Convergence analysis and error estimates of adaptive finite element methods

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1 Outline

- Adaptive finite element methods (AFEM)
- Adaptive conforming finite element methods
- Adaptive nonconforming finite element methods
- Adaptive mixed finite element methods
- Extensions and open problem

Adaptive finite element methods (AFEM)

Let h_0 be an initial triangulation and set $k = 0$.

- SOLVE: Compute the solution u_k of the discrete problem;
- ESTIMATE: Compute an estimator for the error in terms of the discrete solution u_k and given data;
- MARK: Use the estimator to mark a subset $\mathcal{M}_k$ (edges or cells) for refinement.
- REFINE: Refine the marked subset $\mathcal{M}_k$ to obtain the mesh h_{k+1}, increase k and go to step SOLVE.
- Popular for more than 30 years, why?
- How about the convergence and convergence rate of the error?

Convergence history of AFEM (residual-based a posteriori error estimator)

- Babuska and Rheinboldt [1978] (1D)
- Dörfler [1996] (2D): oscillation small enough
- Morin, Nochetto, and Siebert [2000] : mark oscillation in every step by interior node property
- Binev, Dahmen, and DeVore [2004]: complexity estimate (need coarsening)
- Stevenson [2007]: complexity estimate without coarsening
- Cascon, Kreuzer, Nochetto, Siebert [2008]: without marking oscillation and no interior node property

Our contribution: joint with Roland Becker and Zhongci Shi

- For adaptive conforming linear elements: introduce an adaptive marking strategy and an adaptive stopping criterion for the iterative solution of the discrete system
- The obtained refinement will in general be dominated by the edge residuals
- Convergence analysis and quasi-optimal complexity
- Optimal error estimate in 2D
- Extensions to adaptive mixed finite element methods
- Extensions to adaptive nonconforming finite element methods
- Extensions to adaptive finite element methods for Stokes problem

Adaptive conforming finite element methods

Model problem and linear approximations

For simplicity, we consider

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The Ritz projection $u_h \in V_h$ is defined by

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h, \tag{2}$$

where V_h is the standard linear conforming finite element space.

Error indicators

We define the family of admissible meshes $\mathcal{H}$. For any $h \in \mathcal{H}$, the set of interior edges is denoted by $\mathcal{E}_h$ and the set of nodes by $\mathcal{N}_h$.
Let ω_z be the set of cells joining a node $z \in \mathcal{N}_h$ and $\pi_\omega(f) := \int_\omega f\,dx/|\omega|$.
We define

$$\mathrm{osc}_z := |\omega_z|^{1/2}\,\|f - \pi_{\omega_z} f\|_{\omega_z}, \mathrm{osc}_h^2(\mathcal{P}) := \sum_{z\in\mathcal{P}} \mathrm{osc}_z^2$$

$$J_E(v_h) := |E|^{1/2}\,\|[\frac{\partial v_h}{\partial n}]\|_E, J_h^2(v_h,\mathcal{F}) := \sum_{E\in\mathcal{F}} J_E^2(v_h).$$

We set for brevity $\mathrm{osc}_h := \mathrm{osc}_h(\mathcal{N}_h)$ and $J_h(v_h) := J_H(v_h, \mathcal{E}_h)$.

A posteriori error estimate for iteration errors

- Let u_h^m be an iterative solution and $\zeta_h(u_h^m)$ be an estimator satisfying

$$|u_h - u_h^m|_1^2 \le C_{it}\zeta_h^2(u_h^m). \tag{3}$$

- A simple one for some iteration methods (CG, MG):

$$\zeta_h(u_h^m) := |u_h^{m+1} - u_h^m|_1. \tag{4}$$

- A posteriori estimate for CG: [ArioliGeorgoulis09], [StrakovsVohralik09]
- We also developed a practical one for MG:

$$\zeta_h(u_h^m) := \sum_{j=1}^{k} \|h_{j-1} R_j(\widetilde{v}_j)\|, \tag{5}$$

where $R_j(\widetilde{v}_j)$ can be related to the residuals appearing in the multigrid iteration.

Algorithm 1: collective marking

- Choose parameters $0 < \theta, \alpha < 1$ and an initial mesh h_0, and set $k = 0$.
- Do m_k iterations for the discrete system (2) to obtain $u_{h_k}^{m_k}$, m_k is determined by:

$$\zeta_{h_k}^2(u_{h_k}^{m_k}) \leq \alpha\,(J_{h_k}^2(u_{h_k}^{m_k}) + \mathrm{osc}_{h_K}^2). \tag{6}$$

- Mark a set $\mathcal{F} \subset \mathcal{E}_{h_k}$ with minimal cardinality such that

$$J_{h_k}^2(\mathcal{F}) + \mathrm{osc}_{h_k}^2(\mathcal{F}) \geq \theta\,(J_{h_k}^2(u_{h_k}^{m_k}) + \mathrm{osc}_{h_k}^2).$$

- Adapt the mesh : $h_{k+1} := \mathit{Refine}(h_k, \mathcal{F})$.
- Set $k := k + 1$ and go to the next step.

