Convergence analysis and error estimates of adaptive finite element methods

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Outline

- Adaptive finite element methods (AFEM)
- Adaptive conforming finite element methods
- Adaptive nonconforming finite element methods
- Adaptive mixed finite element methods
- Extensions and open problem
Adaptive finite element methods (AFEM)
Let $h_0$ be an initial triangulation and set $k = 0$.

- **SOLVE**: Compute the solution $u_k$ of the discrete problem;
- **ESTIMATE**: Compute an estimator for the error in terms of the discrete solution $u_k$ and given data;
- **MARK**: Use the estimator to mark a subset $M_k$ (edges or cells) for refinement.
- **REFINE**: Refine the marked subset $M_k$ to obtain the mesh $h_{k+1}$, increase $k$ and go to step SOLVE.

Popular for more than 30 years, why?

How about the convergence and convergence rate of the error?
Convergence history of AFEM (residual-based a posteriori error estimator)

- Babuska and Rheinboldt [1978] (1D)
- Dörfler [1996] (2D): oscillation small enough
- Morin, Nochetto, and Siebert [2000]: mark oscillation in every step by interior node property
- Binev, Dahmen, and DeVore [2004]: complexity estimate (need coarsening)
- Stevenson [2007]: complexity estimate without coarsening
- Cascon, Kreuzer, Nochetto, Siebert [2008]: without marking oscillation and no interior node property
Our contribution: joint with Roland Becker and Zhongci Shi

- For adaptive conforming linear elements: introduce an adaptive marking strategy and an adaptive stopping criterion for the iterative solution of the discrete system
- The obtained refinement will in general be dominated by the edge residuals
- Convergence analysis and quasi-optimal complexity
- Optimal error estimate in 2D
- Extensions to adaptive mixed finite element methods
- Extensions to adaptive nonconforming finite element methods
- Extensions to adaptive finite element methods for Stokes problem
Adaptive conforming finite element methods
Adaptive conforming finite element methods

Model problem and linear approximations

For simplicity, we consider

\begin{align*}
-\Delta u &= f, & \text{in } \Omega \subset \mathbb{R}^2, \\
 u &= 0, & \text{on } \partial \Omega.
\end{align*}

(1)

The Ritz projection $u_h \in V_h$ is defined by

\[(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h,\]

where $V_h$ is the standard linear conforming finite element space.

(2)
Error indicators

We define the family of admissible meshes $\mathcal{H}$. For any $h \in \mathcal{H}$, the set of interior edges is denoted by $\mathcal{E}_h$ and the set of nodes by $\mathcal{N}_h$. Let $\omega_z$ be the set of cells joining a node $z \in \mathcal{N}_h$ and $\pi_\omega(f) := \int_\omega f \, dx / |\omega|$. We define

$$\text{osc}_z := |\omega_z|^{1/2} \| f - \pi_\omega z f \|_{\omega_z}, \quad \text{osc}_h^2(\mathcal{P}) := \sum_{z \in \mathcal{P}} \text{osc}_z^2$$

$$J_E(v_h) := |E|^{1/2} \| \left[ \frac{\partial v_h}{\partial n} \right] \|_E, \quad J_h^2(v_h, \mathcal{F}) := \sum_{E \in \mathcal{F}} J_E^2(v_h).$$

We set for brevity $\text{osc}_h := \text{osc}_h(\mathcal{N}_h)$ and $J_h(v_h) := J_H(v_h, \mathcal{E}_h)$. 
Inexact solutions of the discrete problem

A posteriori error estimate for iteration errors

- Let $u_h^m$ be an iterative solution and $\zeta_h(u_h^m)$ be an estimator satisfying
  \[ |u_h - u_h^m|_1^2 \leq C_{it} \zeta_h^2(u_h^m). \]  
  (3)

- A simple one for some iteration methods (CG, MG):
  \[ \zeta_h(u_h^m) := |u_h^{m+1} - u_h^m|_1. \]  
  (4)

- A posteriori estimate for CG: [ArioliGeorgoulis09], [StrakovsVohralik09]

- We also developed a practical one for MG:
  \[ \zeta_h(u_h^m) := \sum_{j=1}^{k} \|h_{j-1} R_j(\tilde{v}_j)\|, \]  
  (5)

  where $R_j(\tilde{v}_j)$ can be related to the residuals appearing in the multigrid iteration.
Algorithm 1: collective marking

