Adaptive finite elements with large aspect ratio: theory and practice

M. Picasso

Institut d'Analyse et Calcul Scientifique
Ecole Polytechnique Fédérale de Lausanne

and

Projet Gamma
INRIA Rocquencourt

INRIA Rocquencourt, january 2010

Adaptive finite elements

- Theory + practice: a posteriori error estimates + adaptive algorithms $\rightarrow$ mathematicians can have the lead.
- A posteriori error estimates for elliptic, parabolic, hyperbolic ? nonlinear ? problems + efficient meshing tools (INRIA, LJLL) $\rightarrow$ can be used for industrial problems.
- Adaptive finite element with large aspect ratio: the ultimate tool to reduce the number of vertices given a prescribed level of accuracy.
- Remark: Sparse tensor product methods are anisotropic methods (shape functions).

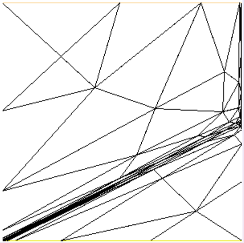
Finite elements with large aspect ratio

- General statement (mathematician): If nothing is known about the solution, then finite elements with large aspect ratio should not be used.
- A priori error estimates: $\|\nabla(u-u_h)\|_{L^2(\Omega)} \leq Ch|u|_{H^2(\Omega)}$ and C is large when the aspect ratio is large.
- But engineers use finite elements with large aspect ratio. Example (Dassault Aviation): viscous compressible flows around aircrafts, aspect ratio 10^3.
- More precise statement: finite elements with large aspect ratio can be used provided the mesh fits the solution.
- This is precisely the goal of adaptive finite elements with large aspect ratio.
- The theory of finite elements has to be revisited to handle meshes with large aspect ratio.

Examples

- Advection-diffusion (academic problem).
- Flows around aircrafts (no theory, allows very fast computations).
- Microfluidics (parabolic problem, theory and practice).

Example 1: 2D advection-diffusion

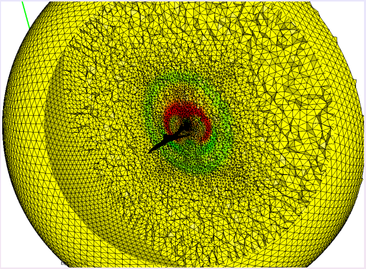


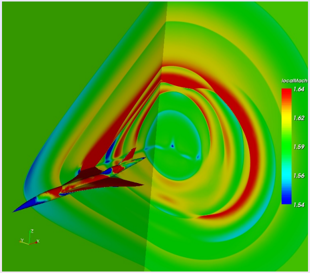
Mesh

Isovalues

- $-\epsilon\Delta u + \vec{a}\cdot\nabla u = f$, boundary layer 10^{-4}, 241 vertices, asp. ratio $O(10^5)$, same precision with isotropic mesh $O(10^5)$ vertices, Picasso SISC 2003.
- Design of the stabilization parameter: anisotropic a priori error estimates, Micheletti Perotto Picasso SINUM 2003.

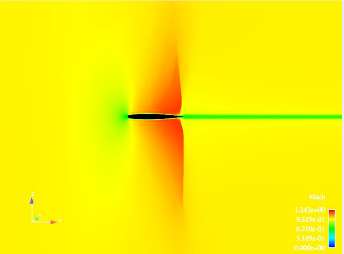
Example 2: 3D flows around aircrafts

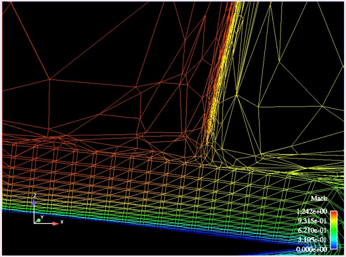




- Bourgault Picasso Alauzet Loseille IJNMF 2009, supported by Dassault Aviation.
- Compressible Euler solver (Alauzet), Anisotropic mesh generator (Dobrzynski Frey), Anisotropic error estimator.
- Animation, $Ma = 1.6$, $AoA = 3^o$, 10^6 vertices, single 32b processor.
- Goal oriented: see A. Loseille's PhD.

Example 2: 3D flows around aircrafts





- With W. Hassan, F. Alauzet, supported by Dassault Aviation.
- Compressible Navier-Stokes solver (Alauzet), Anisotropic mesh generator (Dobrzynski Frey), Anisotropic error estimator.

Example 3: Microfluidics

- Heat equation, Lozinski Picasso Prachittham SISC 2009.
- Unsteady convection-diffusion, Picasso Prachittham JCAM 2009.
- Microfluidics, Picasso Prachittham Gijs IJNMF 2009.
- Adaptive time steps and finite elements with large aspect ratio.
- Optimal a posteriori error estimates for the Crank Nicolson scheme.
- Animation.
- Animation.