Algorithm 2: adaptive marking

- Choose parameters $0 < \theta, \alpha, \sigma < 1$, $\gamma > 0$ and an initial mesh h_0, and set $k = 0$.
- Do m_k iterations for the discrete system (2) to obtain $u_{h_k}^{m_k}$, m_k is determined by (6).
- If
$$\mathrm{osc}_{h_k}^2 \leq \gamma J_{h_k}^2(u_{h_k}^{m_k}),$$
mark a set $\mathcal{F} \subset \mathcal{E}_{h_k}$ with minimal cardinality such that
$$J_{h_k}^2(\mathcal{F}) \geq \theta\, J_{h_k}^2(u_{h_k}^{m_k}). \tag{7}$$
else find a set $\mathcal{P} \subset \mathcal{N}_{h_k}$ with minimal cardinality such that
$$\mathrm{osc}_{h_k}^2(\mathcal{P}) \geq \sigma\, \mathrm{osc}_{h_k}^2. \tag{8}$$
- Adapt the mesh : $h_{k+1} := \mathit{Refine}(h_k, \mathcal{F})$.

Upper bounds

Lemma 1

(upper bounds) *Let $h \in \mathcal{H}$. There exists a constant $C_1 > 0$ depending only on the minimum angle of h_0 such that for $u_h \in V_h$ the solution of (47) and arbitrary $w_h \in V_h$*

$$|u - w_h|_1^2 \leq C_1 (J_h^2(w_h) + osc_h^2) + 2\,|u_h - w_h|_1^2. \tag{9}$$

Suppose in addition that $H \in \mathcal{H}$ and $\mathcal{F} \subset \mathcal{E}_H$ are such that $h = \mathcal{R}_{loc}(H, \mathcal{F})$. Letting $\mathcal{P} \subset \mathcal{N}_H$ the set of nodes included in $\mathcal{F}$ and $u_H \in V_H$ the discrete solution, we have

$$|u_h - w_H|_1^2 \leq C_1 \left(J_H^2(w_H, \mathcal{F}) + osc_H^2(\mathcal{P}) + |u_H - w_H|_1^2 \right) \quad \forall w_H \in V_H, \tag{10}$$

and

$$\#\mathcal{F} \leq C_3\,(N_h - N_H). \tag{11}$$

Lower bounds

Lemma 2

(lower bounds) *There exists a constant $C_2 > 0$ depending only on the minimum angle of h_0 such that for all $v_H \in V_H$*

$$J_H^2(v_H) \le C_2 \left(|u - v_H|_1^2 + osc_H^2 \right). \tag{12}$$

There exists a constant $C_4 > 0$ depending only on the minimum angle of h_0 such that for $\mathcal{F} \subset \mathcal{E}_H$, $h = \mathcal{R}_{loc}(H, \mathcal{F})$ and arbitrary $\delta > 0$

$$J_h^2(v_h) \le (1+\delta) J_H^2(v_H) - \frac{1+\delta}{2} J_H^2(v_H, \mathcal{F}) + C_4 (1 + 1/\delta) |v_h - v_H|_1^2 \quad \forall v_h \in V_h, v_H \in V \tag{13}$$

Convergence of Algorithm 1

Theorem 3

Let $\{h_k\}_{k\geq 0}$ be a sequence of meshes generated by Algorithm 1 and let $\{u_{h_k}^{m_k}\}_{k\geq 0}$ be the corresponding sequence of finite element solutions. Suppose that

$$0 < \alpha < C^*\theta^2, \tag{14}$$

then there exist constants $\beta_1 > 0$, $\beta_2 > 0$, and $\rho < 1$ such that for all $k = 1, 2, \ldots$

$$e(h_{k+1}, m_{k+1}) \leq \rho\, e(h_k, m_k), \tag{15}$$

where $e(h, m) := |u - u_h^m|_1^2 + \beta_1\, osc_h^2 + \beta_2\, J_h^2(u_h^m)$.

