On the discontinuous Galerkin finite element method for nonstationary convection-diffusion problems

Theory and applications to compressible flow

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Goal: to work out a sufficiently accurate, robust and theoretically based method for the numerical solution of compressible flow with a wide range of Mach numbers and Reynolds numbers.

Difficulties:
- nonlinear convection dominating over diffusion
- boundary layers, wakes for large Reynolds numbers
- shock waves, contact discontinuities for large Mach numbers
- instabilities caused by acoustic effects for low Mach numbers
One of promising, efficient methods for the solution of compressible flow is the **discontinuous Galerkin finite element method (DGFEM)** using piecewise polynomial approximation of a sought solution without any requirement on the continuity between neighbouring elements.

- Cockburn & Shu 1989, Bassi & Rebay, Baumann & Oden 1997, ... Hartmann, Houston, ... van der Vegt, ... M.F., Dolejší, Kučera
- theory for elliptic or parabolic problems: Arnold, Brezzi, Marini, et al, Schwab, Suli, ..., Wheeler, Girault, Riviere, ...
- theory for nonstationary (nonlinear) convection-diffusion problems: M.F., Dolejší, Schwab, Sobotíková, Švadlenka, Hájek, Kučera
Here:

- analysis of the DGFEM for the solution of a nonlinear nonstationary convection-diffusion equation (= a simple prototype of the compressible Navier-Stokes system)

- applications to the simulation of compressible flow
Continuous model problem

Find \( u : Q_T = \Omega \times (0, T) \rightarrow \mathbb{R} \) such that

\[
a) \quad \frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} = \varepsilon \Delta u + g \quad \text{in} \quad Q_T, \quad (1) \\
b) \quad u|_{\partial \Omega_D \times (0,T)} = u_D, \\
c) \quad \varepsilon \frac{\partial u}{\partial n}|_{\partial \Omega_N \times (0,T)} = g_N, \\
d) \quad u(x, 0) = u^0(x), \; x \in \Omega.
\]

\( \Omega \subset \mathbb{R}^d, \; d = 2, 3 \) - a bounded polygonal (if \( d = 2 \)) or polyhedral (if \( d = 3 \)) domain with Lipschitz-continuous boundary \( \partial \Omega = \partial \Omega_D \cup \partial \Omega_N, \; \partial \Omega_D \neq \emptyset \) and \( T > 0 \)

\( \varepsilon > 0 \) - a given constant, \( g : Q_T \rightarrow \mathbb{R}, \; u_D : \partial \Omega_D \times (0, T) \rightarrow \mathbb{R}, \; g_N : \partial \Omega_N \times (0, T) \rightarrow \mathbb{R}, \; u^0 : \Omega \rightarrow \mathbb{R} \) - given functions, \( f_s \in C^1(\mathbb{R}), \; s = 1, \ldots, d, \) - prescribed fluxes
**DG space semidiscretization**

Let $T_h$ ($h > 0$) be a *partition* of the closure $\overline{\Omega}$ of the domain $\Omega$ into a finite number of closed triangles ($d = 2$) or tetrahedra ($d = 3$) $K$ with mutually disjoint interiors such that

$$\overline{\Omega} = \bigcup_{K \in T_h} K.$$  \hspace{1cm} (2)

We call $T_h$ a *triangulation* of $\Omega$ and do not require the standard conforming properties from the finite element method.

$h_K = \text{diam}(K), \quad h = \max_{K \in T_h} h_K, \quad \rho_K = \text{largest ball inscribed into } K$

$K, K' \in T_h$ - *neighbours* - they have a common face
\( \mathcal{F}_h = \) the system of all faces of all elements \( K \in \mathcal{T}_h \),

the set of all inner faces:

\[
\mathcal{F}^I_h = \{ \Gamma \in \mathcal{F}_h; \ \Gamma \subset \Omega \},
\]

the set of all “Dirichlet” boundary faces:

\[
\mathcal{F}^D_h = \{ \Gamma \in \mathcal{F}_h; \ \Gamma \subset \partial \Omega_D \},
\]

the set of all “Neumann” boundary faces:

\[
\mathcal{F}^N_h = \{ \Gamma \in \mathcal{F}_h, \ \Gamma \subset \partial \Omega_N \}.
\]

Obviously, \( \mathcal{F}_h = \mathcal{F}^I_h \cup \mathcal{F}^D_h \cup \mathcal{F}^N_h \). For a shorter notation we put

\[
\mathcal{F}^{ID}_h = \mathcal{F}^I_h \cup \mathcal{F}^D_h, \quad \mathcal{F}^{DN}_h = \mathcal{F}^D_h \cup \mathcal{F}^N_h.
\]

For each \( \Gamma \in \mathcal{F}_h \) we define a \textit{unit normal vector} \( n_{\Gamma} \).

For \( \Gamma \subset \partial \Omega \) - \( n_{\Gamma} = \) unit outer normal to \( \partial \Omega \).