A priori and a posteriori error estimates

- A priori
  - Isotropic meshes: $\exists C > 0$ (dep. aspect ratio, indep. u, h)

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \le Ch|u|_{H^2(\Omega)}.$$

  - Anisotropic meshes: Hessian matrix $H(u)$.
    - Frey George 2008, Chen Sun Xu 2007, Mirebeau Cohen 2009.
    - Formaggia Perotto 2001, $\exists C > 0$ (indep. aspect ratio, u, h)

$$\int_K |\nabla(u - r_h u)|^2 \le C\left(\frac{\lambda_{1,K}^4}{\lambda_{2,K}^2}\int_K (\vec{r}_{1,K}^T H(u)\vec{r}_{1,K})^2 + 2\lambda_{1,K}^2\int_K (\vec{r}_{1,K}^T H(u)\vec{r}_{2,K})^2 + \lambda_{2,K}^2\int_K (\vec{r}_{2,K}^T H(u)\vec{r}_{2,K})^2\right).$$

    - Ex: the mesh is aligned with u, $u = u(x_2)$, $\vec{r}_{1,K} = (1\ 0)^T$

$$\int_K |\nabla(u - r_h u)|^2 \le C\lambda_{2,K}^2 \int_K \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2.$$

    - Most anisotropic adaptive algorithms use a priori error estimates and an estimate of $H(u)$.
- A posteriori

A priori and a posteriori error estimates

- A priori
- A posteriori
  - Isotropic meshes: $\exists C_1, C_2 > 0$ (dep. aspect ratio, indep. u, h)

    $$C_1\eta \leq \|\nabla(u - u_h)\|_{L^2(\Omega)} \leq C_2\eta + h.o.t.,$$

    where the error estimator η is a computable quantity dep. on the mesh size, the data and on u_h.
  - Anisotropic meshes:
    - Kunert 2000: C_2 depends on the alignement of the mesh with the (unknown) solution u.
    - Picasso 2003, 2006: C_1 and C_2 indep. aspect ratio, u, h, provided the error estimator is equidistributed in the directions of min. and max. stretching.

Outline

- The anisotropic error estimator for the Laplace equation.
- Adaptive algorithms.
- Extension to parabolic problems.

The anisotropic error estimator for the Laplace problem

- Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

- Let $\mathcal{T}_h$ be a mesh of Ω into triangles K with diameter h_K less than h.
- Find $u_h \in V_h$ (continuous, piecewise linears) such that, for all $v_h \in V_h$

$$\int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h.$$

A posteriori H^1 error estimates - explicit, residual based error estimator

- The classical procedure to obtain an explicit, residual based error estimator is:

$$\int_\Omega |\nabla(u-u_h)|^2 = \int_\Omega f(u-u_h) - \int_\Omega \nabla u_h \cdot \nabla(u-u_h),$$

$$= \int_\Omega f(u-u_h-v_h) - \int_\Omega \nabla u_h \cdot \nabla(u-u_h-v_h) \qquad \forall v_h \in V_h,$$

$$= \sum_{K\in\mathcal{T}_h} \left(\int_K (f+\Delta u_h)(u-u_h-v_h) \right.$$

$$\left. + \frac{1}{2}\int_{\partial K} [\nabla u_h \cdot n](u-u_h-v_h) \right).$$

- Use Cauchy-Schwarz inequality, take $v_h = R_h(u-u_h)$ Clément interpolant, use anisotropic interpolation estimates to obtain ...

A posteriori H^1 error estimates - explicit, residual based error estimator

$$\int_\Omega |\nabla(u-u_h)|^2 \leq C \sum_{K\in\mathcal{T}_h} \eta_K^2,$$

- where C does not depend on the aspect ratio.
- Here $\eta_K^2 =$

$$\left(\|f+\Delta u_h\|_{L^2(K)} + \frac{1}{2}\left(\frac{|\partial K|}{\lambda_{1,K}\lambda_{2,K}}\right)^{1/2} \|[\nabla u_h \cdot n]\|_{L^2(\partial K)} \right)$$

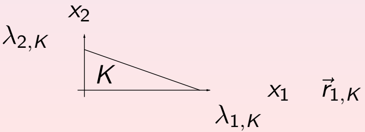
$$\left(\lambda_{1,K}^2 \left(\vec{r}_{1,K}^T G_K(u-u_h)\vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\vec{r}_{2,K}^T G_K(u-u_h)\vec{r}_{2,K} \right) \right)^{1/2},$$

- with $G_K(v) = \begin{pmatrix} \int_{\Delta_K} \left(\frac{\partial v}{\partial x_1}\right)^2 & \int_{\Delta_K} \frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2} \\ \int_{\Delta_K} \frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2} & \int_{\Delta_K} \left(\frac{\partial v}{\partial x_2}\right)^2 \end{pmatrix}.$