- Choose parameters $0 < \theta, \alpha < 1$ and an initial mesh $h_0$, and set $k = 0$.
- Do $m_k$ iterations for the discrete system (2) to obtain $u_{h_k}^{m_k}$, $m_k$ is determined by:

  \[ \zeta_{h_k}^2(u_{h_k}^{m_k}) \leq \alpha (J_{h_k}^2(u_{h_k}^{m_k}) + \text{osc}_{h_k}^2). \]  

- Mark a set $\mathcal{F} \subset \mathcal{E}_{h_k}$ with minimal cardinality such that

  \[ J_{h_k}^2(\mathcal{F}) + \text{osc}_{h_k}^2(\mathcal{F}) \geq \theta (J_{h_k}^2(u_{h_k}^{m_k}) + \text{osc}_{h_k}^2). \]

- Adapt the mesh: $h_{k+1} := \text{Refine}(h_k, \mathcal{F})$.
- Set $k := k + 1$ and go to the next step.
Algorithm 2: adaptive marking

- Choose parameters $0 < \theta, \alpha, \sigma < 1$, $\gamma > 0$ and an initial mesh $h_0$, and set $k = 0$.
- Do $m_k$ iterations for the discrete system (2) to obtain $u_{h_k}^{m_k}$, $m_k$ is determined by (6).
- If
  \[ \text{osc}_{h_k}^2 \leq \gamma J_{h_k}^2(u_{h_k}^{m_k}), \]
  mark a set $\mathcal{F} \subset \mathcal{E}_{h_k}$ with minimal cardinality such that
  \[ J_{h_k}(\mathcal{F}) \geq \theta J_{h_k}(u_{h_k}^{m_k}). \]  
  \[ \text{(7)} \]
- else find a set $\mathcal{P} \subset \mathcal{N}_{h_k}$ with minimal cardinality such that
  \[ \text{osc}_{h_k}^2(\mathcal{P}) \geq \sigma \text{osc}_{h_k}^2. \]
  \[ \text{(8)} \]
- Adapt the mesh: $h_{k+1} := \text{Refine}(h_k, \mathcal{F})$. 
Upper bounds

Lemma 1

*(upper bounds)* Let $h \in \mathcal{H}$. There exists a constant $C_1 > 0$ depending only on the minimum angle of $h_0$ such that for $u_h \in V_h$ the solution of (47) and arbitrary $w_h \in V_h$

$$|u - w_h|^2 \leq C_1(J_h^2(w_h) + \text{osc}_h^2) + 2|u_h - w_h|^2. \quad (9)$$

Suppose in addition that $H \in \mathcal{H}$ and $\mathcal{F} \subset \mathcal{E}_H$ are such that $h = R_{\text{loc}}(H, \mathcal{F})$. Letting $\mathcal{P} \subset \mathcal{N}_H$ the set of nodes included in $\mathcal{F}$ and $u_H \in V_H$ the discrete solution, we have

$$|u_h - w_H|^2 \leq C_1(J_H^2(w_H, \mathcal{F}) + \text{osc}_H^2(\mathcal{P}) + |u_H - w_H|^2) \quad \forall w_H \in V_H, \quad (10)$$

and

$$\#\mathcal{F} \leq C_3(N_h - N_H). \quad (11)$$
**Lower bounds**

**Lemma 2**

**Lower bounds** There exists a constant $C_2 > 0$ depending only on the minimum angle of $h_0$ such that for all $v_H \in V_H$

$$J_H^2(v_H) \leq C_2 \left(|u - v_H|^2_1 + \text{osc}_H^2 \right). \quad (12)$$

There exists a constant $C_4 > 0$ depending only on the minimum angle of $h_0$ such that for $\mathcal{F} \subset \mathcal{E}_H$, $h = R_{loc}(H, \mathcal{F})$ and arbitrary $\delta > 0$

$$J_h^2(v_h) \leq (1+\delta)J_H^2(v_H) - \frac{1}{2} \delta J_H^2(v_H, \mathcal{F}) + C_4(1+1/\delta)|v_h - v_H|^2_1 \quad \forall v_h \in V_h, v_H \in V_H. \quad (13)$$
Convergence of Algorithm 1