\( d(\Gamma) = \) diameter of \( \Gamma \in \mathcal{F}_h \).
Elements with hanging nodes
Broken Sobolev spaces

Over a triangulation $\mathcal{T}_h$ we define the so-called *broken Sobolev space*

$$H^k(\Omega, \mathcal{T}_h) = \{ v; v|_K \in H^k(K) \ \forall \ K \in \mathcal{T}_h \}$$

(7)

with the norm

$$\|v\|_{H^k(\Omega, \mathcal{T}_h)} = \left( \sum_{K \in \mathcal{T}_h} \|v\|_{H^k(K)}^2 \right)^{1/2}$$

(8)

and the seminorm

$$|v|_{H^k(\Omega, \mathcal{T}_h)} = \left( \sum_{K \in \mathcal{T}_h} |v|^2_{H^k(K)} \right)^{1/2}.$$  

(9)
For each $\Gamma \in \mathcal{F}^I_h$ there exist two neighbours adjacent to $\Gamma$: $K_{\Gamma}^{(L)}, K_{\Gamma}^{(R)} \in \mathcal{T}_h$

**convention:** $K_{\Gamma}^{(R)}$ lies in the direction of $n_{\Gamma}$ and $K_{\Gamma}^{(L)}$ lies in the opposite direction to $n_{\Gamma}$

For $\Gamma \subset \partial \Omega$ there exists an element $K_{\Gamma}^{(L)}$ adjacent to $\Gamma$, i.e. $\Gamma \subset \partial \Omega \cap \partial K_{\Gamma}^{(L)}$. 
Let \( v \in H^1(\Omega, T_h) \), \( \Gamma \in \mathcal{F}_h^{I} \) - notation:

\[
v|^{(L)}_{\Gamma} = \text{the trace of } v|^{(L)}_{K_{\Gamma}} \text{ on } \Gamma, \tag{10}
\]

\[
v|^{(R)}_{\Gamma} = \text{the trace of } v|^{(R)}_{K_{\Gamma}} \text{ on } \Gamma,
\]

\[
\langle v \rangle_{\Gamma} = \frac{1}{2} \left( v|^{(L)}_{\Gamma} + v|^{(R)}_{\Gamma} \right), \quad [v]_{\Gamma} = v|^{(L)}_{\Gamma} - v|^{(R)}_{\Gamma}.
\]

The approximate solution — sought in the space of discontinuous piecewise polynomial functions

\[
S_h = S_{h}^{p,-1} = \{ v; v|_{K} \in P^{p}(K) \ \forall K \in T_h \},
\]

\( p > 0 \) — integer, \( P^{p}(K) \) — the space of all polynomials on \( K \) of degree at most \( p \).

Derivation of the discrete problem

Assume that \( u \) — sufficiently regular exact solution

— multiply the PDE by any \( \varphi \in H^2(\Omega, T_h) \)

— integrate over \( K \in T_h \)

— apply Green’s theorem

— sum over all \( K \in T_h \)
After some manipulation we obtain the identity

\[
\int_{\Omega} \frac{\partial u}{\partial t} \varphi \, dx 
\]

\[+ \sum_{K \in T_h} \left( \sum_{\Gamma \in F_h} \int_{\Gamma} \sum_{s=1}^{d} f_s(u) (n_{\partial K})_s \varphi_{|_{\Gamma}} \, dS \right) \]

\[ - \sum_{K \in T_h} \int_{K} \sum_{s=1}^{d} f_s(u) \frac{\partial \varphi}{\partial x_s} \, dx \]

\[+ \sum_{K \in T_h} \int_{K} \varepsilon \nabla u \cdot \nabla \varphi \, dx \]

\[ - \sum_{\Gamma \in F_h^{I}} \int_{\Gamma} \varepsilon \langle \nabla u \rangle \cdot n_{\Gamma}[\varphi] \, dS \]

\[ - \sum_{\Gamma \in F_h^{D}} \int_{\Gamma} \varepsilon \nabla u \cdot n_{\Gamma} \varphi \, dS \]

\[= \int_{\Omega} g \varphi \, dx + \sum_{\Gamma \in F_h^{N}} \int_{\Gamma} \varepsilon \nabla u \cdot n_{\Gamma} \varphi \, dS. \]
In view of the Neumann condition, the second term on the right-hand side of (11) reads

\[ \sum_{\Gamma \in \mathcal{F}_h^N} \int_{\Gamma} \varepsilon \nabla u \cdot n_{\Gamma} \varphi \, dS = \int_{\partial \Omega_N} g_N \varphi \, dS. \]  \hspace{1cm} (12)

To the left-hand side of (11) we add now the terms

\[ -\theta \sum_{\Gamma \in \mathcal{F}_h^I} \int_{\Gamma} \varepsilon \langle \nabla \varphi \rangle \cdot n_{\Gamma}[u] \, dS = 0. \]  \hspace{1cm} (13)

