Qualitative analysis of solutions to discrete contact problems with Coulomb friction

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Outline

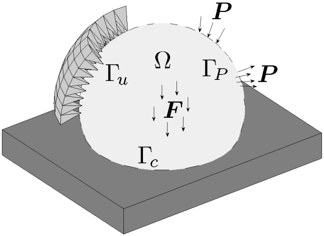
(*i*) uniqueness results — dependence on the mesh norm and the coefficient of friction;

(*ii*) existence of local Lipschitz continuous branches of solutions;

(*iii*) piecewise-smooth Moore-Penrose continuation method;

(*iv*) elementary examples.

3D-contact problems with orthotropic Coulomb friction and solution-dependent coefficients of friction



$\Omega \subset \mathbb{R}^3$, $\partial\Omega = \overline{\Gamma}_u \cup \overline{\Gamma}_P \cup \overline{\Gamma}_c$

Classical formulation

- (equilibrium equations)

$$\frac{\partial \sigma_{ij}}{\partial x_j}(\boldsymbol{u}) + F_i = 0 \quad \text{in } \Omega,\ i = 1, 2, 3,$$

$$\sigma_{ij}(\boldsymbol{u}) = c_{ijkl}\varepsilon_{kl}(\boldsymbol{u});$$

- $\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_u;$
- $\sigma_{ij}(\boldsymbol{u})\nu_j = P_i \quad \text{on } \Gamma_P,\ i = 1, 2, 3;$

- (unilateral conditions)

$$u_\nu := \boldsymbol{u}\cdot\boldsymbol{\nu} \le 0, \quad T_\nu(\boldsymbol{u}) := \sigma_{ij}(\boldsymbol{u})\nu_i\nu_j \le 0,$$
$$T_\nu(\boldsymbol{u})u_\nu = 0 \quad \text{on } \Gamma_c;$$

- (orthotropic Coulomb friction law)

$x \in \Gamma_c \mapsto \boldsymbol{t}_1(x), \boldsymbol{t}_2(x) \ldots$ principal orthotropic axes
$\mathcal{F}_i := \mathcal{F}_i(x, \|\boldsymbol{u}_t(x)\|)$, $i = 1, 2 \ldots$ coefficients of friction
in the direction $\boldsymbol{t}_i$, $\boldsymbol{\mathcal{F}} = \operatorname{diag}(\mathcal{F}_1, \mathcal{F}_2)$

$$\begin{aligned}
\boldsymbol{u}_t(x) = \boldsymbol{0} &\implies \|\boldsymbol{\mathcal{F}}^{-1}(x,0)\boldsymbol{T}_t(\boldsymbol{u})(x)\| \le -T_\nu(\boldsymbol{u})(x), \quad x \in \Gamma_c,\\
\boldsymbol{u}_t(x) \ne \boldsymbol{0} &\implies \boldsymbol{\mathcal{F}}^{-1}(x, \|\boldsymbol{u}_t(x)\|)\boldsymbol{T}_t(\boldsymbol{u})(x)\\
&\quad = T_\nu(\boldsymbol{u})(x)\frac{\boldsymbol{\mathcal{F}}(x, \|\boldsymbol{u}_t(x)\|)\boldsymbol{u}_t(x)}{\|\boldsymbol{\mathcal{F}}(x, \|\boldsymbol{u}_t(x)\|)\boldsymbol{u}_t(x)\|}, \quad x \in \Gamma_c.
\end{aligned}$$

Weak formulation

$$\boldsymbol{V} = \{\boldsymbol{v} \in (H^1(\Omega))^3 \,|\, \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_u\}$$
$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{V} \,|\, v_\nu \leq 0 \text{ on } \Gamma_c\}$$
$$X_\nu = \{\varphi \in L^2(\Gamma_c) \,|\, \exists\, \boldsymbol{v} \in \boldsymbol{V} : \varphi = v_\nu \text{ on } \Gamma_c\}, \quad X_\nu' = \text{dual of } X_\nu$$
$$X_{t+} = \{\varphi \in L^2(\Gamma_c) \,|\, \exists\, \boldsymbol{v} \in \boldsymbol{V} : \varphi = \|\boldsymbol{v}_t\| \text{ on } \Gamma_c\}$$
$$\Lambda_\nu = \{\mu_\nu \in X_\nu' \,|\, \langle \mu_\nu, v_\nu \rangle_\nu \leq 0 \ \forall\, \boldsymbol{v} \in \boldsymbol{K}\}$$

$$a : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}^1, \quad \ell : \boldsymbol{V} \to \mathbb{R}^1, \quad j : X_{t+} \times \Lambda_\nu \times \boldsymbol{V} \to \mathbb{R}^1$$
$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega c_{ijkl}\varepsilon_{ij}(\boldsymbol{u})\varepsilon_{kl}(\boldsymbol{v})\,\mathrm{d}x, \quad \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$$
$$\ell(\boldsymbol{v}) := \int_\Omega \boldsymbol{F} \cdot \boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_P} \boldsymbol{P} \cdot \boldsymbol{v}\,\mathrm{d}s, \quad \boldsymbol{F} \in (L^2(\Omega))^3, \ \boldsymbol{P} \in (L^2(\Gamma_P))^3$$
$$j(\varphi, g, \boldsymbol{v}_t) := \langle g, \|\mathcal{F}(\varphi)\boldsymbol{v}_t\| \rangle_\nu, \quad g \in \Lambda_\nu, \ \varphi \in X_{t+}, \ \boldsymbol{v} \in \boldsymbol{V}$$

Definition A function $\boldsymbol{u} \in \boldsymbol{K}$ is said to be a weak solution to our problem iff

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\|\boldsymbol{u}_t\|, -T_\nu(\boldsymbol{u}), \boldsymbol{v}_t) - j(\|\boldsymbol{u}_t\|, -T_\nu(\boldsymbol{u}), \boldsymbol{u}_t) \geq \ell(\boldsymbol{v} - \boldsymbol{u}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{K} \quad (\mathcal{P})$$

Fixed-point formulation

Let $(\varphi, g) \in X_{t+} \times \Lambda_\nu$ be given and define the auxiliary problem:

$$\left.\begin{array}{l} \text{Find } \boldsymbol{u} := \boldsymbol{u}(\varphi, g) \in \boldsymbol{K} \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\varphi, g, \boldsymbol{v}_t) - j(\varphi, g, \boldsymbol{u}_t) \\ \qquad \geq \ell(\boldsymbol{v} - \boldsymbol{u}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{K}. \end{array}\right\} \quad (\mathcal{P}(\varphi, g))$$

Let $\Psi : X_{t+} \times \Lambda_\nu \to X_{t+} \times \Lambda_\nu$ be defined by

$$\Psi(\varphi, g) = (\|\boldsymbol{u}_t\|, -T_\nu(\boldsymbol{u})), \quad (\varphi, g) \in X_{t+} \times \Lambda_\nu.$$

Then $\boldsymbol{u} \in \boldsymbol{K}$ solves $(\mathcal{P})$ iff $(\|\boldsymbol{u}_t\|, -T_\nu(\boldsymbol{u}))$ is a fixed point of Ψ.