**Theorem 3**

Let \( \{h_k\}_{k \geq 0} \) be a sequence of meshes generated by Algorithm 1 and let \( \{u_{h_k}^m\}_{k \geq 0} \) be the corresponding sequence of finite element solutions. Suppose that

\[
0 < \alpha < C^* \theta^2,
\]

then there exist constants \( \beta_1 > 0, \beta_2 > 0, \text{ and } \rho < 1 \) such that for all \( k = 1, 2, \ldots \)

\[
e(h_{k+1}, m_{k+1}) \leq \rho e(h_k, m_k),
\]

where \( e(h, m) := |u - u_h^m|^2_1 + \beta_1 \text{osc}_h^2 + \beta_2 J_h^2(u_h^m) \).
Optimal Marking cardinality and class of approximation

**Assumption** Let $h_k, k = 0, \ldots n$ be a sequence of locally refined meshes created by the local mesh refinement algorithm, starting from the initial mesh $h_0$. Let $\mathcal{F}_k \subset \mathcal{E}_{h_k}, k = 0, \ldots n - 1$ be the collection of all marked edges in step $k$. Then there exists a mesh-independent constant $C_0$ such that

$$N_{h_n} \leq N_{h_0} + C_0 \sum_{k=0}^{n-1} \# \mathcal{F}_k. \quad (16)$$

(16) is known to be true for the newest vertex bisection algorithm, see [BinevDahmenDeVore04] and [Stevenson08].

Next we define the approximation class

$$\mathcal{W}^s := \left\{ (u, f) \in (H_0^1(\Omega), L^2(\Omega)) : \|(u, f)\|_{\mathcal{W}^s} < +\infty \right\}. \quad (17)$$

with

$$\|(u, f)\|_{\mathcal{W}^s} := \sup_{N \geq N_0} N^s \inf_{h \in H_N} \left( |u - u_h|^2_1 + \text{osc}_h^2 \right).$$
Quasi-optimality and error estimate of Algorithm 1

Theorem 4

Let \( \{ h_k \}_{k \geq 0} \) be a sequence of meshes generated by Algorithm 1 and let \( \{ u_{h_k}^m \}_{k \geq 0} \) be the corresponding sequence of finite element solutions. Suppose that

\[
0 < \alpha < C^* \theta^2, \quad 0 < \theta < \theta^* < 1,
\]

then we have the following estimate on the complexity of the algorithm:

\[
N_k \leq C \varepsilon_k^{-1/s}.
\]

Furthermore, in case of 2D, there exists \( k_0 \geq 1 \), such that for all \( k = k_0, k_0 + 1, \ldots, \) we have

\[
e(h_k, m_k) \leq C (N_k - N_{k_0})^{-1} \| f \|^2.
\]
Convergence of Algorithm 2

Theorem 5

Let $\{h_k\}_{k \geq 0}$ be a sequence of meshes generated by Algorithm 2 and let $\{u_{h_k}^{m_k}\}_{k \geq 0}$ be the corresponding sequence of finite element solutions. Suppose that

$$0 < \alpha < C^* \theta^2,$$  \hspace{1cm} (21)

then there exist constants $\beta_1 > 0$, $\beta_2 > 0$, and $\rho < 1$ such that for all $k = 1, 2, \ldots$

$$e(h_{k+1}, m_{k+1}) \leq \rho e(h_k, m_k).$$  \hspace{1cm} (22)
Quasi-optimality and error estimate of Algorithm 2

**Theorem 6**

Let \( \{h_k\}_{k \geq 0} \) be a sequence of meshes generated by Algorithm 2 and let \( \{u_{h_k}^{m_k}\}_{k \geq 0} \) be the corresponding sequence of FE solutions. Suppose

\[
0 < \alpha < C^* \theta^2, \quad 0 < \theta < \theta^* < 1, \quad 0 < \gamma < \gamma^*,
\]

(23)

then we have the following estimate on the complexity of the algorithm:

\[
N_k \leq C \varepsilon_k^{-1/s}.
\]

(24)

Furthermore, in case of 2D, there exists \( k_0 \geq 1 \), such that for all \( k = k_0, k_0 + 1, \ldots \), we have

\[
e(h_k, m_k) \leq C (N_k - N_{k_0})^{-1} \|f\|^2.
\]

(25)
Features of the results

- In Algorithm 2, the edge residuals alone dominate the error estimation in most cases, which verifies the well-known result in practice.

