Stabilization methods for transient transport problems

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Outline of the talk

- Two families of stabilized finite element methods
  - Petrov-Galerkin methods: (Streamline Upwind Petrov-Galerkin)
  - Symmetric stabilization methods (subgrid viscosity, interior penalty)
- The SU Petrov-Galerkin method and $A$-stable time discretization
- Symmetric stabilization methods and $A$-stable time discretization
- Symmetric stabilization methods and Explicit Runge-Kutta methods
Two families of stabilized finite element methods I

\[ \beta \cdot \nabla u + \mu u = f \quad \text{in } \Omega \]
\[ u = 0 \quad \text{on } \partial \Omega \]

- **Streamline Upwind Petrov Galerkin type methods:** Find \( u_h \in V_h \) such that

\[
\int_{\Omega} (\beta \cdot \nabla u_h + \mu u_h)(v_h + \delta \beta \cdot \nabla v_h) \, dx = \int_{\Omega} f(v_h + \delta \beta \cdot \nabla v_h) \, dx, \quad \forall v_h \in V_h
\]

- **Symmetric stabilization methods:** Find \( u_h \in V_h \) such that

\[
\int_{\Omega} (\beta \cdot \nabla u_h + \mu u_h) v_h \, dx + s(u_h, v_h) = \int_{\Omega} f v_h \, dx, \quad \forall v_h \in V_h
\]

\( s(u_h, v_h) \) is a symmetric, positive semidefinite operator that:
(i) controls fluctuations, (ii) is (weakly) consistent
Two families of stabilized finite element methods I

- both classes allows for estimates of the type

\[ \|u - u_h\| + \|\delta^{1/2} \beta \cdot \nabla (u - u_h)\| \lesssim h^{p+\frac{1}{2}} |u|_{p+1} \]

where \( p \) is the polynomial order and \( \delta \) a stabilization parameter \( \delta \approx h/|\beta| \).

- **SUPG**: Hughes et al. and Johnson et al.

- **Symmetric stabilization**: subgrid viscosity [Guermond ’99], orthogonal subscale stabilization [Codina ’00], local projection stabilization [Braack et al. ’07], interior penalty on gradient jumps (CIP) [EB & Hansbo ’04]

- The **discontinuous Galerkin method** (Raviart & Lesaint) is also a symmetric stabilization method.

- **local estimates** for SUPG (Johnson, Nävert), DG (Guzman), CIP (EB, Guzman, Lekhemann)
What is known for stabilization for transient flow problems

- **SUPG**
  - Space-time FEM [Johnson et al. '84], [Hansbo-Szepessy '90], [Hughes et al '91, '94]
  - Analysis for first order A-stable schemes [Lube-Weiss '95]
  - Proof that the problem is well-posed for each time-step for the \( \theta \)-method [Bochev et al. '04]

- **Symmetric stabilization**
  - Semi-discretization in space, subgrid viscosity [Guermond '01]
  - Backward Euler/orthogonal subscales [Codina & Blasco '02]

- **Runge-Kutta FEM**
  - Runge-Kutta discontinuous Galerkin, [Cockburn et al. '89,'90],
  - Analysis RK2, DG, smooth solutions
    - scalar conservation laws 1D, [Zhang & Shu '04],
    - symmetrizable systems 1D, [Zhang & Shu '06]
Part I

SU Petrov-Galerkin method and $A$-stable FD
Main results (Streamline Upwind Petrov Galerkin)

- All A-stable schemes stable and convergent for smooth solutions
  - Convergence in the $L^2$-norm and in the $h$-weighted material derivative
  - $O(\tau^l + h^{p+1/2})$ energy error estimate, for $l$-order FD in time
  - The analysis requires more regularity than the standard analysis for transient problems.

- Insufficient regularity
  - Optimal estimates still hold under the condition $h \leq |\beta| \tau$.
  - This is what is known as the small time-step limit instability
    - Observed numerically in the form of oscillations close to layers
    - First theoretical explanation

- These results appear to be new
  - Allows to understand SUPG for transient problems
  - Ongoing work:
    - Transient convection–diffusion equations with SUPG
    - Transient Stokes’ with PSPG
The continuous problem: pure transport

Find $u$ such that:

$$
\begin{align}
\partial_t u + \beta \cdot \nabla u &= f & \text{in } \Omega, \ t > 0 \\
 u &= 0 & \text{on } \partial \Omega^-, \ t > 0 \\
 u(\cdot, 0) &= u_0 & \text{in } \Omega,
\end{align}
(1)
$$

velocity vector field $\beta \in W^{1,\infty}(\Omega)$, $f \in C^0(0, T; L^2(\Omega))$ is a source function, and $u_0 \in L^2(\Omega)$ is the initial data, with $u_0|_{\partial \Omega^-} = 0$.

$\partial \Omega^-$ is the inflow boundary, i.e. where $\beta \cdot n < 0$;

$\partial \Omega^+$ is the outflow boundary, i.e. where $\beta \cdot n > 0$. 
SU Petrov-Galerkin + Crank-Nicolson

Finite element spaces:

\[ V_h := \{ v_h \in C(\bar{\Omega}) : v_h|_{\partial \Omega} = 0; v_h|_K \in P_p(K), \forall K \in T_h \}, \]

\[ W_h := \{ w_h : w_h(v_h) = v_h + \delta \beta \cdot \nabla v_h; v_h \in V_h \}, \]

Bilinear forms:

\[ (u, v) := \int_{\Omega} uv \, ds, \quad a(u, v) := \int_{\Omega} \beta \cdot \nabla uv \, ds, \quad F^n(v) := \int_{\Omega} f(t^n)v \, dx. \]

The Crank-Nicolson time-stepping formulation:

\[
\begin{aligned}
\text{For } 1 \leq n \leq N, \text{ find } u^n_h \in V_h \text{ such that:} \\
(\partial_\tau u^n_h, w_h) + a(\bar{u}^n_h, w_h) = F^{n-\frac{1}{2}}(w_h) \quad \forall w_h \in W_h, \text{ with } u^0_h = \pi_h u_0,
\end{aligned}
\]

where \( \partial_\tau u^n_h = \tau^{-1}(u^n_h - u^{n-1}_h) \) and \( \bar{u}^n_h = (u^n_h + u^{n-1}_h)/2, \) \( N := \left\lceil T/\tau \right\rceil. \)
Stability 1

Take $w_h(u^n_h) = \bar{u}_h^n + \delta \beta \cdot \nabla \bar{u}_h^n$,

$$
(\partial_{\tau} u_h^n, \bar{u}_h^n) + \frac{1}{2} \beta^2 \frac{1}{2} \left\| \bar{u}_h^n \right\|_{\partial \Omega}^2 \delta \beta \cdot \nabla \bar{u}_h^n \right\|_0^2 + \delta \beta \cdot \nabla \bar{u}_h^n \right\|_1^2 + (\partial_{\tau} u_h^n, \delta \beta \cdot \nabla \bar{u}_h^n) = F^{n-\frac{1}{2}}(w_h(\bar{u}_h^n)).
$$

The term $l_1$ does not have a sign!