Mixed formulation of $\mathcal{P}(\varphi, g)$

$$\left.\begin{array}{l} \text{Find } (\boldsymbol{u}, \lambda_\nu) \in \boldsymbol{V} \times \Lambda_\nu \text{ such that} \\ a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\varphi, g, \boldsymbol{v}_t) - j(\varphi, g, \boldsymbol{u}_t) \\ \qquad\qquad \geq \ell(\boldsymbol{v} - \boldsymbol{u}) - \langle \lambda_\nu, v_\nu - u_\nu \rangle_\nu \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \\ \langle \mu_\nu - \lambda_\nu, u_\nu \rangle_\nu \leq 0 \quad \forall\, \mu_\nu \in \Lambda_\nu. \end{array}\right\} \quad (\mathcal{M}(\varphi, g))$$

Since $\lambda_\nu = -T_\nu(\boldsymbol{u})$ on Γ_c, one has

$$\Psi(\varphi, g) = (\|\boldsymbol{u}_t\|, \lambda_\nu).$$

Discrete contact problems with Coulomb friction

Based on an appropriate discretization of the mapping Ψ.

$\mathcal{T}_h^{\Omega} \ldots$ partition of $\overline{\Omega}$ into finite elements T, $h =$ norm of $\mathcal{T}_h^{\Omega}$
$\mathcal{T}_H^{\Gamma_c} \ldots$ partition of $\overline{\Gamma}_c$ into finite elements R, $H =$ norm of $\mathcal{T}_H^{\Gamma_c}$

$$V^h = \{v^h \in C(\overline{\Omega}) \,|\, v^h{}_{|T} \in P_k(T) \;\forall\, T \in \mathcal{T}_h^{\Omega},\; v^h = 0 \text{ on } \Gamma_u\}$$
$$L^H = \{\mu^H \in L^2(\Gamma_c) \,|\, \mu^H{}_{|R} \in P_s(R) \;\forall\, R \in \mathcal{T}_H^{\Gamma_c}\}$$
$$\boldsymbol{V}^h = (V^h)^3$$
$$W^h = V^h{}_{|\Gamma_c}, \quad W_+^h = \{\varphi^h \in W^h \,|\, \varphi^h \geq 0 \text{ on } \Gamma_c\}$$
$$\Lambda_\nu^H = \{\mu^H \in L^H \,|\, \mu^H \geq 0 \text{ on } \Gamma_c\}$$

The couple $(\boldsymbol{V}^h, L^H)$ has to satisfy the following condition:

$$\mu^H \in L^H \;\&\; (\mu^H, v_\nu^h)_{0,\Gamma_c} = 0 \;\forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h \quad \Longrightarrow \quad \mu^H = 0. \qquad (1)$$

Mixed finite element discretization of $\mathcal{M}(\varphi, g)$

For $\varphi^h \in W_+^h$, $g^H \in \Lambda_\nu^H$ given define the problem:

$$\left.\begin{aligned}
&\text{Find } (\boldsymbol{u}^h, \lambda_\nu^H) \in \boldsymbol{V}^h \times \Lambda_\nu^H \text{ such that}\\
&a(\boldsymbol{u}^h, \boldsymbol{v}^h - \boldsymbol{u}^h) + j(\varphi^h, g^H, \boldsymbol{v}_t^h)\\
&\qquad - j(\varphi^h, g^H, \boldsymbol{u}_t^h) \geq \ell(\boldsymbol{v}^h - \boldsymbol{u}^h)\\
&\qquad\qquad - (\lambda_\nu^H, v_\nu^h - u_\nu^h)_{0,\Gamma_c} \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h,\\
&(\mu_\nu^H - \lambda_\nu^H, u_\nu^H)_{0,\Gamma_c} \leq 0 \quad \forall\, \mu_\nu^H \in \Lambda_\nu^H.
\end{aligned}\right\} \; (\mathcal{M}_{hH}(\varphi^h, g^H))$$

Proposition (1) $\Longrightarrow$ $(\mathcal{M}_{hH}(\varphi^h, g^H))$ has a unique solution for any $(\varphi^h, g^H) \in W_+^h \times \Lambda_\nu^H$.

Assumptions

- the vector field $x \mapsto (\boldsymbol{t}_1(x), \boldsymbol{t}_2(x))$, $x \in \Gamma_c$, is sufficiently smooth so that

$$\left.\begin{array}{r} \boldsymbol{v}_t^h = (v_{t_1}^h, v_{t_2}^h) \in (H^1(\Gamma_c))^2 \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h, \\ \exists\, c_t > 0 \text{ independent of } \boldsymbol{v}^h \in \boldsymbol{V}^h \text{ and } h > 0 : \\ \|\boldsymbol{v}_t^h\|_{1,\Gamma_c} \le c_t \|\boldsymbol{v}^h\|_{1,\Gamma_c} \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h \end{array}\right\} \tag{2}$$

- $\exists\, r_h \in \mathcal{L}(H^1(\Gamma_c), W^h)$ such that

$$\left.\begin{array}{r} \|\varphi - r_h\varphi\|_{0,\Gamma_c} \le c_r h_{\Gamma_c} \|\varphi\|_{1,\Gamma_c} \quad \forall\, \varphi \in H^1(\Gamma_c), \\ \varphi \in H^1(\Gamma_c),\ \varphi \ge 0 \text{ on } \Gamma_c \implies r_h\varphi \in W_+^h, \end{array}\right\} \tag{3}$$

where h_{Γ_c} = norm of $\mathcal{T}_h^\Omega|_{\bar{\Gamma}_c}$ and $c_r > 0$ does not depend on φ and h_{Γ_c}.

- the satisfaction of the following inverse inequalities for elements of $\boldsymbol{V}^h{}_{|_{\Gamma_c}}$ (see [Ciarlet, 1978]):

$$\left.\begin{aligned} \|\boldsymbol{v}^h\|_{1,\Gamma_c} \leq c_{\mathrm{inv}}^{(1,0)} h_{\Gamma_c}^{-1} \|\boldsymbol{v}^h\|_{0,\Gamma_c} \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h, \\ \|\boldsymbol{v}^h\|_{\infty,\Gamma_c} \leq c_{\mathrm{inv}}^{(\infty)} h_{\Gamma_c}^{-1} \|\boldsymbol{v}^h\|_{0,\Gamma_c} \quad \forall\, \boldsymbol{v}^h \in \boldsymbol{V}^h, \end{aligned}\right\} \tag{4}$$

  where $c_{\mathrm{inv}}^{(1,0)}$, $c_{\mathrm{inv}}^{(\infty)} > 0$ do not depend on h_{Γ_c} and $\boldsymbol{v}^h \in \boldsymbol{V}^h$

- $$\left.\begin{aligned} \mathcal{F}_1, \mathcal{F}_2 \in C(\Gamma_c \times \mathbb{R}^1_+), \\ 0 < \mathcal{F}_{\min} \leq \mathcal{F}_i(x,\xi) \leq \mathcal{F}_{\max}, \\ i = 1,2,\ \forall\,(x,\xi) \in \Gamma_c \times \mathbb{R}^1_+ \end{aligned}\right\} \tag{5}$$

Let $\Psi_{hH} : W_+^h \times \Lambda_\nu^H \to W_+^h \times \Lambda_\nu^H$ be defined by

$$\Psi_{hH}(\varphi^h, g^H) = (r_h \|\boldsymbol{u}_t^h\|, \lambda_\nu^H), \quad (\varphi^h, g^H) \in W_+^h \times \Lambda_\nu^H.$$

Definition A function $\boldsymbol{u}^h \in \boldsymbol{V}^h$ is said to be a solution to the discrete problem iff the couple $(r_h \|\boldsymbol{u}_t^h\|, \lambda_\nu^H)$ is a fixed point of Ψ_{hH}.