A posteriori H^1 error estimates - anisotropic case

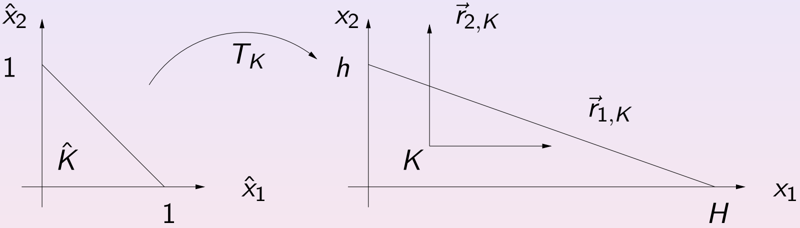
$$\eta_K^2 = \left(\|f + \Delta u_h\|_{L^2(K)} + \frac{1}{2} \left(\frac{|\partial K|}{\lambda_{1,K}\lambda_{2,K}} \right)^{1/2} \|[\nabla u_h \cdot n]\|_{L^2(\partial K)} \right)$$
$$\left(\lambda_{1,K}^2 \left(\vec{r}_{1,K}^T G_K(u-u_h) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\vec{r}_{2,K}^T G_K(u-u_h) \vec{r}_{2,K} \right) \right)^{1/2}.$$

- What are $\lambda_{1,K}$, $\lambda_{2,K}$, $\vec{r}_{1,K}$, $\vec{r}_{2,K}$?
- Lower bound?
- How to estimate $G(u-u_h)$?
- Isotropic case $\lambda_{1,K} \simeq \lambda_{2,K} \simeq h_K$ then the isotropic, explicit, residual based error estimator is recovered.
- Uniform stretching in the x_1 direction, $u(x_2)$, $u_h(x_2)$, then $\lambda_{1,K}$ can be arbitrary large.



Anisotropic interpolation estimates

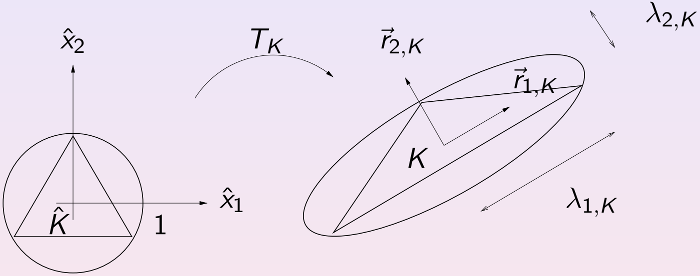
- Formaggia Perotto, Numer. Math. 2001, 2003.
- See also Kunert, Kunert Verfürth, Numer. Math. 2000.



- $\vec{x} = T_K(\hat{\vec{x}}) = M_K \hat{\vec{x}} + \vec{t}_K,$ s. v. d. $M_K = R_K^T \Lambda_K P_K$
- $R_K = \begin{pmatrix} \vec{r}_{1,K}^{\,T} \\ \vec{r}_{2,K}^{\,T} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \Lambda_K = \begin{pmatrix} \lambda_{1,K} & 0 \\ 0 & \lambda_{2,K} \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & h \end{pmatrix}.$

Anisotropic interpolation estimates

- Formaggia Perotto, Numer. Math. 2001, 2003.
- See also Kunert, Kunert Verfürth, Numer. Math. 2000.

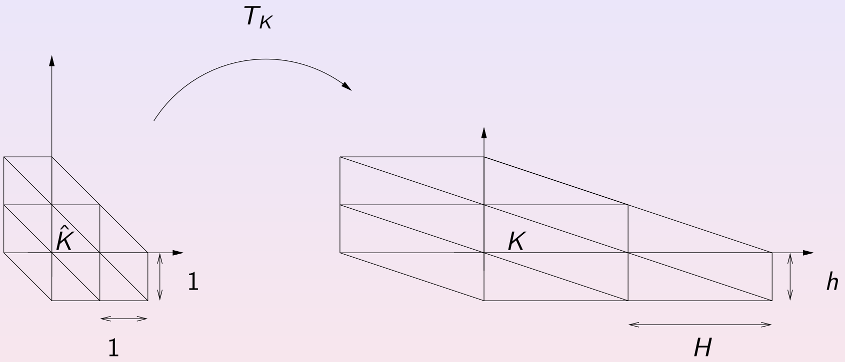


- $\vec{x} = T_K(\hat{\vec{x}}) = M_K \hat{\vec{x}} + \vec{t}_K,$ s. v. d. $M_K = R_K^T \Lambda_K P_K$
- The unit circle $\hat{\vec{x}}^T \hat{\vec{x}} = 1$ is mapped into the ellipse $(\vec{x} - \vec{t}_K)^T R_K^T \Lambda_K^{-2} R_K (\vec{x} - \vec{t}_K) = 1.$