Observation: $l_0$ and $l_1$ forms half of a quadratic form $a^2 + ab$, adding $b^2 + ab$ would give us $(a + b)^2$. 
Stabilized methods

SUPG+ $A$-stable FD

Symmetric stab./ $A$-stable FD

Stability II

- Take $w_h = \delta \partial_\tau u_h^n + \delta \beta \cdot \nabla \partial_\tau u_h^n$,

\[
\underbrace{\delta \| \partial_\tau u_h^n \|^2}_{l_2} + \underbrace{(\beta \cdot \nabla \bar{u}_h^n, \delta \partial_\tau u_h^n)}_{l_3} + (\partial_\tau u_h^n + \beta \cdot \nabla \bar{u}_h^n, \delta^2 \beta \cdot \nabla \partial_\tau u_h^n) = F^n(\delta \partial_\tau u_h^n)
\]

- $l_2$ and $l_3$ are the parts needed to get a quadratic form:

\[
\sum_{j=0}^{3} l_j = \delta \| \partial_\tau u_h^n + \beta \cdot \nabla \bar{u}_h^n \|^2.
\]

- The remaining terms have a sign or telescope

\[
(\partial_\tau u_h^n, \delta^2 \beta \cdot \nabla \partial_\tau u_h^n) = \frac{\delta^2}{2} \| \beta \cdot n \|_{\partial \Omega^+} \| \partial_\tau u_h^n \|_{\partial \Omega^+}^2,
\]

\[
(\beta \cdot \nabla \bar{u}_h^n, \delta^2 \beta \cdot \nabla \partial_\tau u_h^n) = \| \delta \beta \cdot \nabla u_h^n \|^2 - \| \delta \beta \cdot \nabla u_h^{n-1} \|^2.
\]
Stability estimate ($\nabla \cdot \beta = 0$)

Lemma

For the solution of the SUPG/Crank-Nicolson scheme there holds

$$
\| u_h^N \|_{\beta}^2 + \sum_{n=1}^{N} \tau \| u_h^n \|_2^2 \leq C \sum_{n=1}^{N} \tau T (\| \delta \partial_T f (t^{n-\frac{1}{2}}) \|_2^2 + \| f (t^{n-\frac{1}{2}}) \|_2^2) + \| u_h^0 \|_{\beta}^2,
$$

where $\| u_h \|_{\beta}^2 := \| u_h \|_2^2 + \| \delta \beta \cdot \nabla u_h \|_2^2$ and

$$
\| u_h^n \|_2^2 := \delta \| \partial_T u_h^n + \beta \cdot \nabla \bar{u}_h^n \|_2^2 + \text{boundary contribution}
$$

For $f \in C^0(0, T; L^2(\Omega))$ only there holds

$$
\sum_{n=1}^{N} \tau \| \delta \partial_T f (t^{n-\frac{1}{2}}) \|_2^2 + \| f (t^{n-\frac{1}{2}}) \|_2^2 \lesssim \sum_{n=1}^{N} \tau \left(1 + \frac{\delta}{T} \right)^2 \| f (t^{n-\frac{1}{2}}) \|_2^2
$$
Convergence Crank-Nicolson

For all $1 \leq n \leq N$ there holds

$$
\| u_h^N - u(t^N) \|^2 + \sum_{n=1}^{N} \| \bar{u}_h^n - u(t^{n-\frac{1}{2}}) \|^2 \lesssim Th^{2p+1} \int_0^{t^N} (1 + \delta^2) \| u_{tt} \|_{p+1}^2 \, ds
$$

$$
+ T \tau^4 (1 + \delta^2) \int_0^{t^N} (\| u_{tttt} \|^2 + \| \beta \cdot \nabla u_{ttt} \|^2) \, ds
$$

- Only the highest derivatives in time are included.
- Proof: Uses stability and a hyperbolic Ritz-projection.
Convergence Crank-Nicolson, insufficient regularity

For all $1 \leq n \leq N$ there holds

$$
\| u_h^N - u(t^N) \|^2 + \sum_{n=1}^{N} \left\| \bar{u}_h^n - u(t^n - \frac{1}{2}) \right\|^2 \lesssim C_{\delta \tau} T \left( h^{2p+1} \int_0^{t^n} \| u_t \|_{p+1}^2 \, ds \right)
$$

$$
+ \tau^4 \int_0^{t^n} \left( \| u_{ttt} \|^2 + \| \beta \cdot \nabla u_{tt} \|^2 \right) \, ds
$$

where

$$
C_{\delta \tau} = c \left( 1 + \frac{\delta}{\tau} \right).
$$

Optimal convergence requires $\delta \lesssim \tau$. 
Some remarks

- The same analysis holds for any $A$-stable scheme.
- The analysis uses the hyperbolic Ritz-projection obtained from the stationary problem.
- $\beta$ must be constant in time. To work around:
  - perturbation argument
  - show that Ritz-proj. commutes with time derivation
  - use another projection and derive the formulation in time
- It is difficult to construct numerical examples showing the instability.
- Time-discretization dissipation: same form, but in the norm $\|u\|_\beta$
**Consistent vs. non-consistent Crank-Nicolson/SUPG**

Some authors have advocated the use of the **non-consistent** method:

\[
\begin{align*}
\text{For } 1 \leq n \leq N, \text{ find } u^n_h & \in V_h \text{ such that:} \\
(\partial_T u^n_h, v_h) + a(\bar{u}^n_h, v_h + \delta \beta \cdot \nabla v_h) &= 0 \quad \forall v_h \in V_h,
\end{align*}
\]

- Crank-Nicolson, \( p = 1 \)
- 256 elements on the boundary, 1280 timesteps, one revolution

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**initial data**  
**no stabilization**  
**consistent SUPG**  
**non-consist. SUPG**
Part II

Symmetric stabilization/\(A\)-stable FD
The continuous problem

- Find $u$ such that

\[
\partial_t u + \sigma u + \beta \cdot \nabla u - \nu \Delta u = f, \quad \text{in } \Omega, \quad t > 0 \\
u = 0, \quad \text{on } \partial\Omega, \quad t > 0 \\
\left. u \right|_{t=0} = u_0, \quad \text{in } \Omega,
\]

with $\sigma \geq 0$, $\nu > 0$, $\sigma - \nabla \cdot \beta/2 \geq \sigma_0 > 0$ and $u_0 \in L^2(\Omega)$.