To the left-hand side and the right-hand side of (11) we add the identical terms

\[ -\theta \sum_{\Gamma \in \mathcal{F}_h^D} \int_{\Gamma} \varepsilon \nabla \varphi \cdot n_{\Gamma} u \, dS \text{ and } -\theta \sum_{\Gamma \in \mathcal{F}_h^D} \int_{\Gamma} \varepsilon \nabla \varphi \cdot n_{\Gamma} u_D \, dS, \]  \hspace{1cm} (14)

respectively.
Because of the stabilization of the scheme we introduce the \textit{interior penalty}

\[ \varepsilon \sum_{\Gamma \in \mathcal{F}_h^I} \int_{\Gamma} \sigma [u] [\varphi] \, dS \quad (= 0) \quad (15) \]

and the \textit{boundary penalty}

\[ \varepsilon \sum_{\Gamma \in \mathcal{F}_h^D} \int_{\Gamma} \sigma u \varphi \, dS = \varepsilon \sum_{\Gamma \in \mathcal{F}_h^D} \int_{\Gamma} \sigma u_D \varphi \, dS, \quad (16) \]

where \( \sigma \) is a suitable \textit{weight}.

On the basis of above considerations for \( u, \varphi \in H^2(\Omega, \mathcal{T}_h) \) we define the forms:

\((\cdot, \cdot) - L^2(\Omega)\)-scalar product,
\[ a_h(u, \varphi) = \sum_{K \in T_h} \int_K \varepsilon \nabla u \cdot \nabla \varphi \, dx \]  
\[ - \sum_{\Gamma \in \mathcal{F}_h^I} \int_{\Gamma} \varepsilon \langle \nabla u \rangle \cdot n_\Gamma [\varphi] \, dS \]  
\[ -\theta \sum_{\Gamma \in \mathcal{F}_h^I} \int_{\Gamma} \varepsilon \langle \nabla \varphi \rangle \cdot n_\Gamma [u] \, dS \]  
\[ - \sum_{\Gamma \in \mathcal{F}_h^D} \int_{\Gamma} \varepsilon \nabla u \cdot n_\Gamma \varphi \, dS \]  
\[ -\theta \sum_{\Gamma \in \mathcal{F}_h^D} \int_{\Gamma} \varepsilon \nabla \varphi \cdot n_\Gamma u \, dS \]  

**diffusion form**

\[ J_h^\sigma(u, \varphi) = \sum_{\Gamma \in \mathcal{F}_h^I} \int_{\Gamma} \sigma[u] [\varphi] \, dS + \sum_{\Gamma \in \mathcal{F}_h^D} \int_{\Gamma} \sigma u \varphi \, dS \]  

**interior and boundary penalty**
\[ \ell_h(\varphi)(t) = \int_{\Omega} g(t) \varphi \, dx + \int_{\partial \Omega_N} g_N(t) \varphi \, dS \]
\[ -\theta \sum_{\Gamma \in \mathcal{F}_h^D} \int_{\Gamma} \varepsilon \nabla \varphi \cdot n_\Gamma u_D(t) \, dS \]
\[ + \varepsilon \sum_{\Gamma \in \mathcal{F}_h^D} \int_{\Gamma} \sigma u_D(t) \varphi \, dS \]  

right-hand side form

\( \theta = -1 \) nonsymmetric discretization of diffusion terms (NIPG)
\( \theta = 1 \) symmetric discretization of diffusion terms (SIPG)
\( \theta = 0 \) incomplete discretization of diffusion terms (IIPG)
Finally, the **convective terms** are approximated with the aid of a **numerical flux** $H = H(u, v, n)$ by the form

\[
b_h(u, \varphi) = - \sum_{K \in T_h} \int_K \sum_{s=1}^d f_s(u) \frac{\partial \varphi}{\partial x_s} \, dx + \sum_{\Gamma \in F_h^{I}} \int_{\Gamma} H \left( u^{(L)}|_{\Gamma}, u^{(R)}|_{\Gamma}, n_{\Gamma} \right) \left[ \varphi \right]|_{\Gamma} \, dS \\
+ \sum_{\Gamma \in F_h^{DN}} \int_{\Gamma} H \left( u^{(L)}|_{\Gamma}, u^{(R)}|_{\Gamma}, n_{\Gamma} \right) \varphi^{(L)}|_{\Gamma} \, dS
\]  

(20)

**convective form**

**Definition of the boundary state** $u^{(R)}|_{\Gamma}$ for $\Gamma \subset \partial \Omega : u^{(R)}|_{\Gamma} := u^{(L)}|_{\Gamma}$ (extrapolation)
Assumptions (H):