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- In the SOLVE step: CG, MG will be stopped by an adaptive stopping criteria with $\sqrt{\alpha(h + \text{osc})}$, compared with a fixed stopping criterion (e.g., $10^{-8}$) in the usual way.

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- Optimal convergence rate after a finite steps.

- In the SOLVE step: CG, MG will be stopped by an adaptive stopping criteria with $\sqrt{\alpha(h + \text{osc})}$, compared with a fixed stopping criterion (e.g., $10^{-8}$) in the usual way.

- Optimal convergence rate after a finite steps.
Numerical experiment

- We solve Poisson’s equation on the L-shaped domain with Dirichlet boundary condition. The exact solution is \( u(r, \theta) = r^{2/3} \sin\left(\frac{2\theta}{3}\right) \).

- Based on the local multigrid algorithm developed by Chen and Wu [06], which has optimal computational cost for discrete systems of PDE.
Figure 1: Adaptive mesh with 1005 elements (11 step)
Figure 2: Adaptive mesh with 11327 elements (22 step)
Figure 5: Number of iterations of every step: 26 vs 6
Figure 7: The energy error
Figure 8: Total iterations: 795 vs 124
Figure 9: Iteration time of every step: 79 vs 25.5 vs 5.5
Figure 10: Total iteration time: 291 vs 119.5 vs 29.2
Adaptive nonconforming finite element methods
Adaptive nonconforming finite element methods (ANFEM)

- We develop a practical adaptive algorithm for linear nonconforming finite element method.

- It is based on an adaptive marking strategy and an adaptive stopping criteria for iterative solution.

- We prove its convergence and optimal error estimate.

- The main difficulties are the proof of the quasi-orthogonality, local upper and lower bounds of ANFEM.
Let \( V_h \) denote the nonconforming \( P_1 \) finite element space (Crouzeix-Raviart element over \( \mathcal{T}_h \), which is given by

\[
V_h := \left\{ v_h \in L^2(\Omega); \forall K \in \mathcal{K}_h, v_h|_K \in P_1(K); \forall E \in \mathcal{E}_h, \int_E [v_h]_E ds = 0 \right\},
\]

here \([v_h]_E\) stands for the jump of \( v_h \) across \( E \) and vanishes when \( E \subset \partial \Omega \).

Let \( u_h \) denote the solution of the discrete problem

\[
\begin{align*}
\text{Find } u_h \in V_h, \text{ such that } \\
a_h(u_h, v_h) &= (f, v_h), \forall v_h \in V_h,
\end{align*}
\]

where \( a_h(u_h, v_h) = \sum_{K \in \mathcal{K}_h} \int_K \nabla u_h \nabla v_h \, dx \).

We suppose that \( \zeta^2_h(u_h^m) \) satisfies the following upper bound

\[
|u_h - u_h^m|_{1,h}^2 \leq C_{it} \zeta^2_h(u_h^m). \tag{27}
\]

Set

\[
\| \cdot \|_h = \left( \sum_{K \in \mathcal{K}_h} | \cdot |_{1,K}^2 \right)^{\frac{1}{2}}.
\]
We define edge residuals for $E \in \mathcal{E}_h$ and any subset $\mathcal{F} \subset \mathcal{E}_h$

$$
\eta_{h,E}(v_h) := h_E^{1/2} \left\| \left[ \frac{\partial v_h}{\partial s} \right] \right\|_{0,E}, \quad \eta_h(v_h, \mathcal{F}) := \left( \sum_{E \in \mathcal{F}} \eta_{h,E}^2(v_h) \right)^{1/2},
$$

(28)

together with volume residuals for $K \in \mathcal{K}_h$ and any subset $\mathcal{M} \subset \mathcal{K}_h$

$$
\mu_K := |K|^{1/2} \| f \|_{0,K}, \quad \mu_h(\mathcal{M}) := \left( \sum_{K \in \mathcal{M}} \mu_K^2 \right)^{1/2}.
$$

(29)

We next define an oscillation term by

$$
osc_E := |\omega_E|^{1/2} \| f - \pi_{\omega_E} f \|_{0,\omega_E}, \quad osc_h(\mathcal{F}) := \left( \sum_{E \in \mathcal{F}} osc_E^2 \right)^{1/2},
$$

(30)

where $\pi_{\omega_E} f := \int_{\omega_E} f \, dx / |\omega_E|$. We set for brevity $\eta_h(v_h) := \eta_h(v_h, \mathcal{E}_h)$, $osc_h := osc_h(\mathcal{E}_h)$ and $\mu_h := \mu_h(\mathcal{K}_h)$. 