- Variational formulation, for all $t > 0$ find $u \in H^1_0(\Omega)$ such that:

\[
(\partial_t u, v) + a(u, v) = (f, v), \quad \forall v \in H^1_0(\Omega)
\]

with

\[
a(u, v) := \sigma(u, v) + (\beta \cdot \nabla u, v) + (\nu \nabla u, \nabla v).
\]
Space semi-discrete formulation

For \( t \in (0, T) \), find \( u_h \in C^1((0, T); V^p_h) \)

\[
\left\{ \begin{aligned}
(\partial_t u_h, v_h) + a(u_h, v_h) + s(u_h, v_h) &= (f, v_h), \quad \text{in } (0, T), \\
u_h(0) &= u^0_h,
\end{aligned} \right.
\]

for all \( v_h \in V^p_h \),

triple norm

\[
\|v_h\|^2 := \|\sigma^{\frac{1}{2}} v_h\|^2 + \|\nu^{\frac{1}{2}} \nabla v_h\|^2 + s(v_h, v_h)
\]
Space semi-discrete formulation II

- $s(u_h, v_h)$ is a stabilization operator that satisfies:
  - [(H1)] Symmetry and non-negativity:
    
    \[
    s(u_h, v_h) = s(v_h, u_h), \quad s(v_h, v_h) \geq 0 \quad \forall u_h, v_h \in V_h;
    \]

- [(H2)] Control of fluctuations:
  
  Example: \[ \|\delta^{1/2} |\beta|(\nabla u_h - \pi_h \nabla u_h)\|_2^2 \lesssim s(u_h, u_h) \]

- [(H3)] Weak consistency:
  
  \[
  |s(\pi_h v, v_h)| \leq c h^{p+1/2} |\beta|^{1/2} \|\nu\|_{p+1} \|v_h\|, \tag{2}
  \]
  
  for all $v_h \in V_h$. 

Symmetric stabilization $+ \theta$-method

For $0 \leq n \leq N$, find $u_{h}^{n+1} \in V_{h}^{p}$ such that

$$
\frac{1}{\tau} (u_{h}^{n+1} - u_{h}^{n}, v_{h}) + a_{h}(u_{h}^{n+\theta}, v_{h}) + s(u_{h}^{n+\theta}, v_{h}) = (f(t_{n+\theta}), v_{h}),
$$

$u_{h}^{0} = u_{0,h},$

for all $v_{h} \in V_{h}^{p}$ with

$$
t_{n} := n\tau, \quad u_{h}^{n+\theta} := \theta u_{h}^{n+1} + (1 - \theta) u_{h}^{n}, \quad t_{n+\theta} := \theta t_{n+1} + (1 - \theta) t_{n}.
$$
Stability: Backward Euler & Crank-Nicolson

- **Backward-Euler** $\theta = 1$:

$$\|u_h^n\|^2 + 2C_a \sum_{l=0}^{n-1} \tau \|u_{h}^{l+1}\|^2 \leq C_T \left( \|u_0\|^2 + \sum_{l=0}^{n-1} \tau \|f(t_{l+1})\|^2 \right).$$

- **Crank-Nicolson** $\theta = \frac{1}{2}$:

$$\|u_h^n\|^2 + 2C_a \sum_{l=0}^{n-1} \tau \|u_{h}^{l+\frac{1}{2}}\|^2 \leq C_T \left( \|u_0\|^2 + \sum_{l=0}^{n-1} \tau \|f(t_{l+\frac{1}{2}})\|^2 \right).$$

with $C_T > 0$ independent of $\nu$, $h$ and $\tau$. 
Error estimate ($\nu$-uniform): $\theta$-scheme

Let $e^n_h := \pi_h u(t^n) - u^n_h$. Then, for $N \geq 2$, there holds

$$
\| e^n_h \|_2^2 + c \sum_{m=0}^{n-1} \tau \| e^{m+\theta}_h \|_2^2 \leq c T \mathcal{H}^2(T, \nu, \beta, \sigma) \| u \|_{C^0([0, T]; H^{p+1}(\Omega))}^2 
$$

$$
+ T^2 \tau^2 \| \partial_{tt} u \|_{C^0([0, T]; L^2(\Omega))}^2 + T \tau^4 \| \partial_{ttt} u \|_{L^2(0, T; L^2(\Omega))}^2
$$

with

$$
\mathcal{H}(T, \mu, \beta, \sigma) := h^p \left( |\beta|_{\infty} h^{\frac{1}{2}} + T^{\frac{1}{2}} |\beta|_{1, \infty, \Omega} h + |\sigma|_{\infty} h + |\nu|^\frac{1}{2} \right).
$$

A similar result holds for BDF2.
Reducing the stencil for transient flows

- Symmetric stabilization leads to an extended matrix stencil.
- The stabilization can be treated in an explicit fashion.
- BDF1, adds in the left hand side of the energy estimate:

\[
\sum_{n=1}^{N} \tau^{-1} \| u^n - u^{n-1} \|^2.
\]

- BDF2, adds in the left hand side of the energy estimate:

\[
\sum_{n=2}^{N} \tau^{-1} \| u^n - 2u^{n-1} + u^{n-2} \|^2.
\]

- Use time dissipation to stabilize explicit treatment of stabilization.
Explicit treatment of the stabilization term

\textbf{BDF1:} \begin{align*}
\frac{1}{\tau} (u_h^n - u_h^{n-1}, v_h) + a_h(u_h^n, v_h) + s(u_h^{n-1}, v_h) &= (f(t_n), v_h), \\
\quad u_h^0 &= u_{0,h},
\end{align*}