1. **$H(u, v, n)$ is defined in $\mathbb{R}^2 \times B_1$, where $B_1 = \{ n \in \mathbb{R}^d; |n| = 1 \}$, and Lipschitz-continuous with respect to $u, v$:**

   $$|H(u, v, n) - H(u^*, v^*, n)| \leq C_L (|u - u^*| + |v - v^*|),$$
   $$u, v, u^*, v^* \in \mathbb{R}, \ n \in B_1.$$  

2. **$H(u, v, n)$ is consistent:**

   $$H(u, u, n) = \sum_{s=1}^{d} f_s(u) n_s, \quad u \in \mathbb{R}, \ n = (n_1, \ldots, n_d) \in B_1.$$  

3. **$H(u, v, n)$ is conservative:**

   $$H(u, v, n) = -H(v, u, -n), \quad u, v \in \mathbb{R}, \ n \in B_1.$$
The **exact sufficiently regular solution** $u$ satisfies the identity

$$
\left( \frac{\partial u(t)}{\partial t}, \varphi_h \right) + b_h(u(t), \varphi_h) + a_h(u(t), \varphi_h) + \varepsilon J^\sigma_h(u(t), \varphi_h)
= \ell_h(\varphi_h)(t) \quad \text{for all } \varphi_h \in S_h \text{ and for a.e. } t \in (0, T).
$$

**Discrete problem**

We say that $u_h$ is a DG approximate solution of the convection-diffusion problem (1), if

a) \quad $u_h \in C^1([0, T]; S_h)$, \quad (21)

b) \quad $\left( \frac{\partial u_h(t)}{\partial t}, \varphi_h \right) + a_h(u_h(t), \varphi_h) + b_h(u_h(t), \varphi_h) + J^\sigma_h(u_h(t), \varphi_h)
= \ell_h(\varphi_h)(t) \quad \forall \varphi_h \in S_h, \forall t \in (0, T),$

\hspace{1cm} (21)

c) \quad $u_h(0) = u_0^h = S_h$–approximation of $u^0$.\hspace{1cm} (21)
The discrete problem is equivalent to a large system of nonlinear ordinary differential equations.

In practical computations: suitable time discretization is applied, e.g.
- Euler forward or backward scheme,
- Runge–Kutta methods,
- discontinuous Galerkin time discretization

The forward Euler and Runge-Kutta schemes are conditionally stable – time step is strongly restricted by the CFL-stability condition.

Suitable: semi-implicit scheme - leads to a linear algebraic system on each time level

Integrals are evaluated with the aid of numerical integration.
Error analysis

Assumptions

– Assumptions (H)
– The weak solution \( u \) of problem (1) is regular, namely

\[
\frac{\partial u}{\partial t} \in L^2(0, T; H^{p+1}(\Omega)). \tag{22}
\]

Then

\[
\left( \frac{\partial u(t)}{\partial t}, \varphi_h \right) + a_h(u(t), \varphi_h) + \varepsilon J^\sigma_h(u(t), \varphi_h) \\
+ b_h(u(t), \varphi_h) = \ell_h(\varphi_h)(t), \quad \forall \varphi_h \in S_h, \text{ for a.a. } t \in (0, T). \tag{23}
\]

– \( \{\mathcal{T}_h\}_{h \in (0, h_0)}, \ h_0 > 0, \) - **regular system** of triangulations of the domain \( \Omega \): there exists \( C_T > 0 \) such that

\[
\frac{h_K}{\rho_K} \leq C_T \quad \forall K \in \mathcal{T}_h \quad \forall h \in (0, h_0). \tag{24}
\]
Some auxiliary results

Multiplicative trace inequality:
There exists a constant $C_M > 0$ independent of $v$, $h$ and $K$ such that

\[ \frac{\|v\|^2_{L^2(\partial K)}}{\|v\|^2_{L^2(K)}} \leq C_M \left( \|v\|_{L^2(K)} \|v\|_{H^1(K)} + h_K^{-1} \|v\|^2_{L^2(K)} \right), \]

$K \in \mathcal{T}_h$, $v \in H^1(K)$, $h \in (0, h_0)$.

Inverse inequality:
There exists a constant $C_I > 0$ independent of $v$, $h$, and $K$ such that

\[ |v|_{H^1(K)} \leq C_I h_K^{-1} \|v\|_{L^2(K)}, \quad v \in P^p(K), \ K \in \mathcal{T}_h, \ h \in (0, h_0). \]
$S_h$-interpolation:
For $v \in L^2(\Omega)$ we denote by $\Pi_h v$ the $L^2(\Omega)$-projection of $v$ on $S_h$:

$$\Pi_h v \in S_h, \quad (\Pi_h v - v, \varphi_h) = 0 \quad \forall \varphi_h \in S_h. \quad (27)$$