Algorithm ANFEM

0) Choose parameters $0 < \theta, \sigma < 1, \gamma > 0, \alpha > 0$ and an initial mesh $h_0$, and set $k = 0$.

1) Do $m_k$ iterations of the multigrid algorithm applied to the discrete system (26) with $h$ replaced by $h_k$ to obtain the finite element solution $u_{h_k}^{m_k}$. The integer $m_k$ is determined by the condition to be the smallest integer verifying:

$$\zeta_{h_k}^2 (u_{h_k}^{m_k}) \leq \alpha (\eta_{h_k}^2 (u_{h_k}^{m_k}) + \mu_{h_k}^2).$$

(31)

2) If $\mu_{h_k}^2 \leq \gamma \eta_{h_k}^2 (u_{h_k}^{m_k})$ then mark a subset $\mathcal{F}$ of $\mathcal{E}_{h_k}$ with minimal cardinality such that

$$\eta_{h_k}^2 (u_{h_k}^{m_k}, \mathcal{F}) \geq \theta \eta_{h_k}^2 (u_{h_k}^{m_k}).$$

(32)

else find a set $\mathcal{M} \subset \mathcal{K}_{h_k}$ with minimal cardinality such that

$$\mu_{h_k}^2 (\mathcal{M}) \geq \sigma \mu_{h_k}^2.$$ (33)

and define $\mathcal{F}$ to be the set of edges contained in at least one cell $K \in \mathcal{M}$.

3) Adapt the mesh: $h_{k+1} := \mathcal{R}_{loc}(h_k, \mathcal{F})$.

4) Set $k := k + 1$ and go to step (1).
Upper bounds

Lemma 7

*(global upper bound)* There exists a constant $C_1 > 0$ depending only on the minimum angle of $K_{h_0}$ such that for the multigrid solution $u_h^m \in V_h$, we have

$$|u - u_h^m|_{1,h}^2 \leq C_1 \left( \eta_h^2(u_h^m) + \mu_h^2 \right).$$

(34)

Lemma 8

*(local upper bound)* There exist constants $C_4, C_5 > 0$ depending only on the minimum angle of $K_{h_0}$ such that the following holds. For any mesh $H \in \mathcal{H}$ and any local refinement $h \in \mathcal{H}$ of $H$ let $\mathcal{F} \subset E_H$ be the set of refined edges. The corresponding multigrid coarse $u_h^l \in V_H$ and fine-grid solutions $u_h \in V_h$ satisfy

$$|u_h - u_h^l|_{1,h}^2 \leq C_4 \left( \eta_H^2(u_h^l, \mathcal{F}) + \mu_H^2 + \alpha (\eta_H^2(u_h^l) + \mu_H^2) \right).$$

(35)
Lemma 9

**Global lower bounds** There exist constants $C_2, C_3 > 0$ depending only on the minimum angle of $\mathcal{K}_{h_0}$ such that the following estimates hold for the multigrid solution $u^m_h \in V_h$:

$$\eta^2_h(u^m_h) \leq C_2 |u - u^m_h|_{1,h}^2$$

(36)

and

$$\mu^2_h \leq C_3 \left( |u - u^m_h|_{1,h}^2 + \text{osc}_h^2 \right).$$

(37)

Lemma 10

**Local lower bounds** There exist constants $C_6, C_7 > 0$ depending only on the minimum angle of $\mathcal{K}_{h_0}$ such that for $\mathcal{F} \subseteq \mathcal{E}_H$, $h = R_{loc}(H, \mathcal{F})$, there holds:

$$\eta^2_H(u^l_H, \mathcal{F}) \leq C_6 |u^m_h - u^l_H|_{1,h}^2,$$

(38)