\textbf{BDF2:} \begin{align*}
\frac{1}{2\tau} (3u_h^n - 4u_h^{n-1} + u_h^{n-2}, v_h) + a_h(u_h^n, v_h) + s(\tilde{u}_h^n, v_h) &= (f(t_n), v_h), \\
\quad \tilde{u}_h^n &= 2u_h^{n-1} - u_h^{n-2}, \\
\quad u_h^0 &= u_{0,h},
\end{align*}

- Under CFL condition $\tau \leq Ch/|\beta|$: stability and convergence
- Dissipative time-stepping controls the splitting error.
- No extended stencil in the system matrix.
Convergence Rate, smooth solution (Backward-Euler)

Smooth solution \( u(t, x, y) = \sin t \sin x \sin y, \ \Omega = (0.1)^2, \ \sigma = 0, \ \beta = (1, 0)^T, \ h = 1/160, \ \mathbb{P}_1 \ FE \)

standard Galerkin not robust in diffusion parameter
\( \nu \to 0 \) solution with layer (Galerkin vs. IP)

Galerkin

Interior penalty
Convergence Rate: Crank-Nicolson

- Convergence not sensitive to diffusion coefficient
Explicit treatment of the stabilization term, BDF1

(a) CIP stabilized $P_1$/BDF1

(b) CIP stabilized $P_1$/BDF1. Galerkin matrix pattern
Explicit treatment of the stabilization term, BDF2

(c) CIP stabilized $\mathbb{P}_1$/BDF2

(d) CIP stabilized $\mathbb{P}_1$/BDF2. Galerkin matrix pattern
Comparison SUPG - CIP

(e) SUPG/BDF1
(f) SUPG/C-N
(g) CIP/BDF1
(h) CIP/C-N
Part III

Runge-Kutta finite element methods
Seek $u : \Omega \times (0, T) \rightarrow \mathbb{R}^m$, $m \geq 1$, s.t.

$$\partial_t u + Au = f$$

with suitable IC, BC, and source term $f$

- $\Omega$: bounded open Lipschitz domain in $\mathbb{R}^d$
- $T$: finite simulation time
- $A$: first-order linear differential operator in space of Friedrichs type
  - advection, linear wave propagation in electromagnetics or acoustics
  - can also accommodate a zero-order term
Explicit RK-FEM

- **Time approximation**: explicit RK scheme
  - RK2 (two-stage second-order)
  - RK3 (three-stage third-order)

- **Space approximation**: stabilized finite elements
  - covers both continuous FE and DG
  - examples for CFE: interior penalty of gradient jumps [EB & Hansbo '04], local projection [Braack et al. '07, Roos et al. '08], subgrid viscosity [Guermond '99], orthogonal subscales [Codina '00]
  - unified analysis for symmetric stabilization
  - the above CFE and DG share essentially the same stability properties
  - SUPG-like stabilization does not work well with explicit RK methods
The energy argument I

- Hilbert setting: \( L := [L^2(\Omega)]^m, f \in L^2(0, T; L), \text{IC in } L \)

- Key property of evolution problem: \( \exists \lambda_0 \in \mathbb{R} \text{ s.t.} \)
  \[
  (Au, u)_L \geq \frac{1}{2} (Mu, u)_{L, \partial \Omega} - \lambda_0 \| u \|^2_L
  
  \]
  with non-negative matrix boundary field \( M \)

- Stability property: multiplying by \( u \), integrating in time, and using Gronwall’s lemma yields
  \[
  \max_{t \in [0, T]} \| u \|_L^2 + \int_0^T (Mu, u)_{L, \partial \Omega} dt \leq C
  
  \]

- The energy \( \| u \|_L \) is bounded at any time
The energy argument II

- Following the seminal work of Levy & Tadmor ('98), our analysis of explicit RK-FEM hinges on energy estimates.

- RK schemes are anti-dissipative (i.e., produce energy) because of their explicit nature.

- This needs to be compensated by the space stabilization.
The energy argument III

- Levy & Tadmor introduced a coercivity condition satisfied e.g. with artificial viscosity
  - stability of usual RK3 and RK4 schemes under usual CFL condition

- high-order FEM do not satisfy above coercivity condition

- a more subtle analysis of interplay between space stabilization and time anti-dissipation is needed
  - stability analysis for RK2 and RK3 using energy arguments
  - quasi-optimal energy error estimates for smooth solutions
  - fully unstructured meshes
Main results I

- Explicit RK2 and piecewise affine FEM
  - $O(\tau^2 + h^{3/2})$ energy error estimate
  - under usual CFL condition $\tau \lesssim h$

- Explicit RK2 with polynomials of degree $p \geq 2$ in space
  - $O(\tau^2 + h^{p+1/2})$ energy error estimate
  - under strengthened CFL condition $\tau \lesssim h^{4/3}$

- Explicit RK3 with any polynomial degree in space
  - $O(\tau^3 + h^{p+1/2})$ energy error estimate
  - under usual CFL condition $\tau \lesssim h$
Main results II

- For CFE these results appear to be new
  - viable alternative to the method of characteristics since more easily extendible to high order

- For RK2-DG similar results were obtained by Zhang & Shu (’04) for nonlinear conservation laws
  - the present proofs are much simpler
  - and allow to identify the basic stability mechanisms in DG methods

- For RK3-DG these results appear to be new
The continuous setting

- Let \( \{A_i\}_{1 \leq i \leq d} \) be fields in \( [L^\infty(\Omega)]^{m,m} \) s.t. \( \{A_i\}_{1 \leq i \leq d} \) are symmetric a.e. in \( \Omega \)

\[
\Lambda := \sum_{i=1}^{d} \partial_i A_i \in [L^\infty(\Omega)]^{m,m}
\]

- The differential operator \( A \) is

\[
A := \sum_{i=1}^{d} A_i \partial_i
\]

- \( \sigma := \max_{1 \leq i \leq d} \|A_i\|_{L^\infty(\Omega)}^{m,m} \) maximum wave speed
Boundary fields

- Symmetric boundary field \( D := \sum_{i=1}^{d} A_i n_i \)
  - \( n = (n_1, \ldots, n_d) \): unit outward normal to \( \Omega \)

- BC \( (D - M)u|_{\partial\Omega} = 0 \) with non-negative boundary field \( M \)

- Bilinear form (weakly enforcing BC)
  \[
a(v, w) = (Av, w)_L + \frac{1}{2}((M - D)v, w)_{L,\partial\Omega}
\]
  - IPP yields \( a(v, v) = \frac{1}{2}(Mv, v)_{L,\partial\Omega} - \frac{1}{2}(\Lambda v, v)_L \)