Optimal Marking cardinality and class of approximation

Assumption Let $h_k, k = 0, \ldots n$ be a sequence of locally refined meshes created by the local mesh refinement algorithm, starting from the initial mesh h_0. Let $\mathcal{F}_k \subset \mathcal{E}_{h_k}, k = 0, \ldots n-1$ be the collection of all marked edges in step k. Then there exists a mesh-independent constant C_0 such that

$$N_{h_n} \leq N_{h_0} + C_0 \sum_{k=0}^{n-1} \#\mathcal{F}_k. \tag{16}$$

(16) is known to be true for the newest vertex bisection algorithm, see [BinevDahmenDeVore04] and [Stevenson08].
Next we define the approximation class

$$\mathcal{W}^s := \left\{ (u, f) \in (H_0^1(\Omega), L^2(\Omega)) : \|(u, f)\|_{\mathcal{W}^s} < +\infty \right\}. \tag{17}$$

with

$$\|(u, f)\|_{\mathcal{W}^s} := \sup_{N \geq N_0} N^s \inf_{h \in H_N} \left(|u - u_h|_1^2 + \mathrm{osc}_h^2 \right).$$

Quasi-optimality and error estimate of Algorithm 1

Theorem 4

Let $\{h_k\}_{k\geq 0}$ be a sequence of meshes generated by Algorithm 1 and let $\{u_{h_k}^{m_k}\}_{k\geq 0}$ be the corresponding sequence of finite element solutions. Suppose that

$$0 < \alpha < C^*\theta^2, 0 < \theta < \theta^* < 1, \tag{18}$$

then we have the following estimate on the complexity of the algorithm:

$$N_k \leq C\,\varepsilon_k^{-1/s}. \tag{19}$$

Furthermore, in case of 2D, there exists $k_0 \geq 1$, such that for all $k = k_0, k_0+1, \ldots$, we have

$$e(h_k, m_k) \leq C\,(N_k - N_{k_0})^{-1}\|f\|^2. \tag{20}$$

Convergence of Algorithm 2

Theorem 5

Let $\{h_k\}_{k\geq 0}$ be a sequence of meshes generated by Algorithm 2 and let $\{u_{h_k}^{m_k}\}_{k\geq 0}$ be the corresponding sequence of finite element solutions. Suppose that

$$0 < \alpha < C^*\theta^2, \tag{21}$$

then there exist constants $\beta_1 > 0$, $\beta_2 > 0$, and $\rho < 1$ such that for all $k = 1, 2, \ldots$

$$e(h_{k+1}, m_{k+1}) \leq \rho\, e(h_k, m_k). \tag{22}$$

Quasi-optimality and error estimate of Algorithm 2

Theorem 6

Let $\{h_k\}_{k\geq 0}$ be a sequence of meshes generated by Algorithm 2 and let $\{u_{h_k}^{m_k}\}_{k\geq 0}$ be the corresponding sequence of FE solutions. Suppose

$$0 < \alpha < C^*\theta^2, 0 < \theta < \theta^* < 1, 0 < \gamma < \gamma^*, \tag{23}$$

then we have the following estimate on the complexity of the algorithm:

$$N_k \leq C\,\varepsilon_k^{-1/s}. \tag{24}$$

Furthermore, in case of 2D, there exists $k_0 \geq 1$, such that for all $k = k_0, k_0 + 1, \ldots$, we have

$$e(h_k, m_k) \leq C\,(N_k - N_{k_0})^{-1}\|f\|^2. \tag{25}$$

Features of the results

- Optimal convergence rate after a finite steps.
- In the **SOLVE** step: CG, MG will be stopped by an adaptive stopping criteria with $\sqrt{\alpha}(J_h + \text{osc}_h)$, compared with a fixed stopping criterion (e.g., 10^{-8}) in the usual way.
- In the **REFINE** step: no interior node property, which admits almost all the classical refine rules, e.g., newest vertex bisection, reg-green -refinement, etc.
- In Algorithm 2, the edge residuals alone dominate the error estimation in most cases, which verifies the well known result in practice.

Numerical experiment

- We solve Poisson's equation on the L-shaped domain with Dirichlet boundary condition. The exact solution is $u(r,\theta) = r^{2/3}\sin(\frac{2\theta}{3})$.
- Based on the local multigrid algorithm developed by Chen and Wu [06], which has optimal computational cost for discrete systems of PDE.