Properties of the operator $\Pi_h$:
There exists a constant $C_A > 0$ independent of $h, K, v$ such that

$$\|\Pi_h v - v\|_{L^2(K)} \leq C_A h_K^{k+1} |v|_{H^{k+1}(K)}, \quad (28)$$
$$|\Pi_h v - v|_{H^1(K)} \leq C_A h_K^k |v|_{H^{k+1}(K)},$$
$$|\Pi_h v - v|_{H^2(K)} \leq C_A h_K^{k-1} |v|_{H^{k+1}(K)},$$

for all $v \in H^{k+1}(K), \ K \in T_h$ and $h \in (0, h_0)$, where $k \in [1, p]$ is an integer.
**Coercivity:**

An important step in the analysis of error estimates is the *coercivity of the form*

\[ A_h(u, v) = a_h(u, v) + \varepsilon J^\sigma_h(u, v), \quad (29) \]

which reads

\[ A_h(\varphi_h, \varphi_h) \geq \frac{\varepsilon}{2} \left( |\varphi_h|^2_{H^1(\Omega, T_h)} + J^\sigma_h(\varphi_h, \varphi_h) \right), \quad (30) \]

\[ \varphi \in S_h, \ h \in (0, h_0). \]

We shall discuss the validity of estimate (30) in various situations.
(I) Conforming mesh $\mathcal{T}_h$

Let the mesh $\mathcal{T}_h$ have the standard properties from the finite element method:
if $K, K' \in \mathcal{T}_h$, $K \neq K'$, then $K \cap K' = \emptyset$ or $K \cap K'$ is a common vertex or $K \cap K'$ is a common side of $K$ and $K'$.
In this case we set

$$\sigma|_{\Gamma} = \frac{C_W}{d(\Gamma)}, \quad \Gamma \in \mathcal{F}_h. \quad (31)$$

Then the coercivity inequality (30) holds under the following choice of the constant $C_W$:

$$C_W > 0 \text{ (e.g. } C_W = 1) \text{ for NIPG version, } \quad (32)$$
$$C_W \geq 4C_M(1 + C_I) \text{ for SIPG version, } \quad (33)$$
$$C_W \geq 2C_M(1 + C_I) \text{ for IIPG version, } \quad (34)$$

where $C_M$ and $C_I$ are constants from (25) and (26), respectively.
(II) Nonconforming mesh $\mathcal{T}_h$

In this case $\mathcal{T}_h$ is formed by closed triangles with mutually disjoint interiors with hanging nodes in general. Then the coercivity inequality (30) is guaranteed under conditions (32) – (34). However, in this case it is necessary to assume that

$$h_K \leq C_D d(\Gamma), \quad \Gamma \in \mathcal{F}_h, \Gamma \subset \partial K,$$

(35)

in order to prove the estimate

$$J^\sigma_h (u - \Pi_h u, u - \Pi_h u) \leq Ch^p |u|_{H^p+1(\Omega)}. \quad (36)$$
(III) Nonconforming mesh $T_h$ without assumption (35)

It is obvious that condition (35) is rather restrictive in some cases. In order to avoid it, we change the definition of the weight $\sigma$:

\[
\sigma|_{\Gamma} = \begin{cases} 
\frac{2C_W}{h_{K_{\Gamma}^{(L)}} + h_{K_{\Gamma}^{(R)}}}, & \Gamma \in \mathcal{F}_h^I, \\
\frac{C_W}{h_{K_{\Gamma}^{(L)}}}, & \Gamma \in \mathcal{F}_h^D. 
\end{cases}
\]
Due to theoretical analysis, it is necessary to introduce the assumption of a “local quasiuniformity” of the mesh:

\[ h_{K^{(L)}} \leq C_N h_{K^{(R)}}, \quad \Gamma \in F^I_h. \]  

(Hence, \( C_N \geq 1 \).) Then the coercivity inequality (30) holds under the following choice of \( C_W \):

\[ C_W > 0 \quad (\text{e.g. } C_W = 1) \quad \text{for NIPG version,} \]  
\[ C_W \geq 2C_M(1 + C_I)(1 + C_N) \quad \text{for SIPG version,} \]  
\[ C_W \geq C_M(1 + C_I)(1 + C_N) \quad \text{for IIPG version.} \]
If \( u \) and \( u_h \) denote the exact and approximate solutions, then we set
\[
\eta(t) = \Pi_h u(t) - u(t), \quad \xi(t) = u_h(t) - \Pi_h u(t) (\in S_h)
\]
for a.e. \( t \in (0, T) \).