If $\mathcal{M} \subseteq \mathcal{K}_H$ is the set of refined cells, there holds:

$$\mu^2_H(\mathcal{M}) \leq C_7 \left( |u^m_h - u^l_H|_{1,h}^2 + \alpha(\eta^2_H(u^l_H) + \mu^2_H) + \text{osc}_H^2(\mathcal{F}) + \alpha(\eta^2_h(u^m_h) + \mu^2_h) \right).$$

(39)
Quasi-orthogonality

**Lemma 11**

*(quasi-orthogonality)* Let $h, H \in \mathcal{H}$ be two nested meshes and $\mathcal{M} \subset \mathcal{K}_H$ be the set of refined cells. Then there exists a constant $C_8 > 0$ depending only on the minimum angle in $\mathcal{K}_{h_0}$ such that

\[
(\nabla_h (u - u_h^m), \nabla_h (u_h^m - u_H^l)) \leq \left| u - u_h^m \right|_{1,h} \left( C_8 \mu_H(\mathcal{M}) + \sqrt{\alpha} \left( \sqrt{\eta_h^2(u_h^m) + \mu_h^2} + \sqrt{\eta_H^2(u_H^l) + \mu_H^2} \right) \right),
\]

\[
(\nabla_h (u - u_h), \nabla_h (u_h - u_H^l)) \leq \left| u - u_h \right|_{1,h} \left( C_8 \mu_H(\mathcal{M}) + \sqrt{\alpha} \sqrt{\eta_H^2(u_H^l) + \mu_H^2} \right)
\]

and

\[
(\nabla_h (u - u_h), \nabla_h (u_h - u_H)) \leq C_8 \mu_H(\mathcal{M}) \left| u - u_h \right|_{1,h}.
\]
Convergence of ANFEM

**Theorem 12**

Let \( \{ h_k \}_{k \geq 0} \) be a sequence of meshes generated by algorithm ANFEM and let \( \{ u_{h_k} \}_{k \geq 0} \) be the corresponding sequence of finite element solutions. Suppose that

\[
0 < \alpha \leq C^* \theta^2,
\]

with a generic constant \( C^* \) to be defined in the proof. Then there exist \( \beta > 0 \) and \( 0 < \rho < 1 \) such that for all \( k = 1, 2, \ldots \)

\[
e(h_{k+1}) \leq \rho \, e(h_k)
\]

with \( e(h) := |u - u_h^m|_{1,h}^2 + \beta \mu_h^2. \)
Quasi-optimality of ANFEM

Theorem 13

Suppose \((u, f) \in \mathcal{W}^s\). Let \(\{h_k\}_{k \geq 0}\) be a sequence of meshes generated by algorithm ANFEM and let \(\{V_k\}_{k \geq 0}\) and \(\{u_{h_k}\}_{k \geq 0}\) be the corresponding sequences of finite element spaces and solutions. Let

\[\varepsilon_k := \sqrt{|u - u_{h_k}|_1^2 + \beta \mu_{h_k}^2}.\]

Assuming that the parameters \(\gamma, \theta\) and \(\alpha\) satisfy (61) and

\[
\gamma < \frac{1 - 3\alpha C_2}{C_2 (C_4 + 2C_8^2 + 3\alpha)}, \quad \alpha + \theta < \frac{1 - 3\alpha C_2}{C_2 C_4} - \gamma \left(1 + \frac{2C_8^2 + 3\alpha}{C_4}\right). \tag{45}
\]

Then we have the following estimate on the complexity of the algorithm: there exists a constant \(C\) such that for all \(k = 0, 1, 2, \ldots\)

\[N_k \leq C \varepsilon_k^{-1/s}. \tag{46}\]
Adaptive mixed finite element methods
The Raviart-Thomas space $V_h \subset H(\text{div}; \Omega)$ is defined as

$$V_h = \{ \tau_h \in H(\text{div}; \Omega); \tau_h|_K \in P_0(K)^2 \oplus xP_0(K), \ \forall \ K \in K_h \}.$$ 

$Q_h$ is the space of piecewise constant functions. The discrete solution $(\sigma_h, u_h) \in V_h \times Q_h$ approximating $(\nabla u, u)$ in (1) is defined by

$$\langle \sigma_h, \tau_h \rangle + \langle \text{div} \tau_h, u_h \rangle + \langle \text{div} \sigma_h, v_h \rangle = \langle f, v_h \rangle \ \forall (\tau_h, v_h) \in V_h \times Q_h. \quad (47)$$