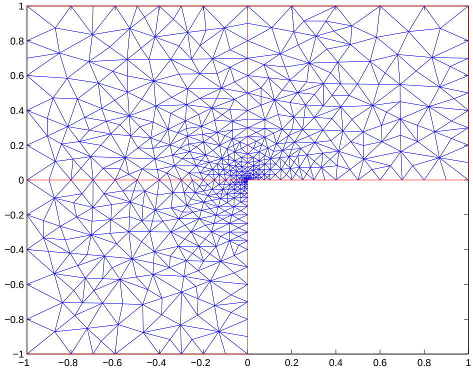

Figure 1: Adaptive mesh with 1005 elements (11 step)

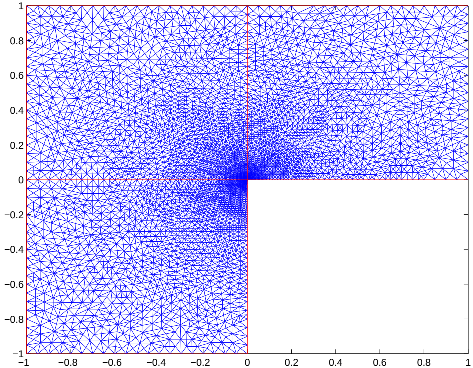

Figure 2: Adaptive mesh with 11327 elements (22 step)

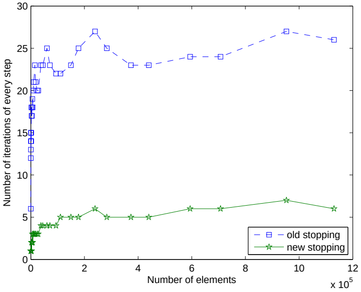

Figure 5: Number of iterations of every step: 26 vs 6

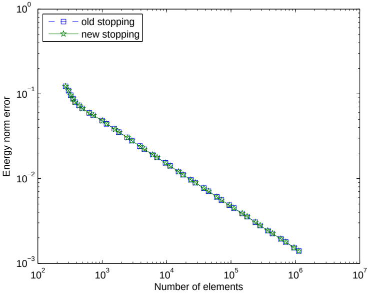

Figure 7: The energy error

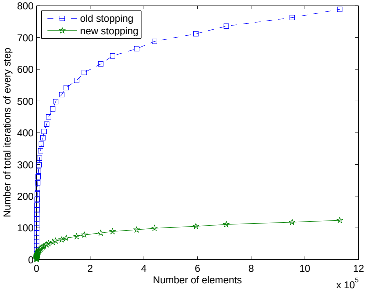

Figure 8: Total iterations: 795 vs 124

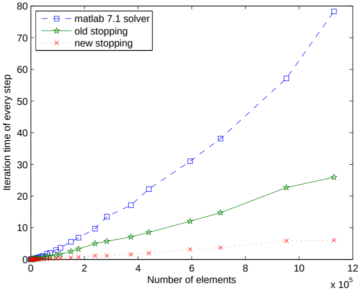

Figure 9: Iteration time of every step: 79 vs 25.5 vs 5.5

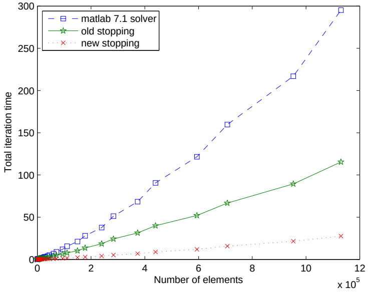

Figure 10: Total iteration time: 291 vs 119.5 vs 29.2

Adaptive nonconforming finite element methods

Adaptive nonconforming finite element methods (ANFEM)

- We develop a practical adaptive algorithm for linear nonconforming finite element method.
- It is based on an adaptive marking strategy and an adaptive stopping criteria for iterative solution.
- We prove its convergence and optimal error estimate
- The main difficulties are the proof of the quasi-orthogonality, local upper and lower bounds of ANFEM.

Let V_h denote the nonconforming P_1 finite element space (Crouzeix-Raviart element over $\mathcal{T}_h$, which is given by

$$V_h := \left\{ v_h \in L^2(\Omega); \forall K \in \mathcal{K}_h, v_h|_K \in P_1(K); \forall E \in \mathcal{E}_h, \int_E [v_h]_E ds = 0 \right\},$$

here $[v_h]_E$ stands for the jump of v_h across E and vanishes when $E \subset \partial\Omega$.
Let u_h denote the solution of the discrete problem

$$\begin{cases} \text{Find } u_h \in V_h, \text{ such that} \\ a_h(u_h, v_h) = (f, v_h), \forall\ v_h \in V_h, \end{cases} \tag{26}$$

where $a_h(u_h, v_h) = \sum_{K \in \mathcal{K}_h} \int_K \nabla u_h \nabla v_h \, dx$.
We suppose that $\zeta_h^2(u_h^m)$ satisfies the following upper bound

$$|u_h - u_h^m|_{1,h}^2 \le C_{it} \zeta_h^2(u_h^m). \tag{27}$$

Set

$$\| \cdot \|_h = \left(\sum_{K \in \mathcal{K}_h} | \cdot |_{1,K}^2 \right)^{\frac{1}{2}}.$$