**Truncation error in the convection form:** If \( \partial \Omega_D = \partial \Omega, \partial \Omega_N = \emptyset \), then
\[
|b_h(u, \xi) - b_h(u_h, \xi)| \leq C \left( \|\xi\|^2_{H^1(\Omega, T_h)} + J_h^\sigma(\xi, \xi) \right)^{1/2} \left( h^{p+1} |u|_{H^{p+1}(\Omega)} + \|\xi\|_{L^2(\Omega)} \right).
\]

If \( \partial \Omega_N \neq \emptyset \), then
\[
|b_h(u, \xi) - b_h(u_h, \xi)| \leq C \left( \|\xi\|^2_{H^1(\Omega, T_h)} + J_h^\sigma(\xi, \xi) \right)^{1/2} \left( h^{p+1/2} |u|_{H^{p+1}(\Omega)} + \|\xi\|_{L^2(\Omega)} \right).
\]
**Error estimates**

**Assumptions:**

- (H),
- regularity of $u$,
- regularity of the mesh,
- $u_h^0 = \Pi_h u^0$,
- $\sigma, d(\Gamma), h_K$ and $C_W$ satisfy assumptions from the cases (I) or (II) or (III).

Then the error $e_h = u - u_h$ satisfies the estimate

$$
\text{max}_{t \in [0,T]} \| e_h(t) \|_{L^2(\Omega)}^2 \\
+ \frac{\varepsilon}{2} \int_0^t \left( |e_h(\vartheta)|_{H^1(\Omega, I_h)}^2 + J^\sigma_h(e_h(\vartheta), e_h(\vartheta)) \right) d\vartheta \\
\leq C h^{2p}, \quad h \in (0, h_0),
$$

with a constant $C > 0$ independent of $h$. 

Sketch of the proof

– Subtract the relations valid for the exact and approximate solutions, set \( \varphi_h = \xi_h \) and use the coercivity inequality:

\[
\frac{1}{2} \frac{d}{dt} \| \xi(t) \|^2_{L^2(\Omega)} + \frac{\varepsilon}{2} |\xi(t)|^2_{H^1(\Omega,T_h)} + \frac{\varepsilon}{2} J^\sigma_h(\xi(t),\xi(t)) \leq b_h(u(t),\xi(t)) - b_h(u_h(t),\xi(t)) - \left( \frac{\partial \eta(t)}{\partial t}, \xi(t) \right) - a_h(\eta(t),\xi(t)) - \varepsilon J^\sigma_h(\eta(t),\xi(t)) \quad \text{for a.a. } (0,T).
\]

– Estimate individual terms in (46):

\[
\frac{d}{dt} \| \xi \|^2_{L^2(\Omega)} + \varepsilon |\xi|^2_{H^1(\Omega,T_h)} + \varepsilon J^\sigma_h(\xi,\xi)
\]

\[
\leq C \left\{ \left( J^\sigma_h(\xi,\xi)^{1/2} + |\xi|_{H^1(\Omega,T_h)} \right) \left\| \xi \right\|_{L^2(\Omega)} + h^{p+1} |u|_{H^{p+1}(\Omega)} \right\}
\]

\[
+ h^{p+1} |\partial u/\partial t|_{H^{p+1}(\Omega)} \| \xi \|_{L^2(\Omega)} + \varepsilon h^p |u|_{H^{p+1}(\Omega)} \left( J^\sigma_h(\xi,\xi)^{1/2} + |\xi|_{H^1(\Omega,T_h)} \right) \}
\]

– Apply Young’s inequality, integrate from 0 to \( t \in [0,T] \) and use Gronwall’s lemma:
\[ \begin{align*}
\|\xi(t)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2} \int_0^t \left( |\xi(\vartheta)|^2_{H^1(\Omega, T_h)} + J_h^\sigma(\xi(\vartheta), \xi(\vartheta)) \right) d\vartheta \\
\leq \mathcal{C} \left( \left( \varepsilon + \frac{h^2}{\varepsilon} \right) \|u\|^2_{L^2(0, T; H^{p+1}(\Omega))} + h^2 \|\partial u / \partial t\|^2_{L^2(0, T; H^{p+1}(\Omega))} \right) \\
\times h^{2p} \exp \left( \frac{\tilde{\mathcal{C}}}{\varepsilon} \left( 1 + \frac{\varepsilon}{t} \right) \right), \quad t \in [0, T],
\end{align*} \]

(\mathcal{C} \text{ and } \tilde{\mathcal{C}} \text{ are constants independent of } t, h, \varepsilon, u).

- **Use** \( e_h = \xi + \eta \) and thus,

\[ \begin{align*}
\|e_h\|^2_{L^2(\Omega)} &\leq 2 \left( \|\xi\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Omega)} \right), \\
|e_h|^2_{H^1(\Omega, T_h)} &\leq 2 \left( |\xi|^2_{H^1(\Omega, T_h)} + |\eta|^2_{H^1(\Omega, T_h)} \right), \\
J_h^\sigma(e_h, e_h) &\leq 2 \left( J_h^\sigma(\xi, \xi) + J_h^\sigma(\eta, \eta) \right).
\end{align*} \]

- **Combine** the above results and **estimate the terms with** \( \eta \).
Optimal error estimates

The error estimate (44) is **optimal** in the $L^2(H^1)$-norm, but **suboptimal** in the $L^\infty(L^2)$-norm. We carried out the analysis of the $L^\infty(L^2)$-optimal error estimate under the following assumptions.