Existence of solutions to discrete problems

Let

$$\|(\varphi^h, \mu^H)\|_{W^h \times L^H} := \|\varphi^h\|_{0,\Gamma_c} + \|\mu^H\|_{-1/2,h}, \quad (\varphi^h, \mu^H) \in W^h \times L^H,$$

where

$$\|\mu^H\|_{-1/2,h} = \sup_{\mathbf{0} \neq \boldsymbol{v}^h \in \boldsymbol{V}^h} \frac{(\mu^H, v_\nu^h)_{0,\Gamma_c}}{\|\boldsymbol{v}^h\|_{1,\Omega}}.$$

Proposition Let (2)–(5) be satisfied. Then Ψ_{hH} has at least one fixed point in $W_+^h \times \Lambda_\nu^H$.

Uniqueness of the solutions

Denote

$$L = \max_{i=1,2}\left\{\sup_{x\in\Gamma_c,\,\xi>0}\left|\frac{\partial\mathcal{F}_i(x,\xi)}{\partial\xi}\right|\right\},$$

$$\kappa(\boldsymbol{\mathcal{F}}) = \sup_{x\in\Gamma_c,\,\xi>0}\frac{\max\{\mathcal{F}_1(x,\xi),\mathcal{F}_2(x,\xi)\}}{\min\{\mathcal{F}_1(x,\xi),\mathcal{F}_2(x,\xi)\}},\quad \boldsymbol{\mathcal{F}} = \mathrm{diag}(\mathcal{F}_1,\mathcal{F}_2).$$

Then Ψ_{hH} is Lipschitz continuous in $W_+^h \times \Lambda_\nu^H \cap B$, where B is a ball in $W^h \times L^H$ with a sufficiently large radius:

$$\exists\, C>0: \quad \|\Psi_{hH}(\varphi^h,g^H)-\Psi_{hH}(\bar\varphi^h,\bar g^H)\|_{W^h\times L^H} \le C\|(\varphi^h,g^H)-(\bar\varphi^h,\bar g^H)\|_{W^h\times L^H} \quad \forall\,(\varphi^h,g^H),(\bar\varphi^h,\bar g^H)\in W_+^h\times\Lambda_\nu^H\cap B,$$

$$C = \max\{C_1(\mathcal{F}_{\max},H),\, C_2(L,\kappa(\boldsymbol{\mathcal{F}}),H,h_{\Gamma_c})\}.$$

It holds:

(*a*) $C_1(\mathcal{F}_{\max}, H) \to 0$ if $\mathcal{F}_{\max} \to 0+$ for any $H > 0$ fixed, $C_2(L, \kappa(\mathcal{F}), H, h_{\Gamma_c}) \to 0$ if $L \to 0+$ for any H, h_{Γ_c} fixed and $\kappa(\mathcal{F})$ bounded;

(*b*) if $\mathcal{F}$ does not depend on $\|\boldsymbol{u}_t^h\|$ (i.e. $L \equiv 0$) then $C_2 \equiv 0$;

(*c*) if $\mathcal{F}$ is fixed then

$$C_1(\mathcal{F}_{\max}, H) \sim H^{-1/2},$$
$$C_2(L, \kappa(\mathcal{F}), H, h_{\Gamma_c}) \sim (H h_{\Gamma_c})^{-1/2}$$

provided that the Babuška-Brezzi condition for $\{\boldsymbol{V}^h, L^H\}$ is satisfied:

$$\sup_{\boldsymbol{0} \neq \boldsymbol{v}^h \in \boldsymbol{V}^h} \frac{(\mu^H, v_\nu^h)_{0,\Gamma_c}}{\|\boldsymbol{v}^h\|_{1,\Omega}} \geq \beta \|\mu^H\|_{*,\Gamma_c} \quad \forall \mu^H \in L^H,$$

where $\beta > 0$ does not depend on h, H and

$$\|\mu^H\|_{*,\Gamma_c} = \sup_{\boldsymbol{0} \neq \boldsymbol{v} \in \boldsymbol{V}} \frac{(\mu^H, v_\nu)_{0,\Gamma_c}}{\|\boldsymbol{v}\|_{1,\Omega}}.$$

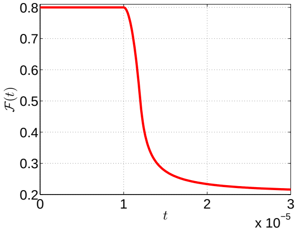
Consequence

Ψ_{hH} is contractive provided that $\mathcal{F}_{\max}$ and L are small enough. However, the bounds $\mathcal{F}_{\text{crit}}$ and L_{crit} are mesh-dependent.

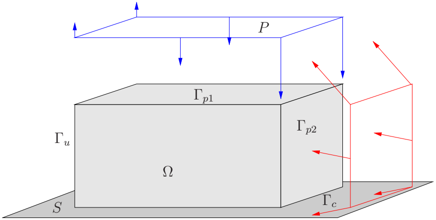
Non-uniqueness of the solution

[Haslinger, Kučera, Vlach, 2008]

$$\Omega = (0, 10) \times (0, 1) \times (0, 1) \ [\mathrm{m}]$$
$$E = 21.19\mathrm{e}10 \ [\mathrm{Pa}], \ \sigma = 0.277$$
$$\boldsymbol{\mathcal{F}} = \mathrm{diag}(\mathcal{F}, \mathcal{F}), \ \mathcal{F} := \mathcal{F}(\|\boldsymbol{u}_t(x)\|)$$