We define edge residuals for $E \in \mathcal{E}_h$ and any subset $\mathcal{F} \subset \mathcal{E}_h$

$$\eta_{h,E}(v_h) := h_E^{1/2} \left\| [\frac{\partial v_h}{\partial s}] \right\|_{0,E}, \quad \eta_h(v_h, \mathcal{F}) := \left(\sum_{E \in \mathcal{F}} \eta_{h,E}^2(v_h) \right)^{1/2}, \tag{28}$$

together with volume residuals for $K \in \mathcal{K}_h$ and any subset $\mathcal{M} \subset \mathcal{K}_h$

$$\mu_K := |K|^{1/2} \|f\|_{0,K}, \quad \mu_h(\mathcal{M}) := \left(\sum_{K \in \mathcal{M}} \mu_K^2 \right)^{1/2}. \tag{29}$$

We next define an oscillation term by

$$\mathrm{osc}_E := |\omega_E|^{1/2} \|f - \pi_{\omega_E} f\|_{0,\omega_E}, \quad \mathrm{osc}_h(\mathcal{F}) := \left(\sum_{E \in \mathcal{F}} \mathrm{osc}_E^2 \right)^{1/2}, \tag{30}$$

where $\pi_{\omega_E} f := \int_{\omega_E} f\, dx / |\omega_E|$. We set for brevity $\eta_h(v_h) := \eta_h(v_h, \mathcal{E}_h)$, $\mathrm{osc}_h := \mathrm{osc}_h(\mathcal{E}_h)$ and $\mu_h := \mu_h(\mathcal{K}_h)$.

Algorithm ANFEM

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(0) Choose parameters $0 < \theta, \sigma < 1$, $\gamma > 0$, $\alpha > 0$ and an initial mesh h_0, and set $k = 0$.

(1) Do m_k iterations of the multigrid algorithm applied to the discrete system (26) with h replaced by h_k to obtain the finite element solution $u_{h_k}^{m_k}$. The integer m_k is determined by the condition to be the smallest integer verifying:

$$\zeta_{h_k}^2(u_{h_k}^{m_k}) \leq \alpha\,(\eta_{h_k}^2(u_{h_k}^{m_k}) + \mu_{h_k}^2). \tag{31}$$

(2)
- If $\mu_{h_k}^2 \leq \gamma\, \eta_{h_k}^2(u_{h_k}^{m_k})$ then mark a subset $\mathcal{F}$ of $\mathcal{E}_{h_k}$ with minimal cardinality such that
$$\eta_{h_k}^2(u_{h_k}^{m_k}, \mathcal{F}) \geq \theta\, \eta_{h_k}^2(u_{h_k}^{m_k}). \tag{32}$$
- else find a set $\mathcal{M} \subset \mathcal{K}_{h_k}$ with minimal cardinality such that
$$\mu_{h_k}^2(\mathcal{M}) \geq \sigma\, \mu_{h_k}^2. \tag{33}$$
and define $\mathcal{F}$ to be the set of edges contained in at least one cell $K \in \mathcal{M}$.

(3) Adapt the mesh : $h_{k+1} := \mathcal{R}_{loc}(h_k, \mathcal{F})$.

(4) Set $k := k + 1$ and go to step (1).

Upper bounds

Lemma 7

(global upper bound) *There exists a constant $C_1 > 0$ depending only on the minimum angle of $\mathcal{K}_{h_0}$ such that for the multigrid solution $u_h^m \in V_h$, we have*

$$|u - u_h^m|_{1,h}^2 \leq C_1 \left(\eta_h^2(u_h^m) + \mu_h^2 \right). \tag{34}$$

Lemma 8

(local upper bound) *There exist constants $C_4, C_5 > 0$ depending only on the minimum angle of $\mathcal{K}_{h_0}$ such that the following holds. For any mesh $H \in \mathcal{H}$ and any local refinement $h \in \mathcal{H}$ of H let $\mathcal{F} \subset \mathcal{E}_H$ be the set of refined edges. The corresponding multigrid coarse $u_H^l \in V_H$ and fine-grid solutions $u_h \in V_h$ satisfy*

$$|u_h - u_H^l|_{1,h}^2 \leq C_4 \left(\eta_H^2(u_H^l, \mathcal{F}) + \mu_H^2 + \alpha \left(\eta_H^2(u_H^l) + \mu_H^2 \right) \right). \tag{35}$$