- Letting \( |v|_M^2 := (Mv, v)_{L,\partial\Omega} \) and \( \lambda_0 := \frac{1}{2}||\Lambda||_{L^\infty(\Omega)} \) yields
  \[
a(v, v) \geq \frac{1}{2} |v|_M^2 - \lambda_0 \|v\|_L^2
\]
Advection

Let $\beta \in [L^\infty(\Omega)]^d$ with $\nabla \cdot \beta \in L^\infty(\Omega)$

PDE $\partial_t u + \beta \cdot \nabla u = f$

Set $m = 1$ and $A_i = \beta_i$, $1 \leq i \leq d$

Boundary fields: $D = \beta \cdot n$ while $M = |\beta \cdot n|$ enforces $u = 0$ on inflow boundary
Examples II

Maxwell’s equations \((\mu, \epsilon > 0)\)

\[
\begin{align*}
\mu \partial_t H + \nabla \times E &= f_1 \\
\epsilon \partial_t E - \nabla \times H &= f_2
\end{align*}
\]

- Set \(m = 6\), \(u = (\mu^{1/2} H, \epsilon^{1/2} E)\), \(c_0 = (\mu \epsilon)^{-1/2}\), and

\[
A_i = c_0 \begin{bmatrix} 0_{3,3} & R_i \\ R_i^t & 0_{3,3} \end{bmatrix} \quad 1 \leq i \leq d
\]

\((R_i)_{jk} = \epsilon_{ijk}\) is the Levi–Civita permutation tensor

- Boundary fields

\[
D = c_0 \begin{bmatrix} 0_{3,3} & N \\ N^t & 0_{3,3} \end{bmatrix} \quad M = c_0 \begin{bmatrix} 0_{3,3} & -N^t \\ N^t & 0_{3,3} \end{bmatrix}
\]

enforcing \(NE = n \times E = 0\) on the boundary
Examples III

Acoustics equations \((c_0 > 0)\)

\[
\begin{align*}
  c_0^{-2} \partial_t p + \nabla \cdot q &= f_1 \\
  \partial_t q + \nabla p &= f_2
\end{align*}
\]

- Set \(m = d + 1\), \(u = (c_0^{-1} p, q)\), and

\[
A_i = c_0 \begin{bmatrix}
  0 & e_i^t \\
  e_i & 0_{d,d}
\end{bmatrix} \quad 1 \leq i \leq d
\]

\((e_1, \ldots, e_d)\) Cartesian basis of \(\mathbb{R}^d\)

- Boundary fields

\[
D = c_0 \begin{bmatrix}
  0 & n^t \\
  n & 0_{d,d}
\end{bmatrix} \quad M = c_0 \begin{bmatrix}
  0 & -n^t \\
  n & 0_{d,d}
\end{bmatrix}
\]

enforcing \(n^t q = 0\) on the boundary
The discrete setting

- $\{T_h\}_{h>0}$ family of simplicial meshes of $\Omega$
  - meshes are kept fixed in time
  - (for simplicity) meshes are quasi-uniform (can be shape regular)

- $V_h$ FE space of either continuous or discontinuous piecewise polynomials of degree $\leq p$

- $\pi_h$ $L$-orthogonal projection onto $V_h$

- Set $V(h) := [H^{p+1}(\Omega)]^m + V_h$
Discrete bilinear forms and operators

- Two discrete bilinear forms defined on $V(h) \times V_h$
  - $a_h \rightarrow$ discrete version of $a$
  - $s_h \rightarrow$ stabilization

- Linear operators $A_h : V(h) \rightarrow V_h$ and $S_h : V(h) \rightarrow V_h$ s.t.

\[
(A_h v, w_h)_L := a_h(v, w_h) \quad (S_h v, w_h)_L := s_h(v, w_h)
\]

- Define $L_h : V(h) \rightarrow V_h$ s.t. $L_h = A_h + S_h$
Design conditions I

\((A1)\) \(\forall v_h \in V_h, a_h(v_h, v_h) = \frac{1}{2}|v_h|^2_M - \frac{1}{2}(\Lambda v_h, v_h)_L\)

\((A2)\) \(s_h\) is symmetric and non-negative on \(V(h) \times V_h\)

\((A3)\) the exact solution \(u\) satisfies \(L_h u = \pi_h(f - \partial_t u)\)

- Assumptions \((A1)-(A2)\) imply discrete dissipativity: \(\forall v_h \in V_h, (L_h v_h, v_h)_L = a_h(v_h, v_h) + s_h(v_h, v_h) = |v_h|^2_S - \frac{1}{2}(\Lambda v_h, v_h)_L\)

  where \(|v_h|^2_S := s_h(v_h, v_h) + \frac{1}{2}|v_h|^2_M\)

- Assumption \((A3)\) is a (strong) consistency property
  - can be weakened
Design conditions II

\((A4)\) there is \(C_S\) s.t. \(\forall v_h \in V_h\)

\[
|v_h|_S \leq C_S^{1/2} \sigma^{1/2} h^{-1/2} \|v_h\|_L
\]

and for smooth \(v\), \(|v - \pi_h v|_S \leq C'_S h^{p+1/2} \|v\|_{[H^{p+1}(\Omega)]^m}\)

\((A5)\) there is \(C_L\) s.t. \(\forall z \in V(h)\)

\[
\|L_hz\|_L \leq C_L(\sigma \|\nabla_h z\|_{L^d} + \sigma^{1/2} h^{-1/2} |z|_S)
\]

- Assumptions \((A4)-(A5)\) imply \(\|L_h v_h\|_L \leq C_L* \sigma h^{-1} \|v_h\|_L\)
Design conditions III

(A6) there is $C_{\pi}$ s.t. $\forall (z, v_h) \in V(h) \times V_h$

$$|(L_h(z - \pi_hz), v_h)_L| \leq C_{\pi}\sigma^{1/2}\|z - \pi_hz\|_\ast(\|v_h\|_S + \|v_h\|_L)$$

with $\|y\|_\ast := h^{1/2}\|\nabla_h y\|_{L^d} + h^{-1/2}\|y\|_L + \|y\|_{L,\mathcal{F}_h} + \sigma^{-1/2}|y|_S$