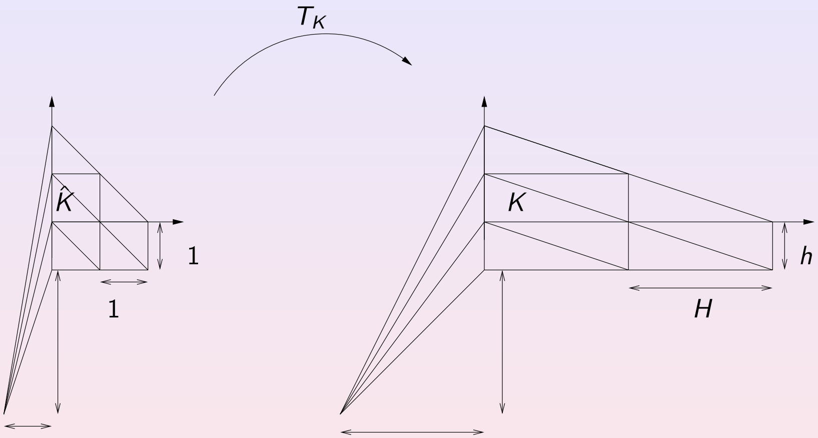
Anisotropic interpolation estimates

- Formaggia Perotto, Numer. Math. 2001, 2003.
- See also Kunert, Kunert Verfürth, Numer. Math. 2000.
- The number of neighbours must be bounded above.
- The reference patch $T_K^{-1}(\Delta_K)$ must be $O(1)$. This excludes meshes having "large curvature". In practice, anisotropic mesh generators satisfy this condition.
- No conditions about the angles.

Acceptable patch: $T_K^{-1}(\Delta_K) = O(1)$



Non acceptable patch: $T_K^{-1}(\Delta_K) \neq O(1)$



Clément interpolant - anisotropic interpolation estimates

- Formaggia Perotto, Numer. Math. 2001, 2003.
- See also Kunert, Kunert Verfürth, Numer. Math. 2000.
- Clément interpolation estimates : $\exists C > 0$ indep. mesh size and aspect ratio s.t. $\forall v \in H^1(\Omega)$, $\forall K \in \mathcal{T}_h$:

$$\|v - R_h v\|^2_{L^2(K)} + \frac{\lambda_{1,K}\lambda_{2,K}}{|\partial K|}\|v - R_h v\|^2_{L^2(\partial K)}$$

$$\leq C\left(\lambda^2_{1,K}\left(\vec{r}^{\,T}_{1,K} G_K(v)\vec{r}_{1,K}\right) + \lambda^2_{2,K}\left(\vec{r}^{\,T}_{2,K} G_K(v)\vec{r}_{2,K}\right)\right),$$

$$G_K(v) = \begin{pmatrix} \int_{\Delta_K}\left(\frac{\partial v}{\partial x_1}\right)^2 dx & \int_{\Delta_K}\frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2}dx \\ \int_{\Delta_K}\frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2}dx & \int_{\Delta_K}\left(\frac{\partial v}{\partial x_2}\right)^2 dx \end{pmatrix}.$$

A posteriori H^1 error estimates - anisotropic case

$$\eta_K^2 = \left(\|f + \Delta u_h\|_{L^2(K)} + \frac{1}{2} \left(\frac{|\partial K|}{\lambda_{1,K}\lambda_{2,K}} \right)^{1/2} \|[\nabla u_h \cdot n]\|_{L^2(\partial K)} \right)$$
$$\left(\lambda_{1,K}^2 \left(\vec{r}_{1,K}^T G_K(u - u_h) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\vec{r}_{2,K}^T G_K(u - u_h) \vec{r}_{2,K} \right) \right)^{1/2}.$$

- Lower bound?
- How to estimate $G(u - u_h)$?

Lower bound?

$$\eta_K^2 = \left(\|f + \Delta u_h\|_{L^2(K)} + \frac{1}{2} \left(\frac{|\partial K|}{\lambda_{1,K}\lambda_{2,K}} \right)^{1/2} \|[\nabla u_h \cdot n]\|_{L^2(\partial K)} \right)$$

$$\left(\lambda_{1,K}^2 \left(\vec{r}_{1,K}^T G_K(u - u_h) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\vec{r}_{2,K}^T G_K(u - u_h) \vec{r}_{2,K} \right) \right)^{1/2}.$$

- Mimicking the proof of Verfürth (bubbles): if the mesh is such that $\exists C_1 > 0$ (indep. mesh size and aspect ratio) s.t.

  $$\lambda_{1,K}^2 \left(\vec{r}_{1,K}^T G_K(u - u_h) \vec{r}_{1,K} \right) \leq C_1 \, \lambda_{2,K}^2 \left(\vec{r}_{2,K}^T G_K(u - u_h) \vec{r}_{2,K} \right) \qquad \forall K \in \mathcal{T}_h$$

  then $\exists C_2 > 0$ (indep. mesh size and aspect ratio) s.t.