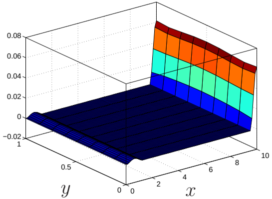
The graph of $\mathcal{F}$



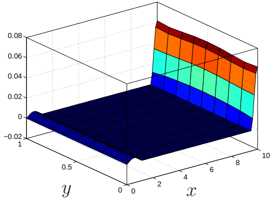
Geometry of the problem

$$\boldsymbol{\mathcal{F}}_\zeta = \zeta \boldsymbol{\mathcal{F}}$$

$\zeta = 1.4$

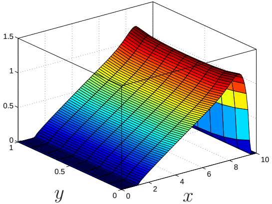


solution 1

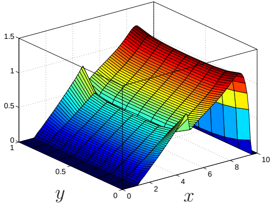


solution 2

Normal displacements



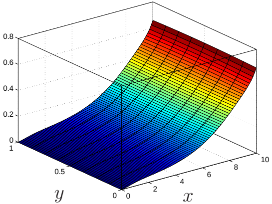
solution 1



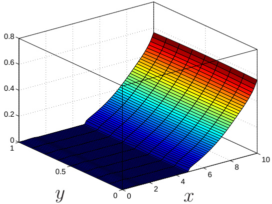
solution 2

Normal stress

$\zeta = 1.4$

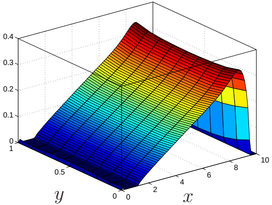


solution 1

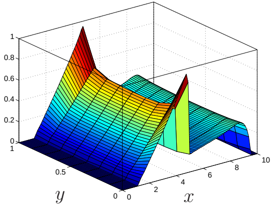


solution 2

The norm of the tangential displacements

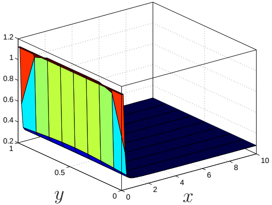


solution 1

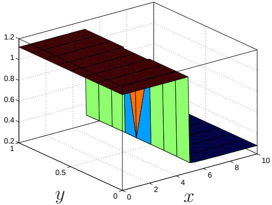


solution 2

The slip bound $\mathcal{F}_\zeta(\|\boldsymbol{u}_t\|)|T_\nu(\boldsymbol{u})|$

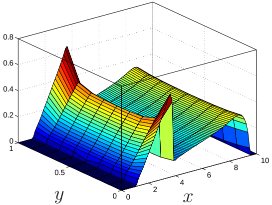


solution 1

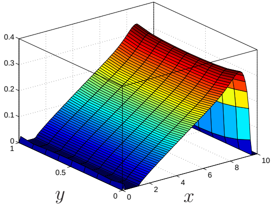


solution 2

The graph of $\mathcal{F}(\|\boldsymbol{u}_t\|)$



$\zeta = 1.37689$



$\zeta = 1.37688$

The slip bound $\mathcal{F}_\zeta(\|\boldsymbol{u}_t\|)|T_\nu(\boldsymbol{u})|$

Existence of local Lipschitz continuous branches of solutions

Algebraic formulation (2D contact problems with isotropic Coulomb friction with the coefficient $\mathcal{F} := \mathcal{F}(x)$)

$$\left.\begin{array}{l}\text{Find } (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t) \in \mathbb{R}^n \times \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_t(\boldsymbol{\lambda}_\nu) \text{ such that} \\ \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{B}_\nu^T \boldsymbol{\lambda}_\nu - \boldsymbol{B}_t^T \boldsymbol{F} \boldsymbol{\lambda}_t, \\ (\boldsymbol{\mu}_\nu - \boldsymbol{\lambda}_\nu) \cdot \boldsymbol{B}_\nu \boldsymbol{u} + \boldsymbol{F}(\boldsymbol{\mu}_t - \boldsymbol{\lambda}_t) \cdot \boldsymbol{B}_t \boldsymbol{u} \leq 0 \\ \qquad\qquad\qquad\qquad \forall\, (\boldsymbol{\mu}_\nu, \boldsymbol{\mu}_t) \in \boldsymbol{\Lambda}_\nu \times \boldsymbol{\Lambda}_t(\boldsymbol{\lambda}_\nu), \end{array}\right\} \qquad (\mathcal{A})$$

- n — the number of degrees of freedom for displacements
- p — the number of the contact nodes
- $\boldsymbol{F} = \operatorname{diag}(\mathcal{F}_1, \ldots, \mathcal{F}_p)$
- $\boldsymbol{\Lambda}_\nu = \mathbb{R}^p_+$
- $\boldsymbol{\Lambda}_t(\boldsymbol{g}) = \{\boldsymbol{\mu} \in \mathbb{R}^p \mid |\mu_i| \leq g_i \;\forall\, i = 1, \ldots, p\}, \quad \boldsymbol{g} \in \boldsymbol{\Lambda}_\nu$

Solution maps

- $\mathscr{S} : \mathbb{R}^n \times \mathbb{R}^p_{++} \rightrightarrows \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p$
  $\mathscr{S}(\bar{\boldsymbol{f}}, \bar{\boldsymbol{\mathcal{F}}}) = \{(\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t)\} \ldots$ the solution set to $(\mathcal{A})$
  with $\boldsymbol{f} := \bar{\boldsymbol{f}}$ and $\boldsymbol{\mathcal{F}} := \bar{\boldsymbol{\mathcal{F}}}$

- $\boldsymbol{\mathcal{S}}_{\bar{\boldsymbol{f}}} : \mathbb{R}^p_{++} \rightrightarrows \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p$
  $\boldsymbol{\mathcal{S}}_{\bar{\boldsymbol{f}}}(\boldsymbol{\mathcal{F}}) = \mathscr{S}(\bar{\boldsymbol{f}}, \boldsymbol{\mathcal{F}}), \quad \boldsymbol{\mathcal{F}} \in \mathbb{R}^p_{++}, \bar{\boldsymbol{f}} \in \mathbb{R}^n$ given

- $\boldsymbol{S}_{\bar{\boldsymbol{\mathcal{F}}}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p$
  $\boldsymbol{S}_{\bar{\boldsymbol{\mathcal{F}}}}(\boldsymbol{f}) = \mathscr{S}(\boldsymbol{f}, \bar{\boldsymbol{\mathcal{F}}}), \quad \boldsymbol{f} \in \mathbb{R}^n, \bar{\boldsymbol{\mathcal{F}}} \in \mathbb{R}^p_{++}$ given

Theorem

Let us suppose that $(\boldsymbol{f}^0, \mathcal{F}^0, \boldsymbol{y}^0) \in \mathbb{R}^n \times \mathbb{R}^p_{++} \times \mathbb{R}^{n+2p}$ is such that $\boldsymbol{y}^0 := (\boldsymbol{u}^0, \lambda^0_\nu, \lambda^0_t) \in \mathcal{S}_{\boldsymbol{f}^0}(\mathcal{F}^0)$ and there exist: a single-valued Lipschitz continuous function $\phi_{\mathcal{F}^0}$ from a neighborhood $\boldsymbol{O}$ of $\boldsymbol{f}^0$ into $\mathbb{R}^{n+2p}$ and a neighborhood $\hat{\boldsymbol{Y}}$ of $\boldsymbol{y}^0$ such that

$$\phi_{\mathcal{F}^0}(\boldsymbol{f}^0) = \boldsymbol{y}^0 \quad \& \quad \phi_{\mathcal{F}^0}(\boldsymbol{f}) = \boldsymbol{S}_{\mathcal{F}^0}(\boldsymbol{f}) \cap \hat{\boldsymbol{Y}} \quad \forall \boldsymbol{f} \in \boldsymbol{O}.$$