(A7) for $p = 1$ there is $C'_{\pi}$ s.t. $\forall (v_h, w_h) \in V_h \times V_h$

$$|(L_h v_h, w_h - \pi^0_h w_h)_L| \leq C'_{\pi}\sigma^{1/2}h^{-1/2}(\|v_h\|_S + \|v_h\|_L)\|w_h - \pi^0_h w_h\|_L$$

$\pi^0_h$: $L$-orthogonal projection onto piecewise constant functions
Examples I

- Mesh faces $\mathcal{F}_h = \mathcal{F}_h^{\text{int}} \cup \mathcal{F}_h^{\text{ext}}$
  - for $F \in \mathcal{F}_h^{\text{int}}$, $F = \partial T^- \cap \partial T^+$
  - $n_F$ is the unit normal to $F$ pointing from $T^-$ to $T^+$
  - jump $[v] := v_{|T^-} - v_{|T^+}$ and average $\{v\} = \frac{1}{2}(v_{|T^-} + v_{|T^+})$
Examples II

- **Continuous FE with gradient jump penalty** [EB & Ern ’07]

  \[
  a_h^{\text{cip}}(v, w) := \sum_{T \in \mathcal{T}_h} (A v, w)_{L,T} + \sum_{F \in \mathcal{F}_h^{\text{ext}}} \frac{1}{2}((M - D)v, w)_{L,F}
  \]

  \[
  s_h^{\text{cip}}(v, w) := \sum_{F \in \mathcal{F}_h^{\text{ext}}} (S^\text{ext}_F v, w)_{L,F} + \sum_{F \in \mathcal{F}_h^{\text{int}}} h_F^2 (S^\text{int}_F n_F \cdot [\nabla v], n_F \cdot [\nabla w])_{L,F}
  \]

- **Discontinuous Galerkin** [Ern & Guermond ’06]

  \[
  a_h^{\text{dg}}(v, w) := a_h^{\text{cip}}(v, w) - \sum_{F \in \mathcal{F}_h^{\text{int}}} (D_F[v], \{w\})_{L,F}
  \]

  \[
  s_h^{\text{dg}}(v, w) := \sum_{F \in \mathcal{F}_h^{\text{ext}}} (S^\text{ext}_F v, w)_{L,F} + \sum_{F \in \mathcal{F}_h^{\text{int}}} (S^\text{int}_F [v], [w])_{L,F}
  \]

  where \( D_F := \sum_{i=1}^d A_i n_{F,i} \) for all \( F \in \mathcal{F}_h^{\text{int}} \)
Examples III

- Design conditions on operators $S_F^{\text{int}}$ and $S_F^{\text{ext}}$

  For exact solution, $S_F^{\text{ext}} u = 0$ on $\partial \Omega$

  $S_F^{\text{ext}}$ and $S_F^{\text{int}}$ are symmetric and non-negative

  $S_F^{\text{ext}} \leq \alpha_1 \sigma I_m$ and $\alpha_2 |D_F| \leq S_F^{\text{int}} \leq \alpha_3 \sigma I_m$

  $|((M - D)y, z)_{L,F}| \leq \alpha_4 \sigma^{1/2} |y|_{S,F} |z|_{L,F}$

  $|((M + D)y, z)_{L,F}| \leq \alpha_5 \sigma^{1/2} |y|_{L,F} |z|_{S,F}$

- Under the above design conditions and if $A_i|_T \in [C^{0,1/2}(T)]^{m,m}$ for all $T \in \mathcal{T}_h$, Assumptions (A1)–(A7) hold
Examples IV

- **Advection** \( S_F^{\text{ext}} = 0 \) and \( S_F^{\text{int}} = \gamma |\beta \cdot n_F| \)

- **Maxwell’s equations**

  \[
  S_F^{\text{ext}} = c_0 \begin{bmatrix}
  0_{3,3} & 0_{3,3} \\
  0_{3,3} & \gamma_1 N^t N
  \end{bmatrix} \quad S_F^{\text{int}} = c_0 \begin{bmatrix}
  \gamma_2 N^t N_F & 0_{3,3} \\
  0_{3,3} & \gamma_3 N^t_F N_F
  \end{bmatrix}
  \]

- **Acoustics equations**

  \[
  S_F^{\text{ext}} = c_0 \begin{bmatrix}
  0 & 0^t_d \\
  0_d & \gamma_1 n \otimes n
  \end{bmatrix} \quad S_F^{\text{int}} = c_0 \begin{bmatrix}
  \gamma_2 & 0^t_d \\
  0_d & \gamma_3 n_F \otimes n_F
  \end{bmatrix}
  \]
RK2-FEM

- We assume $u \in C^3(0, T; L)$ and $f \in C^2(0, T; L)$
- Two-stage explicit RK2 scheme ($f_h := \pi_h f$)

\[
\begin{align*}
    w^n_h &= u^n_h - \tau L_h u^n_h + \tau f^n_h \\
    u^{n+1}_h &= \frac{1}{2}(u^n_h + w^n_h) - \frac{1}{2}\tau L_h w^n_h + \frac{1}{2}\tau \psi^n_h
\end{align*}
\]

with the assumption that

\[
\psi^n_h = f^n_h + \tau \partial_t f^n_h + \delta^n_h, \quad \|\delta^n_h\|_L \lesssim \tau^2
\]

- Examples
  - second-order Heun method $\psi^n_h = f^{n+1}_h$
  - second-order Runge method $\psi^n_h = 2f^{n+1/2}_h - f^n_h$

- In the homogeneous case ($f = 0$)

\[
    u^{n+1}_h = u^n_h - \tau L_h u^n_h + \frac{1}{2}\tau^2 L^2_h u^n_h
\]
The error equation

- Define \( w = u + \tau \partial_t u \) and set

\[
\begin{align*}
\xi_h^n &= u_h^n - \pi_h u^n \\
\zeta_h^n &= w_h^n - \pi_h w^n \\
\xi_\pi^n &= u^n - \pi_h u^n \\
\zeta_\pi^n &= w^n - \pi_h w^n
\end{align*}
\]