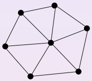
  $$\text{estimator} \leq C_2 \text{ error} + h.o.t.$$

- Suggests to use an adaptive algorithm such that

  $$\lambda_{1,K}^2 \left(\vec{r}_{1,K}^T G_K(e) \vec{r}_{1,K} \right) \simeq \lambda_{2,K}^2 \left(\vec{r}_{2,K}^T G_K(e) \vec{r}_{2,K} \right) \qquad \forall K \in \mathcal{T}_h$$

How to estimate $G(u-u_h)$? Zienkiewicz-Zhu ZZ (post-processing)

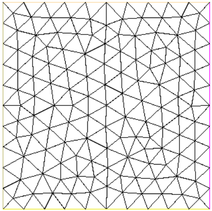
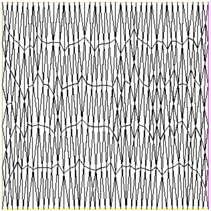
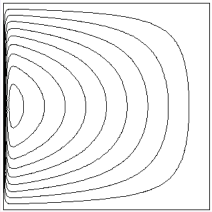
- From ∇u_h, compute a better gradient $\rightarrow \Pi_h \nabla u_h$ (Π_h is an approximation of the L^2 projection of ∇u_h onto V_h).



- Then $\eta^{ZZ} = \left(\int_\Omega |\nabla u_h - \Pi_h \nabla u_h|^2 \right)^{1/2}$ is asymptotically exact on parallel meshes (Rodriguez 1994, Ainsworth Oden 1997), mildly structured meshes (Xu Zhang 2003), equivalent to the true error on general unstructured meshes (Carstensen 2004).
- Superconvergence

$$\|\nabla u - \nabla u_h\|_{L^2(\Omega)} \leq \|\nabla u - \Pi_h \nabla u_h\|_{L^2(\Omega)} + \|\Pi_h \nabla u_h - \nabla u_h\|_{L^2(\Omega)}.$$

How to estimate $G(u - u_h)$? Zienkiewicz-Zhu ZZ (post-processing)

-   

-

$h1 - h2$ (1 : 1)	error	ei
0.01 – 0.01	1.36	0.81
0.005 – 0.005	0.69	0.92
0.0025 – 0.0025	0.35	0.97

$h1 - h2$ (1 : 8)	error	ei
0.005 – 0.04	0.65	0.94
0.0025 – 0.02	0.33	0.98
0.00125 – 0.01	0.16	0.99

- Open question: why ZZ so good on general meshes?

How to estimate $G(u - u_h)$? Zienkiewicz-Zhu ZZ (post-processing)

$$\eta_K^2 = \left(\|f + \Delta u_h\|_{L^2(K)} + \frac{1}{2} \left(\frac{|\partial K|}{\lambda_{1,K}\lambda_{2,K}} \right)^{1/2} \|[\nabla u_h \cdot n]\|_{L^2(\partial K)} \right)$$
$$\left(\lambda_{1,K}^2 \left(\vec{r}_{1,K}^{\,T} G_K(u-u_h) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\vec{r}_{2,K}^{\,T} G_K(u-u_h) \vec{r}_{2,K} \right) \right)^{1/2}.$$

- See also Kunert Nicaise M2AN 2003.
- Zienkiewicz-Zhu

$$\int_K \left(\frac{\partial (u-u_h)}{\partial x_1} \right)^2 \rightarrow \int_K \left(\Pi_h \frac{\partial u_h}{\partial x_1} - \frac{\partial u_h}{\partial x_1} \right)^2.$$

- Question: could we use only ZZ ?