Then there are neighborhoods $\boldsymbol{U}$, $\boldsymbol{Y}$ of $\mathcal{F}^0$ and $\boldsymbol{y}^0$, respectively, and a single-valued Lipschitz continuous function $\boldsymbol{\sigma}_{\boldsymbol{f}^0} : \boldsymbol{U} \to \boldsymbol{Y}$ satisfying

$$\boldsymbol{\sigma}_{\boldsymbol{f}^0}(\mathcal{F}^0) = \boldsymbol{y}^0 \quad \& \quad \boldsymbol{\sigma}_{\boldsymbol{f}^0}(\mathcal{F}) = \mathcal{S}_{\boldsymbol{f}^0}(\mathcal{F}) \cap \boldsymbol{Y} \quad \forall \mathcal{F} \in \boldsymbol{U}.$$

- Locally the dependence of a solution on $\mathcal{F}$ can be deduced from the dependence of the solution on the load vector $\boldsymbol{f}$ keeping $\mathcal{F}$ fixed. This is much simpler since the dependence on the load vector is piecewise affine.

Numerical continuation of solution curves

Taking a smooth path

$$\alpha \in I \mapsto \boldsymbol{\mathcal{F}}(\alpha) = (\mathcal{F}_1(\alpha), \ldots, \mathcal{F}_p(\alpha)) \in \mathbb{R}^p_+, \quad I \subset \mathbb{R}^1 \text{ open},$$

we shall approximate the solution curve of the system:

$$\text{Find } \boldsymbol{x} \in \mathbb{R}^{n+2p} \times I \text{ such that } \boldsymbol{\mathcal{H}}(\boldsymbol{x}) = \boldsymbol{0},$$

where

$$\boldsymbol{\mathcal{H}}(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}_\nu^T \boldsymbol{\lambda}_\nu + \boldsymbol{B}_t^T \boldsymbol{\lambda}_t - \boldsymbol{f} \\ \boldsymbol{\lambda}_\nu - \boldsymbol{P}_{\boldsymbol{\Lambda}_\nu}(\boldsymbol{\lambda}_\nu + r\boldsymbol{B}_\nu \boldsymbol{u}) \\ \boldsymbol{\lambda}_t - \boldsymbol{P}_{\boldsymbol{\Lambda}_t(\boldsymbol{\mathcal{F}}(\alpha)\boldsymbol{\lambda}_\nu)}(\boldsymbol{\lambda}_t + r\boldsymbol{B}_t \boldsymbol{u}) \end{pmatrix},$$

$$\boldsymbol{x} := (\boldsymbol{u}, \boldsymbol{\lambda}_\nu, \boldsymbol{\lambda}_t, \alpha) \in \mathbb{R}^{n+2p} \times I,$$

$$\boldsymbol{\Lambda}_t(\boldsymbol{\mathcal{F}}\boldsymbol{g}) = \{\boldsymbol{\mu} \in \mathbb{R}^p \,|\, |\mu_i| \le \mathcal{F}_i g_i \ \forall\, i = 1, \ldots, p\}, \quad \boldsymbol{g} \in \boldsymbol{\Lambda}_\nu.$$

$\mathcal{H}$ is a piecewise smooth function:
for every $\bar{\boldsymbol{x}} \in \mathbb{R}^{n+2p} \times I$ there exists an open neighborhood $\boldsymbol{O} \subset \mathbb{R}^{n+2p} \times I$, $\bar{\boldsymbol{x}} \in \boldsymbol{O}$, and a finite number of smooth functions $\mathcal{H}^{(i)} : \boldsymbol{O} \to \mathbb{R}^{n+2p}$, $i = 1, \ldots l$, such that $\mathcal{H}(\boldsymbol{x}) \in \{\mathcal{H}^{(1)}(\boldsymbol{x}), \ldots, \mathcal{H}^{(l)}(\boldsymbol{x})\}$ for every $\boldsymbol{x} \in \boldsymbol{O}$.

- $\mathcal{H}^{(1)}, \ldots, \mathcal{H}^{(l)}$ — the selection functions for $\mathcal{H}$ at $\bar{\boldsymbol{x}}$
- $I_{\mathcal{H}}(\bar{\boldsymbol{x}}) \equiv \{i \in \{1, \ldots, l\} \mid \mathcal{H}(\bar{\boldsymbol{x}}) = \mathcal{H}^{(i)}(\bar{\boldsymbol{x}})\}$ — the active index set at $\bar{\boldsymbol{x}}$
- $\mathcal{H}^{(i)}$, $i \in I_{\mathcal{H}}(\bar{\boldsymbol{x}})$ — the active selection functions for $\mathcal{H}$ at $\bar{\boldsymbol{x}}$

Piecewise-smooth variant of the Moore-Penrose continuation

- computes a sequence $\{\boldsymbol{x}^j\}$ with $\|\mathcal{H}(\boldsymbol{x}^j)\| \leq \varepsilon$ and a sequence of the corresponding unit tangential vectors $\{\boldsymbol{\tau}^j\}$:

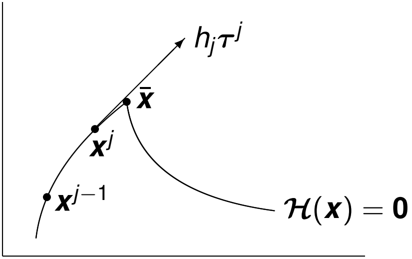
$$\mathcal{H}'(\boldsymbol{x}^j; \boldsymbol{\tau}^j) = \mathbf{0}, \quad \|\boldsymbol{\tau}^j\| = 1,$$

- consists of two steps:
  prediction — an initial approximation of the new point $\boldsymbol{x}^{j+1}$ is given by

$$\boldsymbol{X}^0 := \boldsymbol{x}^j + h\boldsymbol{\tau}^j, \quad h > 0,$$

  corrections — $\boldsymbol{x}^{j+1}$, $\boldsymbol{\tau}^{j+1}$ are obtained by a piecewise smooth Newton-like procedure.