In order to estimate the iteration error, we use an a posteriori error estimator $\zeta_h^2(\sigma^m_h)$ which is supposed to satisfy the upper bound

$$\|\sigma_h - \sigma^m_h\|^2 \leq C_{it} \zeta_h^2(\sigma^m_h). \quad (48)$$

Next we define edge residuals for $E \in \mathcal{E}_h$ and any given subset $\mathcal{F} \subseteq \mathcal{E}_h$

$$\eta_{h,E}(\tau_h) := h_{E}^{1/2} \|[\tau_h \cdot t_E]\|_E, \quad \eta_h(\tau_h, \mathcal{F}) := \left( \sum_{E \in \mathcal{F}} \eta_{h,E}^2(\tau_h) \right)^{1/2}. \quad (49)$$
Algorithm AMFEM

(0) Choose parameters $0 < \theta, \sigma < 1, \gamma > 0, \alpha > 0$ and an initial mesh $h_0$, and set $k = 0$.

(1) Do $m_k$ iterations of the discrete system (47) to obtain $\sigma_{h_k}^{m_k}$, $m_k$ is determined by the condition to be the smallest integer verifying:

$$
\zeta_{h_k}^2(\sigma_{h_k}^{m_k}) \leq \alpha \eta_{h_k}^2(\sigma_{h_k}^{m_k}).
$$

(2) Compute the a posteriori error estimator $\eta_{h_k}(\sigma_{h_k}^{m_k})$ and the oscillation term $\text{osc}_{h_k}$.

(3) If $\text{osc}_{h_k}^2 \leq \gamma \eta_{h_k}^2(\sigma_{h_k}^{m_k})$ then mark a set $\mathcal{F}$ of $\mathcal{E}_{h_k}$ with minimal cardinality such that

$$
\eta_{h_k}^2(\sigma_{h_k}^{m_k}, \mathcal{F}) \geq \theta \eta_{h_k}^2(\sigma_{h_k}^{m_k}).
$$

else find a set $\mathcal{M} \subset \mathcal{K}_{h_k}$ with minimal cardinality such that

$$
\text{osc}_{h_k}^2(\mathcal{M}) \geq \sigma \text{osc}_{h_k}^2.
$$

and define $\mathcal{F}$ to be the set of edges contained in at least one cell $K \in \mathcal{M}$.

(4) Adapt the mesh: $h_{k+1} := \mathcal{R}_{loc}(h_k, \mathcal{F})$.

(5) Set $k := k + 1$ and go to step (1).
Upper bounds

Lemma 14

**Global upper bound** There exists a constant $C_1 > 0$ depending only on the minimum angle of $h_0$ such that for the multigrid solution $\sigma^m_h \in V_h$, we have

$$\| \sigma - \sigma^m_h \|^2 \leq C_1 \left( \eta_h^2(\sigma^m_h) + \text{osc}_h^2 \right). \tag{53}$$

Lemma 15

**Local upper bound** There exist constants $C_3, C_5 > 0$ depending only on the minimum angle of $h_0$ such that the following holds. For any subset $\mathcal{F} \subset \mathcal{E}_H$, $h = \mathcal{R}_{\text{loc}}(H, \mathcal{F})$, and $M$ the set of refined cells, the iterative solutions $\sigma^l_H \in V_H$ and $\sigma_h \in V_h$, we have

$$\| \sigma_h - \sigma^l_H \|^2 \leq C_3 \left( \eta_H^2(\sigma^l_H, \mathcal{F}) + \text{osc}_H^2(M) \right) + \alpha \eta_H^2, \tag{54}$$

and

$$\# \mathcal{F} \leq C_5 (N_h - N_H). \tag{55}$$
Lower bounds

**Lemma 16**

*(global lower bounds)* There exists a constant $C_2 > 0$ depending only on the minimum angle of $h_0$ such that the multigrid solution $\sigma^I_H \in V_H$ satisfies

$$\eta^2_H(\sigma^I_H) \leq C_2 \|\sigma - \sigma^I_H\|^2. \quad (56)$$

**Lemma 17**

*(local lower bounds)* There exists a constant $C_4 > 0$ depending only on the minimum angle of $h_0$ such that for $F \subset \mathcal{E}_H$, $h = R_{loc}(H, F)$ and $\mathcal{M} \subset \mathcal{K}_H$ the set of refined cells there holds:

$$\eta^2_H(\sigma^I_H, F) \leq C_4 \left( \|\sigma^m_h - \sigma^I_H\|^2 + \text{osc}^2_H(\mathcal{M}) + \alpha \eta^2_H(\sigma^I_H) \right). \quad (57)$$
Quasi-orthogonality