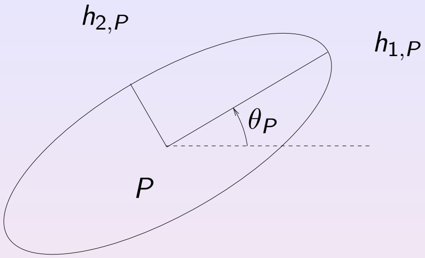
$$\eta_K^2 = \vec{r}_{1,K}^{\,T} G_K(u-u_h) \vec{r}_{1,K} + \vec{r}_{2,K}^{\,T} G_K(u-u_h) \vec{r}_{2,K}$$

Anisotropic, adaptive finite elements

- Goal: find $\mathcal{T}_h$ s.t. $0.75\ TOL \leq \dfrac{\left(\sum_{K\in\mathcal{T}_h} \eta_K^2\right)^{1/2}}{\left(\int_\Omega |\nabla u_h|^2\right)^{1/2}} \leq 1.25\ TOL$
- Sufficient condition (N_K is the number of triangles):
  $$\frac{0.75^2\, TOL^2 \int_\Omega |\nabla u_h|^2}{N_K} \leq \eta_K^2 \leq \frac{1.25^2\, TOL^2 \int_\Omega |\nabla u_h|^2}{N_K}$$
- Sufficient condition in the directions of maximum ($i = 1$) and minimum ($i = 2$) stretching:

$$\frac{0.75^2\, TOL^2 \int_\Omega |\nabla u_h|^2}{\sqrt{2} N_K}$$
$$\leq \left(\|f\|_{L^2(K)} + \frac{1}{2}\left(\frac{|\partial K|}{\lambda_{1,K}\lambda_{2,K}}\right)^{1/2} \|\,[\nabla u_h \cdot \vec{n}]\,\|_{L^2(\partial K)} \right) \left(\lambda_{i,K}^2 \left(\vec{r}_{i,K}^{\,T} G_K(e) \vec{r}_{i,K}\right)\right)^{1/2}$$
$$\leq \frac{1.25^2\, TOL^2 \int_\Omega |\nabla u_h|^2}{\sqrt{2} N_K}$$

Anisotropic, adaptive finite elements



- Equidistribute the error in directions 1 and 2
- Align the triangle with the eigenvectors of $G_K(u - u_h)$
- 2D: use the `BL2D` mesh generator (INRIA, Borouchaki, Laug) or the `BAMG` mesh generator (Hecht).
- 3D: use the `MeshAdapt` mesh generator (INRIA, George Hecht Saltel, Distene) or the `MMG3D` mesh generator (Dobrzynski Frey).

Adaptive meshes for the Laplace problem in 2D

- 2D: $TOL = 0.25$, 30 mesh generations, animation, zoom.
- The effectivity index is aspect ratio independent on adapted meshes

TOL	vertices	error	ei	ei^{ZZ}	asp. ratio
0.125	854	0.25	2.70	1.00	262
0.0625	2793	0.13	2.75	0.99	288
0.03125	10812	0.062	2.79	0.95	425
0.015625	42562	0.031	2.79	0.98	1199

- 3D: $TOL = 0.25$, 30 mesh generations, animation, zoom.

The heat equation with space discretization only

- $\dfrac{\partial u}{\partial t} - \Delta u = f$ in $\Omega \times (0, T)$.
- Find $u_h : t \rightarrow u_h(\cdot, t) \in V_h$ such that, for all $t \in (0, T)$, for all $v_h \in V_h$

$$\int_\Omega \frac{\partial u_h}{\partial t} v_h dx + \int_\Omega \nabla u_h \cdot \nabla v_h dx = \int_\Omega f v_h dx.$$

- $e = u - u_h$

$$\left\langle \frac{\partial e}{\partial t}, e \right\rangle + \int_\Omega |\nabla e|^2 dx = \int_\Omega \left(f - \frac{\partial u_h}{\partial t} \right) e \, dx - \int_\Omega \nabla u_h \cdot \nabla e dx$$
$$= \int_\Omega \left(f - \frac{\partial u_h}{\partial t} \right) (e - v_h) \, dx - \int_\Omega \nabla u_h \cdot \nabla (e - v_h) dx$$

- Use Cauchy-Schwarz inequality, take $v_h = R_h e$ Clément interpolant, use anisotropic interpolation estimates to obtain ...

The heat equation with space discretization only

$$\frac{1}{2}\int_\Omega (u-u_h)^2(T)dx + \int_0^T \int_\Omega |\nabla(u-u_h)(t)|^2 dxdt \leq \frac{1}{2}\int_\Omega (u-u_h)^2(0)dx + C\int_0^T \sum_{K\in\mathcal{T}_h} \eta_K^2,$$

- where C does not depend on the mesh size and aspect ratio.
- Here $\eta_K^2 =$

$$\left(\|f - \frac{\partial u_h}{\partial t} + \Delta u_h\|_{L^2(K)} + \frac{1}{2}\left(\frac{|\partial K|}{\lambda_{1,K}\lambda_{2,K}}\right)^{1/2} \|[\nabla u_h \cdot n]\|_{L^2(\partial K)} \right)$$

$$\left(\lambda_{1,K}^2 \left(\vec{r}_{1,K}^T G_K(u-u_h)\vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\vec{r}_{2,K}^T G_K(u-u_h)\vec{r}_{2,K} \right) \right)^{1/2}.$$

