Robin Boundary Conditions in Mixed Finite Element Methods

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Background

My talk here on December 7, 2007.

The displacement method with Robin conditions near the Dirichlet limit

- The error estimate detoriates
- The stiffness matrix becomes ill-conditioned

Cure suggested: generalizing Nitsche's method

Darcy's equations

- Problem

$$\begin{aligned} \boldsymbol{\sigma} - \nabla u &= \mathbf{0} && \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\sigma} + f &= 0 && \text{in } \Omega. \end{aligned}$$

- The boundary conditions we parametrize with $\varepsilon \geq 0$

$$\varepsilon \boldsymbol{\sigma} \cdot \mathbf{n} = u_0 - u + \varepsilon g \qquad \text{on } \partial\Omega.$$

- For $\varepsilon = 0$ we have Dirichlet conditions: $u = u_0$.
- In the limit $\varepsilon \to \infty$ we obtain Neumann conditions: $\boldsymbol{\sigma} \cdot \mathbf{n} = g$.

Variational formulation

- Solve for u in the boundary condition equation and insert into the boundary integral term

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, u) + \varepsilon\langle \boldsymbol{\sigma} \cdot \mathbf{n}, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\partial\Omega} = \langle u_0 + \varepsilon g, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\partial\Omega} \qquad \forall \boldsymbol{\tau} \in H(\mathrm{di}$$

$$(\operatorname{div} \boldsymbol{\sigma}, v) + (f, v) = 0 \qquad \forall v \in L^2(\Omega$$

- The bilinear form $a(\cdot, \cdot)$ is ε-dependent

$$a_\varepsilon(\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\boldsymbol{\sigma}, \boldsymbol{\tau}) + \langle \varepsilon \boldsymbol{\sigma} \cdot \mathbf{n}, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\partial\Omega}.$$

- The Dirichlet condition $\varepsilon = 0$ is now natural.
- The essential Neumann conditions are obtained by "penalizing", i.e. choosing ε "large".
- Question: does this lead to ill-conditioning? Answer: No!

Mixed methods FE

Formulation

$$
\begin{aligned}
a_\varepsilon(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\mathrm{div}\ \boldsymbol{\tau}, u_h) &= \langle u_0 + \varepsilon g, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\partial\Omega} & \forall \boldsymbol{\tau} \in \mathbf{S}_h, \\
(\mathrm{div}\ \boldsymbol{\sigma}_h, v) + (f, v) &= 0 & \forall v \in V_h.
\end{aligned}
$$

Pressure space

$$
V_h = \{ u \in L^2(\Omega) \,|\, u|_K \in P_{k-1}(K),\ K \in \mathcal{K}_h \}.
$$

Velocity

$$
\mathbf{S}_h^{RT} = \{ \boldsymbol{\sigma} \in H(\mathrm{div}, \Omega) \,|\, \boldsymbol{\sigma}|_K \in [P_{k-1}(K)]^n \oplus \mathbf{x}\widetilde{P}_{k-1}(K),\ K \in \mathcal{K}_h \},
$$

$$
\mathbf{S}_h^{BDM} = \{ \boldsymbol{\sigma} \in H(\mathrm{div}, \Omega) \,|\, \boldsymbol{\sigma}|_K \in [P_k(K)]^n,\ K \in \mathcal{K}_h \},
$$

Mesh dependent norms

- For the flux $\boldsymbol{\sigma}_h$ we use

$$|||\boldsymbol{\sigma}_h|||_{\varepsilon,h}^2 = \|\boldsymbol{\sigma}_h\|_0^2 + \sum_{E\in\mathcal{E}_h^\partial} (\varepsilon + h_E)\|\boldsymbol{\sigma}_h \cdot \mathbf{n}\|_{0,E}^2.$$

- And for the displacement u_h

$$|||u_h|||_{\varepsilon,h}^2 = \sum_{K\in\mathcal{K}_h} \|\nabla u_h\|_{0,K}^2 + \sum_{E\in\mathcal{E}_h^0} \frac{1}{h_E}\|[\![u_h]\!]\|_{0,E}^2 + \sum_{E\in\mathcal{E}_h^\partial} \frac{1}{\varepsilon + h_E}\|u_h\|_{0,E}^2.$$

Stability

Coercivity (not in the kernel).

Lemma

There is a constant $C > 0$ *such that*

$$a_\varepsilon(\boldsymbol{\sigma}, \boldsymbol{\sigma}) \geq C \|\boldsymbol{\sigma}\|_{\varepsilon,h}^2 \qquad \forall \boldsymbol{\sigma} \in \mathbf{S}_h.$$

Inf-sup

Lemma

There exists a constant $C > 0$ *such that*

$$\sup_{\boldsymbol{\sigma} \in \mathbf{S}_h} \frac{(\mathrm{div}\ \boldsymbol{\sigma}, u)}{\|\boldsymbol{\sigma}\|_{\varepsilon,h}} \geq C \|u\|_{\varepsilon,h} \qquad \forall u \in V_h.$$

Note: Since $\mathbf{S}_h^{RT} \subset \mathbf{S}_h^{BDM}$, it suffices to prove this for Raviart-Thomas-Nedelec.