Lemma 9

(global lower bounds) *There exist constants $C_2, C_3 > 0$ depending only on the minimum angle of $\mathcal{K}_{h_0}$ such that the following estimates hold for the multigrid solution $u_h^m \in V_h$:*

$$\eta_h^2(u_h^m) \leq C_2 \, |u - u_h^m|_{1,h}^2 \tag{36}$$

and

$$\mu_h^2 \leq C_3 \left(|u - u_h^m|_{1,h}^2 + osc_h^2 \right) . \tag{37}$$

Lemma 10

(local lower bounds) *There exist constants $C_6, C_7 > 0$ depending only on the minimum angle of $\mathcal{K}_{h_0}$ such that for $\mathcal{F} \subset \mathcal{E}_H$, $h = \mathcal{R}_{loc}(H, \mathcal{F})$, there holds:*

$$\eta_H^2(u_H^l, \mathcal{F}) \leq C_6 |u_h^m - u_H^l|_{1,h}^2, \tag{38}$$

If $\mathcal{M} \subset \mathcal{K}_H$ is the set of refined cells, there holds:

$$\mu_H^2(\mathcal{M}) \leq C_7 \Big(|u_h^m - u_H^l|_{1,h}^2 + \alpha(\eta_H^2(u_H^l) + \mu_h^2) + osc_H^2(\mathcal{F}) + \alpha(\eta_h^2(u_h^m) + \mu_H^2) \Big). \tag{39}$$

Quasi-orthogonality

Lemma 11

(quasi-orthogonality) *Let $h, H \in \mathcal{H}$ be two nested meshes and $\mathcal{M} \subset \mathcal{K}_H$ be the set of refined cells. Then there exists a constant $C_8 > 0$ depending only on the minimum angle in $\mathcal{K}_{h_0}$ such that*

$$\begin{aligned}&(\nabla_h(u-u_h^m), \nabla_h(u_h^m - u_H^l)) \le \\ &|u-u_h^m|_{1,h}\left(C_8\,\mu_H(\mathcal{M}) + \sqrt{\alpha}\left(\sqrt{\eta_h^2(u_h^m)+\mu_h^2} + \sqrt{\eta_H^2(u_H^l)+\mu_H^2}\right)\right),\end{aligned} \tag{40}$$

$$(\nabla_h(u-u_h), \nabla_h(u_h - u_H^l)) \le |u-u_h|_{1,h}\left(C_8\,\mu_H(\mathcal{M}) + \sqrt{\alpha}\sqrt{\eta_H^2(u_H^l)+\mu_H^2}\right) \tag{41}$$

and

$$(\nabla_h(u-u_h), \nabla_h(u_h - u_H)) \le C_8\,\mu_H(\mathcal{M})|u-u_h|_{1,h}. \tag{42}$$

Convergence of ANFEM

Theorem 12

Let $\{h_k\}_{k\geq 0}$ be a sequence of meshes generated by algorithm ***ANFEM*** *and let $\{u_{h_k}\}_{k\geq 0}$ be the corresponding sequence of finite element solutions. Suppose that*

$$0 < \alpha \leq C^* \theta^2, \tag{43}$$

with a generic constant C^* to be defined in the proof. Then there exist $\beta > 0$ and $0 < \rho < 1$ such that for all $k = 1, 2, \ldots$

$$e(h_{k+1}) \leq \rho \, e(h_k) \tag{44}$$

with $e(h) := |u - u_h^m|_{1,h}^2 + \beta \, \mu_h^2$.

Quasi-optimality of ANFEM

Theorem 13

*Suppose $(u, f) \in \mathcal{W}^s$. Let $\{h_k\}_{k\geq 0}$ be a sequence of meshes generated by algorithm **ANFEM** and let $\{V_k\}_{k\geq 0}$ and $\{u_{h_k}\}_{k\geq 0}$ be the corresponding sequences of finite element spaces and solutions. Let $\varepsilon_k := \sqrt{|u - u_{h_k}|^2_{1,h_k} + \beta\mu^2_{h_k}}$. Assuming that the parameters γ, θ and α satisfy (61) and*

$$\gamma < \frac{1 - 3\alpha C_2}{C_2(C_4 + 2C_8^2 + 3\alpha)}, \quad \alpha + \theta < \frac{1 - 3\alpha C_2}{C_2 C_4} - \gamma(1 + \frac{2C_8^2 + 3\alpha}{C_4}). \quad (45)$$

Then we have the following estimate on the complexity of the algorithm: there exists a constant C such that for all $k = 0, 1, 2, \ldots$

$$N_k \leq C\,\varepsilon_k^{-1/s}. \quad (46)$$

Adaptive mixed finite element methods

The Raviart-Thomas space $V_h \subset H(\text{div};\Omega)$ is defined as

$$V_h = \{\tau_h \in H(\text{div};\Omega); \tau_h|_K \in P_0(K)^2 \oplus \mathbf{x}P_0(K), \ \forall \ K \in \mathcal{K}_h\}.$$