Difficulty with the points of non-differentiability



- to handle the points of non-differentiability of $\mathcal{H}$, the so-called test functions θ^k, $k = 1, 2, 3$, are employed:

$$\theta_i^1(\boldsymbol{x}) = (\boldsymbol{\lambda}_\nu + r\boldsymbol{B}_\nu\boldsymbol{u})_i,$$
$$\theta_i^2(\boldsymbol{x}) = (\boldsymbol{\lambda}_t + r\boldsymbol{B}_t\boldsymbol{u})_i - \mathcal{F}_i(\alpha)\lambda_{\nu,i},$$
$$\theta_i^3(\boldsymbol{x}) = (\boldsymbol{\lambda}_t + r\boldsymbol{B}_t\boldsymbol{u})_i + \mathcal{F}_i(\alpha)\lambda_{\nu,i},$$
$$i = 1, \ldots, p,\ \boldsymbol{x} \in \mathbb{R}^{n+2p} \times I.$$

Their signs and vanishing components characterize uniquely the selection functions for $\mathcal{H}$ which are active at $\boldsymbol{x}$:

$$\theta_i^1(\boldsymbol{x}) \geq 0 \ldots \text{contact}, \quad \theta_i^1(\boldsymbol{x}) < 0 \ldots \text{no contact}$$
$$\theta_i^2(\boldsymbol{x}) > 0 \vee \theta_i^3(\boldsymbol{x}) < 0\ (\theta_i^1(\boldsymbol{x}) \geq 0) \ldots \text{contact-slip}$$
$$\theta_i^2(\boldsymbol{x}) < 0 < \theta_i^3(\boldsymbol{x})\ (\theta_i^1(\boldsymbol{x}) \geq 0) \ldots \text{contact-stick}$$

at the i^{th} contact node.

Algorithm

Data: $\varepsilon, \varepsilon' > 0$, $h \geq h_{\min} > 0$, $k_{\max} > 0$ and $\boldsymbol{x}^0 \in \mathbb{R}^{n+2p} \times I$, $\boldsymbol{\tau}^0 \in \mathbb{R}^{n+2p+1}$ satisfying:

$$\|\mathcal{H}(\boldsymbol{x}^0)\| < \varepsilon, \quad \mathcal{H}'(\boldsymbol{x}^0; \boldsymbol{\tau}^0) = \mathbf{0}, \quad \|\boldsymbol{\tau}^0\| = 1.$$

Step 1: Set $j := 0$.

Step 2 (*prediction*): Set $\boldsymbol{X}^0 := \boldsymbol{x}^j + h\boldsymbol{\tau}^j$, $\boldsymbol{T}^0 := \boldsymbol{\tau}^j$.

Step 3 (*corrections*): Compute the iterates $\boldsymbol{X}^k$ and $\boldsymbol{T}^k$ until

$$(\|\mathcal{H}(\boldsymbol{X}^k)\| < \varepsilon \ \& \ \|\boldsymbol{X}^k - \boldsymbol{X}^{k-1}\| < \varepsilon') \ \vee \ k = k_{\max}.$$

Step 4: If the corrections have converged, set

$$\boldsymbol{x}^{j+1} := \boldsymbol{X}^k, \ \boldsymbol{\tau}^{j+1} := \boldsymbol{T}^k$$

and go to Step 7.

Step 5: If $h > h_{\min}$, decrease h and go to Step 2.

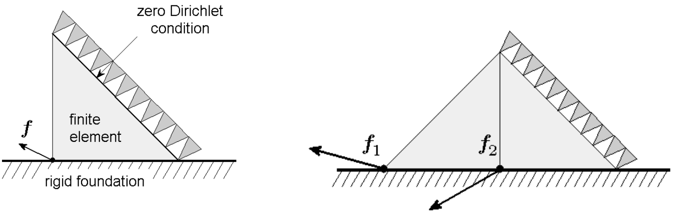
Step 6: Vanishing components of $\theta^1(\boldsymbol{x}^j)$, $\theta^2(\boldsymbol{x}^j)$, $\theta^3(\boldsymbol{x}^j)$ determine a new selection function $\mathcal{H}^{(i)}$ for $\mathcal{H}$ which is likely to be active in a vicinity of $\boldsymbol{x}^j$. Compute $\boldsymbol{\tau}^j$ satisfying

$$\nabla \mathcal{H}^{(i)}(\boldsymbol{x}^j)\boldsymbol{\tau}^j = \mathbf{0}, \quad \|\boldsymbol{\tau}^j\| = 1$$

preserving the so-called orientation. Initialize h and go to Step 2.

Step 7: Define h for the next iteration according to the rate of convergence of the corrections, set $j := j + 1$ and go to Step 2.

Numerical examples



$\lambda > 0$, $\mu > 0$ ... the Lamé coefficients

Geometry of the models

- One contact node: $p = 1$, $n = 2$
- Two contact nodes: $p = 2$, $n = 4$

The Algorithm: Parameter settings

$\varepsilon = \varepsilon' = 10^{-6}$, $h_{\min} = 10^{-5}$, $h = 0.05$, $k_{\max} = 10$.

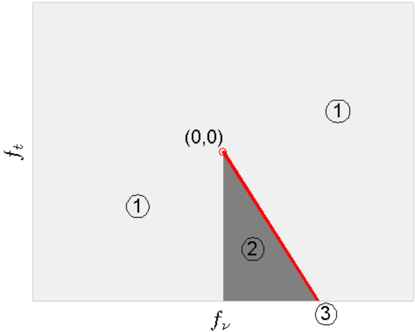
Model: $p = 1$, $n = 2$

Analysis: [Hild, Renard, 2005]

1. if $\{(\lambda + 3\mu)f_\nu + (\lambda + \mu)f_t \leq 0 \wedge f_\nu \leq 0\} \vee \{(\lambda + 3\mu)f_\nu + (\lambda + \mu)f_t > 0\}$
   then $\exists$ one solution branch
2. if $\{(\lambda + 3\mu)f_\nu + (\lambda + \mu)f_t < 0 \wedge f_\nu > 0\}$
   then $\exists$ two solution branches
3. if $\{(\lambda + 3\mu)f_\nu + (\lambda + \mu)f_t = 0 \wedge f_\nu > 0\}$
   then $\exists$ one bifurcating branch

... the explicit formulae available

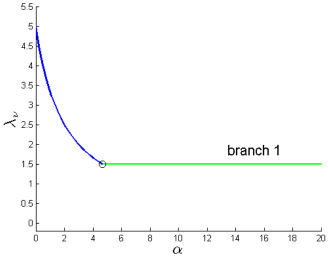
Transition sets



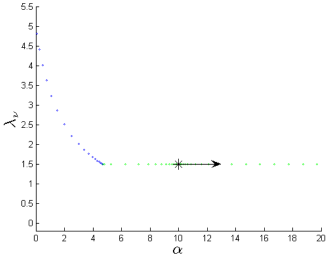
Example 1

One solution branch: $f_\nu = 1.5$, $f_t = 7$, $\lambda = \mu = 1$.