A priori analysis I

Theorem

Let $P_h : L^2(\Omega) \to V_h$ be the L^2-projection and $\mathbf{R}_h : [H^1(\Omega)]^n \to \mathbf{S}_h$ be the RT/BDM interpolation operator. There exists a positive constant $C \neq C(\varepsilon)$ such that

$$\|\!|\boldsymbol{\sigma}_h - \mathbf{R}_h\boldsymbol{\sigma}|\!\|_{\varepsilon,h} + \|\!|P_h u - u_h|\!\|_{\varepsilon,h} \leq C\|\boldsymbol{\sigma} - \mathbf{R}_h\boldsymbol{\sigma}\|_0.$$

Proof. By the stability and consistency there is $(\boldsymbol{\tau}, v) \in \mathbf{S}_h \times V_h$ such that $\|\!|\boldsymbol{\tau}|\!\|_{\varepsilon,h} + \|\!|v|\!\|_{\varepsilon,h} \leq C$ and

$$\begin{aligned}
&\|\!|\boldsymbol{\sigma}_h - \mathbf{R}_h\boldsymbol{\sigma}|\!\|_{\varepsilon,h} + \|\!|u_h - P_h u|\!\|_{\varepsilon,h} \\
&\leq a_\varepsilon(\boldsymbol{\sigma} - \mathbf{R}_h\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\ \boldsymbol{\tau}, u - P_h u) + (\mathrm{div}\ (\boldsymbol{\sigma} - \mathbf{R}_h\boldsymbol{\sigma}), v) \\
&= (\boldsymbol{\sigma} - \mathbf{R}_h\boldsymbol{\sigma}, \boldsymbol{\tau}) + \sum_{E \in \mathcal{E}_h^\partial} \varepsilon\langle(\boldsymbol{\sigma} - \mathbf{R}_h\boldsymbol{\sigma})\cdot\mathbf{n}, \boldsymbol{\tau}\cdot\mathbf{n}\rangle_E \\
&= (\boldsymbol{\sigma} - \mathbf{R}_h\boldsymbol{\sigma}, \boldsymbol{\tau}).
\end{aligned}$$

A priori analysis II. Postprocessing

The postprocessed displacement $u_h^* \in V_h^* \supset V_h$ is defined on each element K by the conditions

$$\begin{aligned} P_h u_h^* &= u_h, \\ (\nabla u_h^*, \nabla v)_K &= (\boldsymbol{\sigma}_h, \nabla v)_K \qquad \forall v \in (I - P_h) V_h^*|_K, \end{aligned}$$

where we recall that P_h is the L^2-projection onto V_h.

Theorem

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|u - u_h^*\|_{\varepsilon,h} \leq C(\|\boldsymbol{\sigma} - \mathbf{R}_h \boldsymbol{\sigma}\|_0 + \inf_{v^* \in V_h^*} \|u - v^*\|_{\varepsilon,h}).$$

With $V_h = \{\, u \in L^2(\Omega) \,|\, u|_K \in P_{k-1}(K),\ K \in \mathcal{K}_h \,\}$, $k \geq 1$, the new space is:

$$V_h^* = \begin{cases} \{\, u^* \in L^2(\Omega) \,|\, u^*|_K \in P_k(K),\ K \in \mathcal{K}_h \,\} & \text{for RT elements,} \\ \{\, u^* \in L^2(\Omega) \,|\, u^*|_K \in P_{k+1}(K),\ K \in \mathcal{K}_h \,\} & \text{for BDM elements.} \end{cases}$$

Residual-based local error estimators.

- In the interior

$$\eta_{1,K}^2 = \|\nabla u_h^* - \boldsymbol{\sigma}_h\|_{0,K}^2, \quad \eta_{2,K}^2 = h_K^2 \|f - P_h f\|_{0,K}^2.$$

- On the interior edges

$$\eta_E^2 = h_E^{-1} \|[\![u_h^*]\!]\|_{0,E}^2.$$

- On the boundary edges

$$\eta_E^2 = \frac{1}{\varepsilon + h_E} \|\varepsilon(\boldsymbol{\sigma}_h \cdot \mathbf{n} - g_h) + u_h^* - u_0\|_{0,E}^2.$$

and

$$\eta_{g,E}^2 = h_E \|g - g_h\|_{0,K}^2,$$

where g_h is the L^2-projection of g to the trace space of $\mathbf{S}_h$.