Q_h is the space of piecewise constant functions.
The discrete solution $(\sigma_h, u_h) \in V_h \times Q_h$ approximating $(\nabla u, u)$ in (1) is defined by

$$\langle \sigma_h, \tau_h \rangle + \langle \text{div}\, \tau_h, u_h \rangle + \langle \text{div}\, \sigma_h, v_h \rangle = \langle f, v_h \rangle \quad \forall (\tau_h, v_h) \in V_h \times Q_h. \tag{47}$$

In order to estimate the iteration error, we use an a posteriori error estimator $\zeta_h^2(\sigma_h^m)$ which is supposed to satisfy the upper bound

$$\|\sigma_h - \sigma_h^m\|^2 \le C_{it}\zeta_h^2(\sigma_h^m). \tag{48}$$

Next we define edge residuals for $E \in \mathcal{E}_h$ and any given subset $\mathcal{F} \subseteq \mathcal{E}_h$

$$\eta_{h,E}(\tau_h) := h_E^{1/2}\|[\tau_h \cdot t_E]\|_E, \quad \eta_h(\tau_h, \mathcal{F}) := \left(\sum_{E\in\mathcal{F}} \eta_{h,E}^2(\tau_h)\right)^{1/2}. \tag{49}$$

Algorithm **AMFEM**

(0) Choose parameters $0 < \theta, \sigma < 1$, $\gamma > 0$, $\alpha > 0$ and an initial mesh h_0, and set $k = 0$.

(1) Do m_k iterations of the discrete system (47) to obtain $\sigma_{h_k}^{m_k}$, m_k is determined by the condition to be the smallest integer verifying:

$$\zeta_{h_k}^2(\sigma_{h_k}^{m_k}) \leq \alpha\, \eta_{h_k}^2(\sigma_{h_k}^{m_k}). \tag{50}$$

(2) Compute the a posteriori error estimator $\eta_{h_k}(\sigma_{h_k}^{m_k})$ and the oscillation term osc_{h_k}.

(3)
- If $\mathrm{osc}_{h_k}^2 \leq \gamma\, \eta_{h_k}^2(\sigma_{h_k}^{m_k})$ then mark a set $\mathcal{F}$ of $\mathcal{E}_{h_k}$ with minimal cardinality such that
$$\eta_{h_k}^2(\sigma_{h_k}^{m_k}, \mathcal{F}) \geq \theta\, \eta_{h_k}^2(\sigma_{h_k}^{m_k}). \tag{51}$$
- else find a set $\mathcal{M} \subset \mathcal{K}_{h_k}$ with minimal cardinality such that
$$\mathrm{osc}_{h_k}^2(\mathcal{M}) \geq \sigma\, \mathrm{osc}_{h_k}^2. \tag{52}$$
and define $\mathcal{F}$ to be the set of edges contained in at least one cell $K \in \mathcal{M}$.

(4) Adapt the mesh : $h_{k+1} := \mathcal{R}_{loc}(h_k, \mathcal{F})$.

(5) Set $k := k + 1$ and go to step (1).

Upper bounds

Lemma 14

(global upper bound) *There exists a constant $C_1 > 0$ depending only on the minimum angle of h_0 such that for the multigrid solution $\sigma_h^m \in V_h$, we have*

$$\|\sigma - \sigma_h^m\|^2 \le C_1 \left(\eta_h^2(\sigma_h^m) + osc_h^2\right). \tag{53}$$

Lemma 15

(local upper bound) *There exist constants $C_3, C_5 > 0$ depending only on the minimum angle of h_0 such that the following holds. For any subset $\mathcal{F} \subset \mathcal{E}_H$, $h = \mathcal{R}_{loc}(H, \mathcal{F})$, and $\mathcal{M}$ the set of refined cells, the iterative solutions $\sigma_H^l \in V_H$ and $\sigma_h \in V_h$, we have*

$$\|\sigma_h - \sigma_H^l\|^2 \le C_3 \left(\eta_H^2(\sigma_H^l, \mathcal{F}) + osc_H^2(\mathcal{M})\right) + \alpha\, \eta_H^2, \tag{54}$$

and

$$\#\mathcal{F} \le C_5\,(N_h - N_H). \tag{55}$$

Lower bounds

Lemma 16

(global lower bounds) *There exists a constant $C_2 > 0$ depending only on the minimum angle of h_0 such that the multigrid solution $\sigma^l_H \in V_H$ satisfies*

$$\eta^2_H(\sigma^l_H) \le C_2 \,\|\sigma - \sigma^l_H\|^2. \tag{56}$$

Lemma 17

(local lower bounds) *There exists a constant $C_4 > 0$ depending only on the minimum angle of h_0 such that for $\mathcal{F} \subset \mathcal{E}_H$, $h = \mathcal{R}_{loc}(H, \mathcal{F})$ and $\mathcal{M} \subset \mathcal{K}_H$ the set of refined cells there holds:*

$$\eta^2_H(\sigma^l_H, \mathcal{F}) \le C_4 \left(\|\sigma^m_h - \sigma^l_H\|^2 + osc^2_H(\mathcal{M}) + \alpha\, \eta^2_H(\sigma^l_H) \right). \tag{57}$$