The heat equation with Euler backward scheme

- $\dfrac{\partial u}{\partial t} - \Delta u = f$ in $\Omega \times (0, T)$.
- For $n = 1, ..., N$, find $u_h^n \in V_h$ such that, for all $v_h \in V_h$

$$\frac{1}{\tau}\int_\Omega (u_h^n - u_h^{n-1}) v_h dx + \int_\Omega \nabla u_h^n \cdot \nabla v_h dx = \int_\Omega f^n v_h v dx.$$

- Isotropic meshes: Picasso 1998, Verfürth 2003, Bergam Bernardi Mghazli 2004.
- $u_{h\tau}(x,t) = \dfrac{t - t^{n-1}}{\tau} u_h^n(x) + \dfrac{t^n - t}{\tau} u_h^{n-1}(x)$.
- $\displaystyle\int_\Omega \frac{\partial u_{h\tau}}{\partial t} v_h dx + \int_\Omega \nabla u_{h\tau} \cdot \nabla v_h dx$
  $\displaystyle= \int_\Omega f v_h dx + \int_\Omega (f^n - f) v_h dx + (t^n - t)\int_\Omega \nabla \frac{\partial u_{h\tau}}{\partial t} \cdot \nabla v_h dx.$

The heat equation with Euler backward scheme

- $e = u - u_{h\tau}$

$$\left\langle \frac{\partial e}{\partial t}, e \right\rangle + \int_\Omega |\nabla e|^2 dx$$
$$= \int_\Omega \left(f - \frac{\partial u_{h\tau}}{\partial t} \right) e \; dx - \int_\Omega \nabla u_{h\tau} \cdot \nabla e dx$$
$$= \int_\Omega \left(f - \frac{\partial u_{h\tau}}{\partial t} \right) (e - v_h) \; dx - \int_\Omega \nabla u_{h\tau} \cdot \nabla (e - v_h) dx$$
$$+ \int_\Omega (f - f^n) v_h dx + (t^n - t) \int_\Omega \nabla \frac{\partial u_{h\tau}}{\partial t} \cdot \nabla v_h dx.$$

- Take $v_h = R_h e$, use anisotropic interpolation estimates:

$$\frac{1}{2} \int_\Omega (u - u_{h\tau})^2 (T) dx + \int_0^T \int_\Omega |\nabla (u - u_{h\tau})(t)|^2 dx dt$$
$$\leq \frac{1}{2} \int_\Omega (u - u_{h\tau})^2 (0) dx + C \sum_{n=1}^{N} \sum_{K \in \mathcal{T}_h} \eta_{K,n}^2.$$

- with C independent of u, the mesh size and aspect ratio and

$$\eta^2_{K,n}$$

$$= \int_{t^{n-1}}^{t^n} \left\{ \left(\left\| f - \frac{\partial u_{h\tau}}{\partial t} + \Delta u_{h\tau} \right\|_{L^2(K)} + \frac{1}{2\lambda_{2,K}^{1/2}} \|[\nabla u_{h\tau} \cdot \vec{n}]\|_{L^2(\partial K)} \right) \right.$$

$$\left(\lambda_{1,K}^2 \left(\vec{r}_{1,K}^{\,T} G_K(u - u_{h\tau}) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\vec{r}_{2,K}^{\,T} G_K(u - u_{h\tau}) \vec{r}_{2,K} \right) \right)^{1/2}$$

$$\left. + \|f - f^n\|^2_{L^2(K)} + \tau_n^2 \left\| \nabla \frac{\partial u_{h\tau}}{\partial t} \right\|^2_{L^2(K)} \right\} dt.$$

The heat equation with Crank-Nicolson scheme

- Optimal $L^2(H^1)$ a posteriori error estimates for Crank-Nicolson time discretization: Akrivis Makridadkis Nochetto, Math. Comp. 2006.
- For $n = 1, ..., N$, find $u_h^n \in V_h$ such that, for all $v_h \in V_h$

$$\frac{1}{\tau}\int_\Omega (u_h^n - u_h^{n-1}) v_h dx + \frac{1}{2}\int_\Omega \nabla(u_h^{n-1} + u_h^n)\cdot\nabla v_h dx = \frac{1}{2}\int_\Omega (f^{n-1} + f^n) v_h dx.$$

- $u_{h\tau}(x,t) = \dfrac{t - t^{n-1}}{\tau} u_h^n(x) + \dfrac{t^n - t}{\tau} u_h^{n-1}(x).$
- $$\int_\Omega \frac{\partial u_{h\tau}}{\partial t} v_h dx + \int_\Omega \nabla u_{h\tau}\cdot\nabla v_h dx = \frac{1}{2}\int_\Omega (f^{n-1} + f^n) v_h dx + (t - t^{n-1/2})\int_\Omega \nabla\frac{\partial u_{h\tau}}{\partial t}\cdot\nabla v_h dx.$$