Exact solution



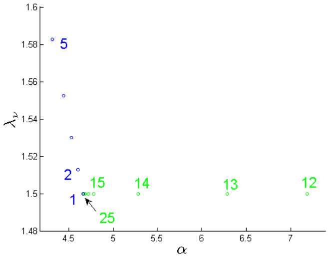
Computed solution



green = contact-stick, blue = contact-slip
○ = the transition point

$*$ initial points of the path-following,
the arrows always mark the positive directions

Computed solution: zoom



test functions:

12: $\theta^1 = 1.5000$, $\theta^2 = -3.7695$, $\theta^3 = 17.7695$

15: $\theta^1 = 1.5000$, $\theta^2 = -0.1754$, $\theta^3 = 14.1754$

25: $\theta^1 = 1.5000$, $\theta^2 = 0.0000$, $\theta^3 = 14.0000$

1 : $\theta^1 = 1.5128$, $\theta^2 = 0.0128$, $\theta^3 = 13.9613$

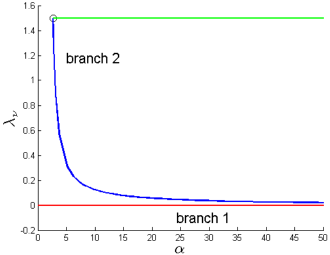
2 : $\theta^1 = 1.5299$, $\theta^2 = 0.0299$, $\theta^3 = 13.9102$

green = `contact-stick`, blue = `contact-slip`

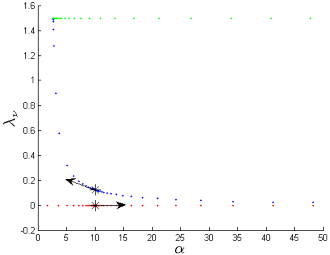
Example 2

Two solution branches: $f_\nu = 1.5,\ f_t = -4,\ \lambda = \mu = 1.$

Exact solution



Computed solution



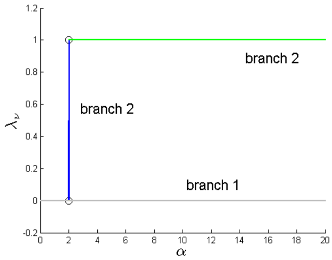
red = no contact, green = contact-stick, blue = contact-slip
o = the transition point

$*$ initial points of the path-following,
the arrows always mark the positive directions

Example 3

One bifurcating branch: $f_\nu = 1,\ f_t = -2,\ \lambda = \mu = 1.$

Exact solution

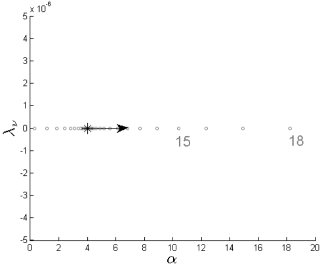


branch 1: gray = `grazing contact`
branch 2: green = `contact-stick`, blue = `contact-slip`

- For $\alpha = (\lambda + 3\mu)/(\lambda + \mu) = 2$ the branch 1 *bifurcates*.
- The bifurcating branch 2 contains the continuum of solutions represented by the vertical segment.

Computed solutions:

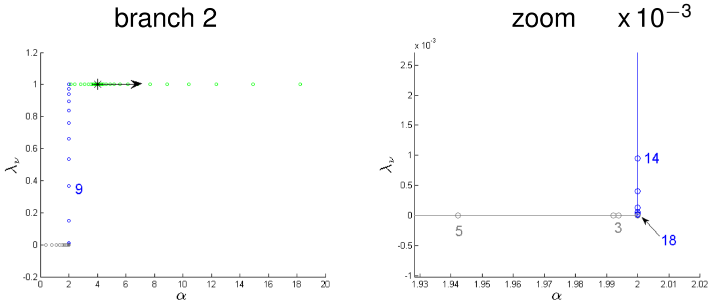
branch 1



15: $\theta^1 = 0.0000$, $\theta^2 = -1.0000$, $\theta^3 = -1.0000$
19: $\theta^1 = 0.0000$, $\theta^2 = -1.0000$, $\theta^3 = -1.0000$

gray = `grazing contact`

Computed solutions:



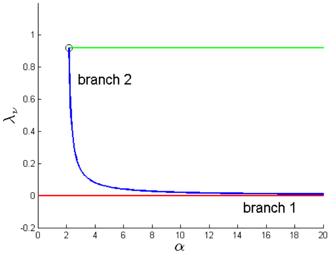
9 : $\theta^1 = 0.3669$, $\theta^2 = -2.1007$, $\theta^3 = -0.6331$
14: $\theta^1 = 0.0001$, $\theta^2 = -1.0004$, $\theta^3 = -0.9999$
18: $\theta^1 = 0.0000$, $\theta^2 = -1.0000$, $\theta^3 = -0.9999$
3 : $\theta^1 = 0.0000$, $\theta^2 = -1.0000$, $\theta^3 = -1.0000$
5 : $\theta^1 = 0.0000$, $\theta^2 = -1.0000$, $\theta^3 = -1.0000$

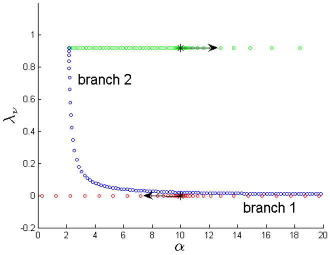
green = `contact-stick`, blue = `contact-slip`,
gray = `grazing contact`

Example 4

A small data perturbation destroys the bifurcation:
$f_\nu = 1 - 0.08$, $f_t = -2$, $\lambda = \mu = 1$

Exact solution | Computed solution





two solution branches

red = `no contact`, green = `contact-stick`, blue = `contact-slip`

Model: $p = 2$, $n = 4$

Data:

- $\boldsymbol{f} = (\boldsymbol{f}_1, \boldsymbol{f}_2)$, $\boldsymbol{f}_1 = (f_{\nu,1}, f_{t,1})$, $\boldsymbol{f}_2 = (f_{\nu,2}, f_{t,2})$
- $\mathcal{F}(\alpha) = (\alpha, \alpha) \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$
- $$\boldsymbol{A} = \begin{pmatrix} \frac{\mu}{2} & 0 & -\frac{\mu}{2} & -\frac{\mu}{2} \\ 0 & \frac{\lambda+2\mu}{2} & -\frac{\lambda}{2} & -\frac{\lambda+2\mu}{2} \\ -\frac{\mu}{2} & -\frac{\lambda}{2} & \lambda+3\mu & 0 \\ -\frac{\mu}{2} & -\frac{\lambda+2\mu}{2} & 0 & \lambda+3\mu \end{pmatrix}$$

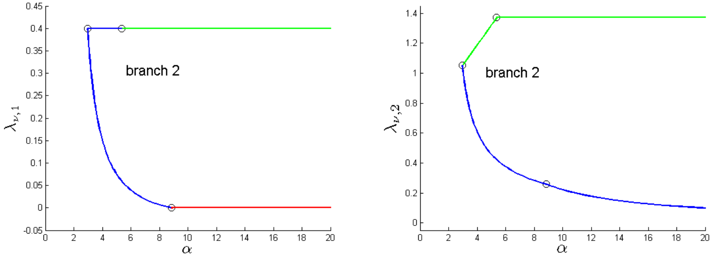
  ... the stiffness matrix

Observation:

$\exists$ at most two solution branches

Example 1

Two solution branches: $f_{\nu,1} = 0.4000$, $f_{t,1} = -2.1417$, $f_{\nu,2} = 1.3717$, $f_{t,2} = -2.1417$, $\lambda = \mu = 1$.