- Owing to consistency

\[
\begin{align*}
\zeta_h^n &= \xi_h^n - \tau L_h \xi_h^n + \tau \alpha_h^n \\
\xi_h^{n+1} &= \frac{1}{2} (\xi_h^n + \zeta_h^n) - \frac{1}{2} \tau L_h \zeta_h^n + \frac{1}{2} \tau \beta_h^n
\end{align*}
\]

with

\[
\begin{align*}
\alpha_h^n &= L_h \xi_\pi^n \\
\beta_h^n &= L_h \zeta_\pi^n - \pi_h \eta^n + \delta_h^n
\end{align*}
\]

and \( \eta^n = \tau^{-1} \int_{t_n}^{t_{n+1}} (t_{n+1} - t)^2 \partial_{ttt} u \, dt \)
Energy identity and stability

> Energy identity: Discrete dissipativity yields \((\Lambda = 0)\)

\[
\|\xi_h^{n+1}\|_L^2 - \|\xi_h^n\|_L^2 + \tau|\xi_h^n|_S^2 + \tau|\zeta_h^n|_S^2 = \|\xi_h^{n+1} - \zeta_h^n\|_L^2 \\
+ \tau(\alpha_h^n, \xi_h^n)_L + \tau(\beta_h^n, \zeta_h^n)_L
\]

> The quantity \(\|\xi_h^{n+1} - \zeta_h^n\|_L^2\) (\(\approx\) second-order derivative in time) is the anti-dissipative term produced by the explicit RK2 scheme
Energy identity and stability II

- Owing to Assumption (A6)

\[ \tau(\alpha_h^n, \xi_h^n) = \tau(L_h \xi^n_\pi, \xi_h^n) \lesssim \tau \| \xi^n_\pi \|_* (\| \xi_h^n \|_S + \| \xi_h^n \|_L) \]

- Owing to Assumptions (A4)–(A5) under usual CFL condition

\[ \| \zeta_h^n \|_L = \| \xi_h^n - \tau L_h \xi_h^n + \tau \alpha_h^n \|_L \leq \| \xi_h^n \|_L + \tau \| L_h \xi_h^n \|_L + \tau \| L_h \xi^n_\pi \|_L \]
\[ \lesssim \| \xi_h^n \|_L + \tau h^{-1} \| \xi_h^n \|_L + \tau h^{-1/2} \| \xi^n_\pi \|_* \lesssim \| \xi_h^n \|_L + \tau^{1/2} \| \xi^n_\pi \|_* \]

- Stability estimate

\[ \| \xi_h^{n+1} \|_L^2 - \| \xi_h^n \|_L^2 + \frac{1}{2} \tau | \xi_h^n |_S^2 + \frac{1}{2} \tau | \zeta_h^n |_S^2 \leq \| \xi_h^{n+1} - \zeta_h^n \|_L^2 \]
\[ + C \tau (\tau^4 + \| \xi^n_\pi \|_*^2 + \| \zeta^n_\pi \|_*^2 + \| \xi_h^n \|_L^2) \]

- There are two ways to bound the term \( \| \xi_h^{n+1} - \zeta_h^n \|_L^2 \)
\( p \geq 2: \) 4/3-CFL condition

- Since \( \xi_h^{n+1} - \zeta_h^n = \frac{1}{2} \tau^2 L_h^2 \xi_h^n + \frac{1}{2} \tau (\beta_h^n - \alpha_h^n - \tau L_h \alpha_h^n) \)

\[
\|\xi_h^{n+1} - \zeta_h^n\|_L^2 \lesssim \tau^4 \|L_h^2 \xi_h^n\|_L^2 + \tau (\tau^5 + \|\xi_n\|_\pi^2 + \|\zeta_n\|_\pi^2)
\]

- The 4/3-CFL condition yields \( \tau^4 \|L_h^2 \xi_h^n\|_L^2 \lesssim \tau \|\xi_h^n\|_L^2 \)

- Hence,

\[
\|\xi_h^{n+1}\|_L^2 - \|\xi_h^n\|_L^2 + \frac{1}{2} \tau |\xi_h^n|_S^2 + \frac{1}{2} \tau |\zeta_h^n|_S^2 \lesssim \tau (\tau^4 + \|\xi_n\|_\pi^2 + \|\zeta_n\|_\pi^2 + \|\xi_h^n\|_L^2)
\]

- Sum over \( n \), use Gronwall’s lemma and approximation property

\[
\|\xi_n\|_\pi + \|\zeta_n\|_\pi \lesssim h^{p+1/2}
\]
\( p = 1: \) usual CFL condition

\begin{itemize}
\item Start with \( \xi_h^{n+1} - \zeta_h^n = \frac{1}{2} \tau L_h (\xi_h^n - \zeta_h^n) + \frac{1}{2} \tau (\beta_h^n - \alpha_h^n) \)
\item Set \( x_h^n := \xi_h^n - \zeta_h^n \) and \( y_h^n := x_h^n - \pi_0^h x_h^n \) so that
\[
\|y_h^n\|_L^2 = (x_h^n, y_h^n)_L = \tau (L_h \xi_h^n, y_h^n)_L - \tau (\alpha_h^n, y_h^n)_L
\]
\item Use Assumption (A7) to bound \( \|y_h^n\|_L \)
\item Infer a bound on \( \| \nabla_h x_h^n \|_{L^d} \) using inverse inequality
\[
\| \nabla_h x_h^n \|_{L^d} \leq C_i h^{-1} \|y_h^n\|_L
\]
\item Use Assumption (A5) to bound \( \| L_h x_h^n \|_L \) and hence \( \| \xi_h^{n+1} - \zeta_h^n \|_L \)
\end{itemize}
We assume \( u \in C^4(0, T; L) \) and \( f \in C^3(0, T; L) \)

Three-stage explicit RK3 scheme

\[
\begin{align*}
    w_h^n &= u_h^n - \tau L_h u_h^n + \tau f_h^n \\
    y_h^n &= \frac{1}{2} (u_h^n + w_h^n) - \frac{1}{2} \tau L_h w_h^n + \frac{1}{2} \tau (f_h^n + \tau \partial_t f_h^n) \\
    u_h^{n+1} &= \frac{1}{3} (u_h^n + w_h^n + y_h^n) - \frac{1}{3} \tau L_h y_h^n + \frac{1}{3} \tau \psi_h^n
\end{align*}
\]

with the assumption that

\[
\psi_h^n = f_h^n + \tau \partial_t f_h^n + \frac{1}{2} \tau^2 \partial_{tt} f_h^n + \delta_h^n \quad \|\delta_h^n\|_L \lesssim \tau^3
\]
Example