**Assumptions (B):**
- the discrete diffusion form $a_h$ is symmetric (i.e. we consider the SIPG version),
- consider a regular system of conforming meshes without hanging nodes,
- $\sigma|_\Gamma = C_W/d(\Gamma)$ and $C_W \geq 4(C_M(1+C_I))$,
- the polygonal domain $\Omega$ is convex,
- the exact solution $u$ satisfies the regularity condition,
- conditions (H) are satisfied,
- $u^0_h = \Pi_h u^0$,
- $\partial\Omega_D = \partial\Omega$ and $\partial\Omega_N = \emptyset$. 
The application of the Aubin-Nitsche technique based on the use of the elliptic dual problem considered for each \( z \in L^2(\Omega) \):

\[
-\Delta \psi = z \quad \text{in } \Omega, \quad \psi|_{\partial \Omega} = 0.
\]  

(50)

Then the weak solution \( \psi \in H^2(\Omega) \) and there exists a constant \( C > 0 \), independent of \( z \), such that

\[
\|\psi\|_{H^2(\Omega)} \leq C \|z\|_{L^2(\Omega)}.
\]

(51)

For each \( h \in (0, h_0) \) and \( t \in [0, T] \) we define the function \( u_h^*(t) \) as the “\( A_h \)-projection” of \( u(t) \) on \( S_h \), i.e. a function satisfying the conditions

\[
u_h^*(t) \in S_h, \quad A_h(u_h^*(t), \varphi_h) = A_h(u(t), \varphi_h) \quad \forall \varphi_h \in S_h, \quad (52)
\]

and set \( \chi = u - u_h^* \).

Using the elliptic dual problem (50), we prove the existence of a constant \( C > 0 \) such that

\[
\|\chi\|_{L^2(\Omega)} \leq C h^{p+1} |u|_{H^p+1(\Omega)},
\]

(53)

\[
\|\chi_t\|_{L^2(\Omega)} \leq C h^{p+1} |u_t|_{H^p+1(\Omega)}, \quad h \in (0, h_0).
\]

(54)
This, the estimate of the truncation error in the form $b_h$ (43), multiple application of Young's inequality and Gronwall's lemma \[\Rightarrow\]

**Theorem.** Let assumptions $\mathcal{B}$ be fulfilled. Then the error $e_h = u - u_h$ satisfies the estimate

\[
\|e_h\|_{L^\infty(0,T;L^2(\Omega))} \leq C h^{p+1},
\]  

(55) with a constant $C > 0$ independent of $h$. 
Numerical examples

2D viscous Burgers equation

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x_1} + u \frac{\partial u}{\partial x_2} = \varepsilon \Delta u + g \quad \text{in} \quad \Omega \times (0, T), \]

\( \Omega = (0, 1)^2, T = 10, \varepsilon = 0.01 \) and define the function \( g \) and the initial and boundary conditions in such a way that the exact solution has the form

\[ u(x_1, x_2, t) = (1 - e^{-10t}) \hat{u}(x_1, x_2), \]

\[ \hat{u}(x_1, x_2) = 2r^{\alpha}x_1x_2(1 - x_1)(1 - x_2) \]

\[ = r^{\alpha + 2} \sin(2\varphi)(1 - x_1)(1 - x_2), \]

\( (r, \varphi) \ (r \equiv (x_1^2 + x_2^2)^{1/2}) \) are the polar coordinates and \( \alpha \in \mathbb{R} \) is a constant.

\( \alpha = 4 \) - regular solution

\( \alpha = -3/2 \) - singularity at 0
Computational errors and orders of convergence of $P_1$, $P_2$ and $P_3$ approximations in $L^2$-norm for

$$\alpha = 4 \text{ (left) and } \alpha = -3/2 \text{ (right) at } t = 10$$

**Remark** The constant $C$ in the error estimates is of the order $O(\exp(\tilde{C}T/\varepsilon))$, which **blows up** for $\varepsilon \to 0^+$. 

$\Rightarrow$ a consequence of the application of necessary tools for overcoming the nonlinear convective terms, namely Young’s inequality and Gronwall’s lemma.
Improved estimates for a linear model convection-diffusion-reaction problem

Find $u : Q_T = \Omega \times (0, T) \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + v \cdot \nabla u - \varepsilon \Delta u + cu = g \quad \text{in } Q_T,$$

$$u = u_D \quad \text{on } \Gamma_D \times (0, T),$$

$$\varepsilon \frac{\partial u}{\partial n} = u_N \quad \text{on } \Gamma_N \times (0, T),$$

$$u(x, 0) = u_0(x), \quad x \in \Omega.$$

$\Gamma_D = \text{inlet, where } v \cdot n < 0$

In the case $\varepsilon = 0$ we put $u_N = 0$ and ignore the Neumann condition; $\Gamma_D = \text{inlet: } v \cdot n < 0$