A posteriori estimates

Let

$$\eta^2 = \sum_{K \in \mathcal{K}_h} \left(\eta_{1,K}^2 + \eta_{2,K}^2\right) + \sum_{E \in \mathcal{E}_h} \left(\eta_E^2 + \eta_{g,E}^2\right).$$

Theorem

There exists constants C_1, C_2, independent of ε, such that

$$C_1 \eta \leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + |||u - u_h^*|||_{\varepsilon,h} \leq C_2 \eta.$$

Remark. Looking at the boundary edges, we see

$$\eta_E^2 = \frac{1}{\varepsilon + h_E} \|\varepsilon(\boldsymbol{\sigma}_h \cdot \mathbf{n} - g_h) + u_h^* - u_0\|_{0,E}^2$$

we see that

$$\eta_E \approx \sqrt{\varepsilon} \|\boldsymbol{\sigma}_h \cdot \mathbf{n} - g_h\|_{0,E}$$

for ε "large". Does this go to infinity?

No!
Since a priori we have:

$$\|\!|\boldsymbol{\sigma}_h - \mathbf{R}_h\boldsymbol{\sigma}|\!\|_{\varepsilon,h} \leq C\|\boldsymbol{\sigma} - \mathbf{R}_h\boldsymbol{\sigma}\|_0$$

from which

$$\sqrt{\varepsilon}\|(\boldsymbol{\sigma}_h - \mathbf{R}_h\boldsymbol{\sigma}) \cdot \mathbf{n}\|_{0,\partial\Omega} \leq C\|\boldsymbol{\sigma} - \mathbf{R}_h\boldsymbol{\sigma}\|_0$$

and $\mathbf{R}_h\boldsymbol{\sigma} \cdot \mathbf{n} \approx g_h$, when ε is "large".

Hybridization

We introduce the multiplier spaces

$$M_h^{BDM} = \{\, m \in L^2(E) \,|\, m \in P_k(E),\ E \in \mathcal{E}_h^0, \quad m|_E = 0,\ E \in \mathcal{E}_h^\partial \,\}.$$
$$M_h^{RT} = \{\, m \in L^2(E) \,|\, m \in P_{k-1}(E),\ E \in \mathcal{E}_h^0, \quad m|_E = 0,\ E \in \mathcal{E}_h^\partial \,\}.$$

Remove the requirement $\boldsymbol{\sigma}_h \in H(\mathrm{div} : \Omega)$. Find $(\boldsymbol{\sigma}_h, u_h, m_h) \in \mathbf{S}_h \times V_h \times M_h$ such that

$$\sum_{K \in \mathcal{K}_h} \Big(a_{K,\varepsilon}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\mathrm{div}\ \boldsymbol{\tau}, u_h)_K + \langle \boldsymbol{\tau} \cdot \mathbf{n}, m_h \rangle_{\partial K} \Big) = \sum_{E \in \mathcal{E}_h^\partial} \langle u_0 + \varepsilon g, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_E,$$

$$\sum_{K \in \mathcal{K}_h} (\mathrm{div}\ \boldsymbol{\sigma}_h, v)_K + (f, v) = 0,$$

$$\sum_{K \in \mathcal{K}_h} \langle \boldsymbol{\sigma}_h \cdot \mathbf{n}, p \rangle_{\partial K} = 0,$$

Well-conditioning

For K with an edge E on the boundary, let $\boldsymbol{\sigma}_h = \boldsymbol{\sigma}_E + \boldsymbol{\sigma}_0$ with $\boldsymbol{\sigma}_0 \cdot \mathbf{n}|_E = 0$.
By testing by the corresponding $\boldsymbol{\tau}_E$, we get

$$\varepsilon\langle \boldsymbol{\sigma}_E \cdot \mathbf{n}, \boldsymbol{\tau}_E \cdot \mathbf{n}\rangle_E + (\boldsymbol{\sigma}_E, \boldsymbol{\tau}_E)_K + (\cdots) = \langle u_0 + \varepsilon g, \boldsymbol{\tau}\rangle_E,$$

and hence

$$\langle \boldsymbol{\sigma}_E \cdot \mathbf{n}, \boldsymbol{\tau}_E \cdot \mathbf{n}\rangle_E + \frac{1}{\varepsilon}(\boldsymbol{\sigma}_E, \boldsymbol{\tau}_E)_K + \frac{1}{\varepsilon}(\cdots) = \frac{1}{\varepsilon}\langle u_0, \boldsymbol{\tau}\rangle_E + \langle g, \boldsymbol{\tau} \cdot \mathbf{n}\rangle_E,$$

giving

$$\langle \boldsymbol{\sigma}_E \cdot \mathbf{n}, \boldsymbol{\tau}_E \cdot \mathbf{n}\rangle_E \approx \langle g, \boldsymbol{\tau} \cdot \mathbf{n}\rangle_E.$$

That is $\mathbf{R}_h\boldsymbol{\sigma} \cdot \mathbf{n} \approx g_h$.