The three points time reconstruction

- Lozinski Prachittham Picasso SISC 2009.
- Lagrange interpolant of degree 2: for $t^{n-1} \leq t \leq t^n$

$$\begin{aligned}\widetilde{u}_{h\tau}(x,t) &= \frac{(t-t^{n-2})(t-t^{n-1})}{(t^n-t^{n-2})(t^n-t^{n-1})} u_h^n \\ &+ \frac{(t-t^n)(t-t^{n-2})}{(t^{n-1}-t^n)(t^{n-1}-t^{n-2})} u_h^{n-1} \\ &+ \frac{(t-t^n)(t-t^{n-1})}{(t^{n-2}-t^n)(t^{n-2}-t^{n-1})} u_h^{n-2}.\end{aligned}$$

- Or equivalently

$$\widetilde{u}_{h\tau}(x,t) = u_{h\tau}(x,t) + \frac{1}{2}(t-t^{n-1})(t-t^n)\partial_n^2 u_h,$$

$$\partial_n^2 u_h = \frac{\dfrac{u_h^n - u_h^{n-1}}{\tau} - \dfrac{u_h^{n-1} - u_h^{n-2}}{\tau}}{\tau}.$$

Error indicator

Approach $\int_{t^1}^{T} \int_{\Omega} |\nabla(u - u_{h\tau})|^2$ by $\sum_{n=2}^{N} \sum_{K \in \mathcal{T}_h} \eta_{K,n}^2$ with $\eta_{K,n}^2 = \int_{t^{n-1}}^{t^n} \Bigg\{$

$$\left(\left\| f - \frac{\partial u_{h\tau}}{\partial t} + \Delta u_{h\tau} \right\|_{L^2(K)} + \frac{1}{2\lambda_{2,K}^{1/2}} \, \|[\nabla u_{h\tau} \cdot \vec{n}]\|_{L^2(\partial K)} \right)$$

$$\left(\lambda_{1,K}^2 \left(\vec{r}_{1,K}^{\,T} G_K(u - \widetilde{u}_{h\tau}) \vec{r}_{1,K} \right) + \lambda_{2,K}^2 \left(\vec{r}_{2,K}^{\,T} G_K(u - \widetilde{u}_{h\tau}) \vec{r}_{2,K} \right) \right)^{1/2}$$

$$+ \left\| f - \left(f^{n-1/2} + (t - t^{n-1/2}) \frac{f^n - f^{n-2}}{2\tau} \right) \right\|_{L^2(K)}^2$$

$$+ \tau^4 \left\| \nabla \partial_n^2 u_h \right\|_{L^2(K)}^2 \Bigg\} dt.$$

Adaptive space-time algorithm

- Choose the time step and the mesh size so that

$$0.875\,TOL \leq \frac{\eta^{space} + \eta^{time}}{\left(\int_0^T \int_\Omega |\nabla u_{h\tau}|^2 \; dx \; dt\right)^{1/2}} \leq 1.125\,TOL.$$

- Test case 1 Animation
- Test case 2 Animation

TOL	$error$	ei^{ZZ}	ei^{space}	ei^{time}	$Vert$	N	$Gen.$	AR
0.125	0.03	0.99	2.87	2.68	155	142	84	63
0.0625	0.015	0.99	2.89	2.91	348	201	52	108
0.03125	0.0078	0.99	2.96	2.99	892	285	52	165
0.015625	0.0040	1.00	2.88	2.71	4408	401	40	118

- Conclusion: $Vert = O(TOL^{-2})$ and $N = O(TOL^{-1/2})$.

Conclusions

- Adaptive finite elements with large aspect ratio $\rightarrow$ reduce the number of vertices.
- Anisotropic interpolation estimates (Formaggia-Perotto, Kunert) + ZZ postprocessing $\rightarrow$ residual based, explicit anisotropic error estimator for the H^1 norm.
- Effectivity index aspect ratio independent whenever the estimator is equidistributed in the direction of maximum and minimum stretching.
- Applied to a wide range of elliptic and parabolic problems, very well suited for problems with boundary or internal layers.

Perspectives

- Goal oriented estimations $\rightarrow$ Formaggia, Micheletti, Perotto et al.
- Polynomial Preserving Recovery (Naga Zhang 2004) instead of ZZ post-processing.
- Residual based, implicit error estimators?
- Numerical linear algebra with anisotropic meshes?
- Hyperbolic problems? Compressible Navier-Stokes?
- Interpolation error induced by remeshing?
- Convergence of anisotropic adaptive finite element algorithms?