▶ Third-order Heun method

\[ k_1 = -L_h u_h^n + f_h^n \]
\[ k_2 = -L_h (u_h^n + \frac{1}{3} \tau k_1) + f_h^{n+1/3} \]
\[ k_3 = -L_h (u_h^n + \frac{2}{3} \tau k_2) + f_h^{n+2/3} \]
\[ u_h^{n+1} = u_h^n + \frac{1}{4} \tau (k_1 + 3k_3) \]

▶ Straightforward algebra yields

\[ \psi_h^n = -\frac{5}{4} f_h^n + \frac{9}{4} f_h^{n+2/3} - \frac{1}{2} \tau \partial_t f_h^n - \frac{3}{2} \tau L_h (f_h^{n+1/3} - f_h^n - \frac{1}{3} \tau \partial_t f_h^n) \]

▶ If \( f \in C^2(0, T; [H^1(\Omega)]^m) \),

\[ \psi_h^n = f_h^n + \tau \partial_t f_h^n + \frac{1}{2} \tau^2 \partial_{tt} f_h^n + O(\tau^3) \]
The error equation

- Additionally define \( y = u + \tau \partial_t u + \frac{1}{2} \tau^2 \partial_{tt} u \) and set

\[
\theta^n = y^n - \pi_h y^n \quad \theta^n = y^n - \pi_h y^n
\]

- Owing to consistency

\[
\zeta^n = \xi^n - \tau \mathcal{L}_h \xi^n + \tau \alpha^n_h \\
\theta^n = \frac{1}{2} (\xi^n + \zeta^n) - \frac{1}{2} \tau \mathcal{L}_h \zeta^n + \frac{1}{2} \tau \beta^n_h \\
\xi^{n+1} = \frac{1}{3} (\xi^n + \zeta^n + \theta^n) - \frac{1}{3} \tau \mathcal{L}_h \theta^n + \frac{1}{3} \tau \gamma^n_h
\]

with

\[
\alpha^n_h = \mathcal{L}_h \xi^n \quad \beta^n_h = \mathcal{L}_h \zeta^n \quad \gamma^n_h = \mathcal{L}_h \theta^n - \pi_h \eta^n + \delta^n_h
\]

and \( \eta^n = \tau^{-1} \int_{t_n}^{t_{n+1}} \frac{1}{2} (t_n + 1 - t)^3 \partial_{tttt} u \, dt \)
Energy identity and stability I

- Energy identity: Discrete dissipativity yields ($\Lambda = 0$)

\[
\frac{1}{2} \|\xi_h^{n+1}\|_L^2 - \frac{1}{2} \|\xi_h^n\|_L^2 + \frac{1}{2} \tau |\xi_h^n|_S^2 + \frac{1}{6} \tau |\zeta_h^n|_S^2 + \frac{1}{3} \tau |\theta_h^n|_S^2 + \frac{1}{6} \|\theta_h^n - \zeta_h^n\|_L^2 \\
= \frac{1}{6} \tau |\zeta_h^n - \xi_h^n|_S^2 + \frac{1}{2} \|\xi_h^{n+1} - \theta_h^n\|_L^2 + T(\alpha_h^n, \beta_h^n, \gamma_h^n)
\]

- There is a dissipative term in the LHS!

- The two anti-dissipative terms in the RHS need to be controlled
Energy identity and stability II

- Under the usual CFL condition, using Young inequalities and Assumption (A4) leads to

\[
\|\xi_{h}^{n+1}\|^2_L - \|\xi_{h}^n\|^2_L + \frac{1}{48}\tau|\xi_{h}^n|^2_S + \frac{1}{12}\tau|\zeta_{h}^n|^2_S + \frac{1}{48}\tau|\theta_{h}^n|^2_S \\
\leq C\tau(\tau^6 + \|\xi_{\pi}^n\|^2_\ast + \|\zeta_{\pi}^n\|^2_\ast + \|\theta_{\pi}^n\|^2_\ast + \|\xi_{h}^n\|^2_L)
\]

- Sum over \(n\), use Gronwall’s lemma and approximation property

\[
\|\xi_{\pi}^n\|^\ast + \|\zeta_{\pi}^n\|^\ast + \|\theta_{\pi}^n\|^\ast \lesssim h^{p+1/2}
\]
Numerical results

- Convergence rates for smooth solutions
- Advection equation (similar results for wave equation)
- Rotating Gaussian benchmark
  \[ \beta = (y, x)^t, \quad f = 0, \quad \Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}, \text{ and IC} \]
  \[ u_0(x, y) = e^{10[(x-0.3)^2+(y-0.3)^2]} \]
- Energy error after one complete rotation
- CIP (gradient jump penalty) and DG with \( p = 1 \) and \( p = 2 \)
- Stabilization parameter: \( \gamma = 0.005 \) and 0.001 for CIP with \( p = 1 \) and \( p = 2 \), \( \gamma = 0.5 \) for DG (upwinding)
Discretization parameters

- for \( p = 1 \), \( \tau = Ch \) (constant usual CFL)
- for \( p = 2 \), \( \tau = Ch^{4/3} \) (constant 4/3-CFL)

Recover theoretical results
RK3-FEM

- Discretization parameters
  - for $p = 1$, $\tau = Ch^{1/2}$ (increasing usual CFL to reduce spatial error)
  - for $p = 2$, $\tau = Ch$ (constant usual CFL)

Recover theoretical results
  - slightly higher CV rates for RK3-P1
Rough solutions I

- Rotating benchmark with IC

$$u_0(x, y) = \frac{1}{2} \left[ \tanh \left( \frac{e^{-10[(x-0.3)^2+(y-0.3)^2]} - 0.5}{0.001} \right) + 1 \right]$$

- Spurious oscillations for under-resolved layer

- IC and discrete solution without stabilization
Rough solutions II

- Results for RK2
  - CIP $p = 1$
  - DG $p = 1$
  - CIP $p = 2$
  - DG $p = 2$

- Results for RK3
  - CIP $p = 1$
  - DG $p = 1$
  - CIP $p = 2$
  - DG $p = 2$
Conclusions

- Analysis of explicit RK schemes with symmetric stabilized FEM
  - balance between space stabilization and time-antidissipation
  - continuous and discontinuous FEM in a unified framework
  - possibility to use high-order explicit methods

- Perspectives
  - extension to RK4 schemes
  - more general evolution problem $A_0 \partial_t u + Au = f$
  - error estimates in graph norm
  - implicit-explicit schemes for evolution problems with diffusion