Quasi-orthogonality

Lemma 18

Let $h, H \in \mathcal{H}$ be two nested meshes and $\mathcal{M} \subset \mathcal{K}_H$ be the set of refined cells. Then there exists a constant $C_6 > 0$ depending only on the minimum angle of h_0 such that

$$\begin{aligned} \langle \sigma - \sigma_h^m, \sigma_h^m - \sigma_H^l \rangle &\leq \sqrt{\alpha}\, \eta_h(\sigma_h^m) \|\sigma_h^m - \sigma_H^l\| \\ &+ \|\sigma - \sigma_h^m\| \left(C_6 osc_H(\mathcal{M}) + \sqrt{\alpha}(\eta_h(\sigma_h^m) + \eta_H(\sigma_H^l)) \right), \end{aligned} \tag{58}$$

and

$$\langle \sigma - \sigma_h, \sigma_h - \sigma_H^l \rangle \leq \|\sigma - \sigma_h\| \left(C_6 osc_H(\mathcal{M}) + \sqrt{\alpha}\, \eta_H(\sigma_H^l) \right). \tag{59}$$

If we solve both of the discretized equations exactly on the meshes h and H, then we have

$$\langle \sigma - \sigma_h, \sigma_h - \sigma_H \rangle \leq C_6 osc_H(\mathcal{M}) \|\sigma - \sigma_h\|. \tag{60}$$

Convergence of algorithm **AMFEM**

Theorem 19

*Let $\{h_k\}_{k\geq 0}$ be a sequence of meshes generated by algorithm **AMFEM** and let $\{\sigma_{h_k}^{m_k}\}_{k\geq 0}$ be the corresponding sequence of iterative finite element solutions. Suppose that*

$$0 < \alpha \leq C^* \theta^2, \tag{61}$$

Then there exist $\beta > 0$ and $\rho < 1$ such that for all $k = 1, 2, \ldots$

$$e(h_{k+1}) \leq \rho\, e(h_k) \tag{62}$$

with

$$e(h) := \|\sigma - \sigma_h^m\|^2 + \beta\, osc_h^2. \tag{63}$$

We define the approximation class

$$\mathcal{W}^s := \Big\{(\sigma, f) \in (H(\mathrm{div}, \Omega), L^2(\Omega)) : \|(\sigma, f)\|_{\mathcal{W}^s} < +\infty\Big\}. \qquad (64)$$

with

$$\|(\sigma, f)\|_{\mathcal{W}^s} := \sup_{N \geq N_0} N^s \inf_{h \in H_N} \Big(\|\sigma - \sigma_h\| + \mu_h\Big).$$

Theorem 20

*Let $\{h_k\}$ be a sequence of meshes generated by algorithm **AMFEM**and $\{\sigma_{h_k}^{m_k}\}_{k\geq 0}$ be the corresponding iterative FE solutions. Assuming*

$$0 < \gamma < \frac{1}{C_2(C_3 + 2C_6^2)}, \quad \theta + \frac{3\alpha}{C_3} < \frac{1}{C_2 C_3} - \gamma(1 + \frac{2C_6^2}{C_3}), \qquad (65)$$

then there exists a constant C such that

$$N_k \leq C\,\varepsilon_k^{-1/s}. \qquad (66)$$

In case of 2D, there exists $k_0 \geq 1$, such that for all $k = k_0, k_0 + 1, \ldots$, we have

$$\|\sigma - \sigma_{h_k}^{m_k}\|^2 + \beta osc_k^2 \leq C(N_k - N_0)^{-1}. \qquad (67)$$

Extensions and open problem

Extensions and open problem

- Adaptive mixed (conforming and nonconforming) FEM for the Stokes problem (submitted).
- Adaptive FEM for the optimal control problem (submitted).
- Adaptive H(curl) FEM for Maxwell problem, based on the local MG method [HiptmairZheng09] (in preparation).
- Open problem: Adaptive hp FEM, exponential convergence rate?

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Thank you for your attention!