \Omega.$$

Moreover $\frac{\partial v}{\partial \nu} = 0$ *(in the viscosity sense) at* $\partial\Omega$.

Application to the control pb.

Back to the control problem

$$\begin{cases} -\Delta u + \frac{1}{2}\,|\nabla u|^2 + u = f(x) & \text{in } \Omega\,, \\ u(x) \to +\infty \quad \text{as} \quad d(x) \to 0 \end{cases}$$

The strategy is:

- Introduce an explicit singular solution S such that $u - S$ is bounded (ex: $S = -2\log d(x)$)
- Consider the equation of $v = u - S$
  (roughly speaking, a linearization around S)
- Prove that
  (i) v is Lipschitz $\Rightarrow \nabla u = \nabla S + O(1)$
  (ii) Deduce from the equation the behaviour of $\nabla v \cdot \nu$ on $\partial\Omega$
  ($\Rightarrow$ description of the normal trace of $\nabla u - \nabla S$)

More precising , if

$$v := u + 2\log d(x)$$

then it solves

$$-\Delta v + v - 2\frac{\nabla d(x)\nabla v}{d(x)} + \frac{1}{2}\left|\nabla v\right|^2 = 2\log d(x) + 2\frac{\Delta d}{d(x)}$$

Roughly, the steps of our approach become:

(i) v is bounded (easy through comparison)

(ii) v bounded $\Rightarrow$ v is Lipschitz

(iii) Recover the Neumann condition :

$$\frac{\partial v}{\partial \nu} \to \Delta d = -(N-1)\,H(x) \qquad \text{(curvature term)}$$

Curvature effects in the control mechanism

We denote by $\overline{x}$ the projection of x on $\partial\Omega$ and with $H(\overline{x})$ the mean curvature of $\partial\Omega$ computed at $\overline{x}$. Ω is assumed to be smooth.

Theorem

Let u solve

$$\begin{cases} -\Delta u + \frac{1}{2}\,|\nabla u|^2 + u = f(x) & \text{in } \Omega\,, \\ u(x) \to +\infty \quad \text{as} \quad d(x) \to 0 \end{cases}$$

and let $a(x) = -\nabla u(x)$ *be the optimal control for the constrained dynamics. Then we have, as* $d(x) \to 0$,

$$a_\nu(x) = -\frac{2}{d(x)} + (N-1)\; H(\overline{x}) + o(1) \qquad [a_\nu = a \cdot \nu]$$

$$a_\tau \in L^\infty \qquad [a_\tau = a \cdot \tau]$$

We observe in particular:

(i) The control is tangentially bounded.

(ii) On the hypersurfaces parallel to $\partial\Omega$, the control is maximum on points of maximal curvature

The “constrained dynamics”

This qualitative result confirms our intuition on which should be the optimal way to control the dynamics.

- In first approximation, only the distance is important. At same distance, the curvature plays a role.
- The control (i.e. the drift) must be stronger where the domain is more curved.
- The optimal control distinguishes between points having positive or negative curvature.