Lemma 18

Let $h, H \in \mathcal{H}$ be two nested meshes and $\mathcal{M} \subset \mathcal{K}_H$ be the set of refined cells. Then there exists a constant $C_6 > 0$ depending only on the minimum angle of $h_0$ such that

$$\langle \sigma - \sigma^m_h, \sigma^m_h - \sigma^l_H \rangle \leq \sqrt{\alpha} \eta_h(\sigma^m_h) \| \sigma^m_h - \sigma^l_H \| \ (58)$$

$$+ \| \sigma - \sigma^m_h \| \left( C_6 \text{osc}_H(\mathcal{M}) + \sqrt{\alpha} (\eta_h(\sigma^m_h) + \eta_H(\sigma^l_H)) \right),$$

and

$$\langle \sigma - \sigma_h, \sigma_h - \sigma^l_H \rangle \leq \| \sigma - \sigma_h \| \left( C_6 \text{osc}_H(\mathcal{M}) + \sqrt{\alpha} \eta_H(\sigma^l_H) \right). \ (59)$$

If we solve both of the discretized equations exactly on the meshes $h$ and $H$, then we have

$$\langle \sigma - \sigma_h, \sigma_h - \sigma_H \rangle \leq C_6 \text{osc}_H(\mathcal{M}) \| \sigma - \sigma_h \|. \ (60)$$
Convergence of algorithm AMFEM

**Theorem 19**

Let \( \{h_k\}_{k \geq 0} \) be a sequence of meshes generated by algorithm AMFEM and let \( \{\sigma_{h_k}^m\}_{k \geq 0} \) be the corresponding sequence of iterative finite element solutions. Suppose that

\[
0 < \alpha \leq C^* \theta^2, \quad (61)
\]

Then there exist \( \beta > 0 \) and \( \rho < 1 \) such that for all \( k = 1, 2, \ldots \)

\[
e(h_{k+1}) \leq \rho \, e(h_k) \quad (62)
\]

with

\[
e(h) := \|\sigma - \sigma_{h}^m\|^2 + \beta \, \text{osc}_h^2. \quad (63)
\]
We define the approximation class

$$\mathcal{W}^s := \left\{ (\sigma, f) \in (H(\text{div}, \Omega), L^2(\Omega)) : \|(\sigma, f)\|_{\mathcal{W}^s} < +\infty \right\}. \quad (64)$$

with

$$\|(\sigma, f)\|_{\mathcal{W}^s} := \sup_{N \geq N_0} \inf_{h \in H_N} \left( \|\sigma - \sigma_h\| + \mu_h \right).$$

**Theorem 20**

Let \( \{h_k\} \) be a sequence of meshes generated by algorithm AMFEM and \( \{\sigma_{h_k}^m\}_{k \geq 0} \) be the corresponding iterative FE solutions. Assuming

$$0 < \gamma < \frac{1}{C_2(C_3 + 2C_6^2)}, \quad \theta + \frac{3\alpha}{C_3} < \frac{1}{C_2C_3} - \gamma \left(1 + \frac{2C_6^2}{C_3}\right), \quad (65)$$

then there exists a constant \( C \) such that

$$N_k \leq C \varepsilon_k^{-1/s}. \quad (66)$$

In case of 2D, there exists \( k_0 \geq 1 \), such that for all \( k = k_0, k_0 + 1, \ldots \), we have

$$\|\sigma - \sigma_{h_k}^m\|^2 + \text{osc}_k^2 \leq C(N_k - N_0)^{-1}. \quad (67)$$
Extensions and open problem
Extensions and open problem

- Adaptive mixed (conforming and nonconforming) FEM for the Stokes problem (submitted).

- Adaptive FEM for the optimal control problem (submitted).


- Open problem: Adaptive hp FEM, exponential convergence rate?
Concerning publications


Thank you for your attention!