**Assumptions on data (A)**

a) some regularity of $g, u^0, u_D, u_N, v, c$

b) $c - \frac{1}{2} \text{div} v \geq \gamma_0 > 0$ in $Q_T$ with a constant $\gamma_0$.

c) $\varepsilon \geq 0$. 
M.F. & K. Švadlenka: Error estimate

\[ \max_{t \in [0, T]} \| e_h(t) \|_{L^2(\Omega)}^2 \]
\[ + \, \varepsilon \int_0^T \left( |e_h(\vartheta)|^2_{H^1(\Omega, T_h)} + J_h^\sigma(e_h(\vartheta), e_h(\vartheta)) \right) d\vartheta \]
\[ \leq C(\varepsilon + h)h^{2p}, \]

with \( C \) independent of \( \varepsilon \to 0^+ \).
Further results:
– the effect of numerical integration (M.F., V. Sobotíková)
– optimal error estimates on nonconforming meshes (M.F.,
  V. Dolejší, V. Kučera, V. Sobotíková)
– analysis of problem with nonlinear convection and diffusion
  (M.F., V. Kučera)
– analysis of the hp-version of the DGFEM (V. Dolejší)
Applications to compressible flow with a wide range of Mach numbers

Standard numerical methods have difficulties with the solution of low Mach number flows

⇒ various modifications of the Euler (Navier-Stokes) equations are introduced (e.g. R. Klein, C.-D. Munz,...) allowing the solution of low Mach number flows

M.F., V. Dolejší, V. Kučera: DG unconditionally stable scheme for the solution of compressible flow using conservative variables – allowing the solution of flow with all positive Mach numbers

Main ingredients:

- semi-implicit time stepping based on homogeneity of fluxes
  Vijayasundaram numerical flux
- characteristic treatment of the boundary conditions
- limiting of order of accuracy in order to avoid the Gibbs phenomenon
- isoparametric elements at curved boundaries
Examples
quadratic triangular elements

1) Inviscid flow
a) Low Mach number flow at incompressible limit
Stationary flow past a Joukowski profile
constant far field quantities $\implies$ the flow is irrotational and homoentropic

**complex function method**: exact solution of incompressible inviscid irrotational flow satisfying the Kutta–Joukowski trailing condition, provided the velocity circulation around the profile, related to the magnitude of the far field velocity, $\gamma_{\text{ref}} = 0.7158$

**Compressible flow**:

$M_{\infty} = 10^{-4}$, $\#T_h = 5418$

The maximum density variation in compressible flow $\rho_{\text{max}} - \rho_{\text{min}} = 1.04 \cdot 10^{-8}$.

Computed velocity circulation related to the magnitude of the far field velocity: $\gamma_{\text{refcomp}} = 0.7205$, $\implies$ the relative error 0.66%
Compressible low Mach flow past a Joukowski profile, approximate solution, streamlines
Velocity distribution along the profile: ◦ ◦ ◦ – exact solution of incompressible flow, ——— – approximate solution of compressible low Mach flow
b) **Transonic and hypersonic flow with shock waves past the Joukowski profile**

with far field Mach number $M_\infty = 0.8$ and $M_\infty = 2.0$, respectively

The maximum density variation: $\rho_{\text{max}} - \rho_{\text{min}} = 0.94$ for $M_\infty = 0.8$ and $\rho_{\text{max}} - \rho_{\text{min}} = 2.61$ for $M_\infty = 2.0$
Entropy isolines of the flow past a Joukowski profile with $M_\infty = 0.8$ (left) and $M_\infty = 2.0$ (right)
c) Transonic nonstationary flow past NACA0012 profile

far field Mach number \( M_\infty = 0.8 \)

angle of attack \( \alpha \) oscillating according to the formula

\[
\alpha = \alpha_0 \sin \left( \frac{2\pi t}{\omega} \right),
\]

with \( \alpha_0 = 1.25^\circ \) and \( \omega = 10 \)

initial condition: stationary solution for \( \alpha = 0 \).
Distribution of the Mach number at $t_k = 50, 51, 52, \ldots, 61$

$\rho_{\text{max}} - \rho_{\text{min}} = 0.76$
2) Hypersonic viscous compressible flow

Flow past NACA0012 profile:
Far field Mach number $M_\infty = 2, \alpha = 10^\circ$
Reynolds number = 1000

Mesh for viscous flow - constructed by AMA
Mach number isolines for viscous flow

Distribution of the Mach number for viscous flow
Conclusion

- DGFEM is rather robust and efficient technique for the numerical solution of convection-diffusion problems and compressible flow
- developed method allows to solve compressible flow with all Mach numbers without any modification of governing equations

Further goals

- optimal error estimates for mixed Dirichlet-Neumann boundary conditions
- analysis of error estimates for solutions with a weak regularity
- applications to fluid-structure interaction problems (in progress)