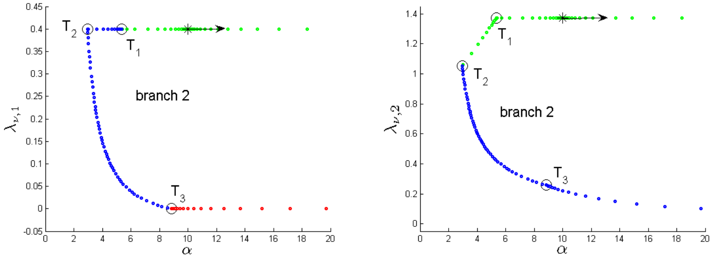
and

$$\lambda_{\nu,1} = \lambda_{\nu,2} = 0\,, \quad \alpha \in \mathbb{R}$$

the "trivial" branch 1 of the `no contact` points

red = `no contact`, green = `contact-stick`, blue = `contact-slip`

Computed solution: branch 2



$\circ$ = the transition points
red = no contact, green = contact-stick, blue = contact-slip

Classification of the transition points

T_1: 19

$$
\begin{array}{ll}
17: & \theta_1^1 = 0.4000,\ \theta_1^2 = -4.2838,\ \theta_1^3 = \ \ 0.0005 \\
17: & \theta_2^1 = 1.3717,\ \theta_2^2 = -9.4874,\ \theta_2^3 = \ \ 5.2041 \\
19: & \theta_1^1 = 0.4000,\ \theta_1^2 = -4.2834,\ \theta_1^3 = \ \ 0.0000 \\
19: & \theta_2^1 = 1.3717,\ \theta_2^2 = -9.4859,\ \theta_2^3 = \ \ 5.2025 \\
21: & \theta_1^1 = 0.4000,\ \theta_1^2 = -4.2542,\ \theta_1^3 = -0.0146 \\
21: & \theta_2^1 = 1.3644,\ \theta_2^2 = -9.3940,\ \theta_2^3 = \ \ 5.0668
\end{array}
$$

Hence,

green = `contact-stick` $\longrightarrow$ blue = `contact-slip`

green = `contact-stick` $\longrightarrow$ green = `contact-stick`

Classification of the transition points

T_2: 38

$$
\begin{array}{ll}
36: & \theta_1^1 = 0.4000,\ \theta_1^2 = -3.0020,\ \theta_1^3 = -0.6407 \\
36: & \theta_2^1 = 1.0513,\ \theta_2^2 = -6.2058,\ \theta_2^3 = \ \ 0.0003 \\
38: & \theta_1^1 = 0.4000,\ \theta_1^2 = -3.0019,\ \theta_1^3 = -0.6407 \\
38: & \theta_2^1 = 1.0513,\ \theta_2^2 = -6.2056,\ \theta_2^3 = \ \ 0.0000 \\
40: & \theta_1^1 = 0.3918,\ \theta_1^2 = -2.9950,\ \theta_1^3 = -0.6689 \\
40: & \theta_2^1 = 1.0372,\ \theta_2^2 = -6.1748,\ \theta_2^3 = -0.0165
\end{array}
$$

Hence,

blue = `contact-slip` $\longrightarrow$ blue = `contact-slip`

green = `contact-stick` $\longrightarrow$ blue = `contact-slip`

Classification of the transition points

T_3: 94

$$
\begin{aligned}
&89: \quad \theta_1^1 = 0.0002,\ \theta_1^2 = -2.2294,\ \theta_1^3 = -2.2265 \\
&89: \quad \theta_2^1 = 0.2584,\ \theta_2^2 = -5.3652,\ \theta_2^3 = -0.7997 \\
&94: \quad \theta_1^1 = 0.0000,\ \theta_1^2 = -2.2278,\ \theta_1^3 = -2.2278 \\
&94: \quad \theta_2^1 = 0.2578,\ \theta_2^2 = -5.3667,\ \theta_2^3 = -0.8000 \\
&96: \quad \theta_1^1 = -0.0024,\ \theta_1^2 = -2.2302,\ \theta_1^3 = -2.2302 \\
&96: \quad \theta_2^1 = 0.2554,\ \theta_2^2 = -5.3595,\ \theta_2^3 = -0.8024
\end{aligned}
$$

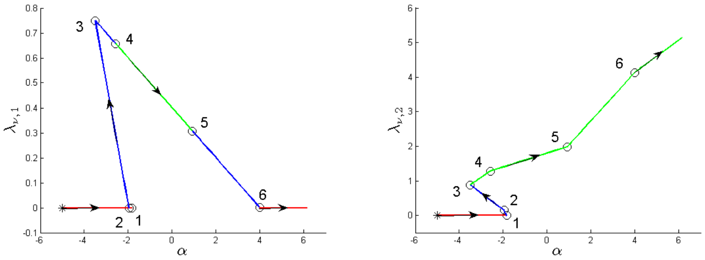
Hence,

blue = `contact-slip` $\longrightarrow$ red = `no contact`
blue = `contact-slip` $\longrightarrow$ blue = `contact-slip`

Consider a smooth loading path $\alpha \in I \mapsto \boldsymbol{f}(\alpha)$ for $\mathcal{F}$ fixed

Example 2

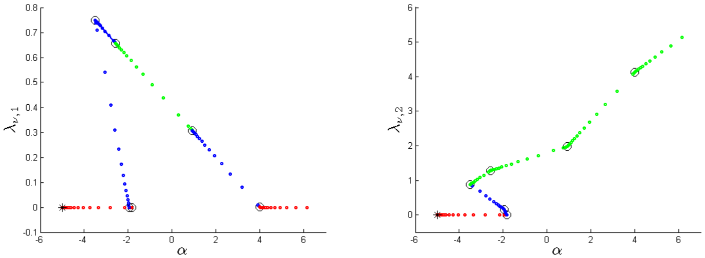
$\mathcal{F} = (4,4) \in \mathbb{R}^2, \quad f_{\nu,1} = -0.1\alpha + 0.4,\ f_{\nu,2} = 0.2\alpha + 1.8,$
$f_{t,1} = 1.1\alpha + 0.2,\ f_{t,2} = 0.8\alpha - 0.1, \quad \lambda = \mu = 1.$



red = `no contact`, green = `contact-stick`, blue = `contact-slip`

∘ = `the transition point`

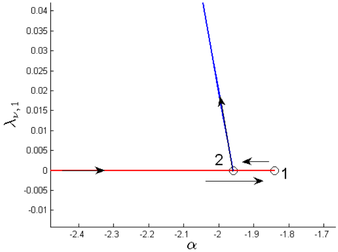
Computed solution



red = no contact, green = contact-stick, blue = contact-slip

o = the transition point

Zoom: transition points 1 and 2



red = `no contact`, blue = `contact-slip`

○ = `the transition point`

Thank you for your attention.

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