Iterative approximation of fixed points

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Approximation itérative des points fixes

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In the last three decades many papers have been published on the iterative approximation of fixed points for certain classes of operators, by using Picard, Krasnoselskij, Mann and Ishikawa iteration methods, see, for example,
Berinde, V., *Iterative Approximation of Fixed Points*, Springer, 2007

and its first edition

Berinde, V., *Iterative Approximation of Fixed Points*, Editura Efemeride, Baia Mare, 2002

that include more than 1000 titles at the bibliography (1575 titles, in the 2nd revised and up-dated edition 2007 at Springer).

In those papers are given various **fixed point theorems**.

The importance of metrical fixed point theory consists mainly in the fact that for most functional equations

$$F(x) = y$$

we can equivalently transform them in a fixed point problem

$$x = Tx$$

and then apply a fixed point theorem to get information on the existence or existence and uniqueness of the fixed point, that is, of a *solution* for the original equation.

Moreover, fixed point theorems usually provide a **method** for constructing such a solution.

Main applications of fixed point theorems:

to obtain existence or existence and uniqueness theorems for various classes of operator equations (differential equations, integral equations, integro-differential equations, variational inequalities etc.)

1. An existence theorem (A. Constantin, *Annali di Matematica* **184** (2005), 131-138) - based on Schauder fixed point theorem

Theorem E1. Assume that

$$|f(t,u)| \leq g(t,|u|), t \geq 0, u \in \mathbb{R},$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ is such that the map $r \mapsto g(t,r)$ is non-decreasing on $\mathbb{R}^+$ for every fixed $t \geq 0$. Then for every $c > 0$ for which

$$\int_0^\infty g(t,2ct)dt < c,$$

the equation

$$u'' + f(t,u) = 0, t > 0, \quad \text{where } f \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$$

has a global solution u_c with $u_c(0) = 0$, $u_c(t) > 0$ for $t > 0$, and

$$\lim_{t\to\infty} \frac{u_c(t)}{t} = c.$$

2. An existence theorem (R.P. Agarwal, D. O'Regan, *Proc. AMS* **128** (2000), No. 7, 2085–2094) - based on Krasnoselskij's fixed point theorem

Theorem E2.

Consider the singular $(n; p)$ problem

$$y^{(n)}(t) + \Phi(t)[g(y(t) + h(y(t)] = 0, 0 < t < 1$$

$$y^{(i)}(0) = 0, 0 \leq i \leq n$$

$$y^{(p)}(1) = 0$$

Under several assumptions on g, h and Φ, the problem has a solution $y \in C^{n-1}[0,1] \cap C^n(0,1]$ with $y > 0$.

3. An existence and uniqueness theorem (A. Ben-Naoum, *Non-linear Differ. Equ. Appl.* **5** (1998) 407-426) - based on a continuation theorem of Leray-Schauder type and Mawhin's coincidence degree theory

4. An existence and uniqueness theorem (Classical)

Consider the integral equation

$$y(x) = f(x) + \lambda \int_a^x K(x, s, y(s))\, ds, \quad x \in [0, T], \tag{1}$$

Using appropriately the contraction mapping principle, we get for equation (1) the following conclusions:

1) existence and uniqueness of the solution;

2) A method for constructing the solution: $\{y_n\}$:

$$y_{n+1}(x) = f(x) + \int_a^x K(x, s, y_n(s))\, ds,$$

3) Error estimate, rate of convergence;

4) Stability results.

In order to prove all the previous theorems, the key tool is to equivalently write the problem (equation) as a fixed point problem

$$x = Tx$$

and then apply a certain fixed point theorem.

Judged from the perspective of its concrete applications, that is, from a **numerical point of view**, a fixed point theorem is valuable if, apart from the conclusion regarding the existence (and, possible, uniqueness) of the fixed point, it also satisfies some minimal *numerical* requirements, amongst which we mention:

(a) it provides a method (generally, *iterative*) for constructing the fixed point (s);

(b) it is able to provide information on the error estimate (rate of convergence) of the iterative process used to approximate the fixed point, and

(c) it can give concrete information on the stability of this procedure, that is, on the data dependence of the fixed point (s).

Only a few fixed point theorems in literature do fulfill all three requirements above. Moreover, the error estimate and data dependence of fixed points appear to have been given systematically mainly for the Picard iteration (sequence of successive approximations), in conjunction with various contraction conditions.

Example. If $T : X \to X$ is an $a-$contraction on a complete metric space (X, d), that is, there exists a constant $0 \leq a < 1$ such that

$$d(Tx, Ty) \leq ad(x, y), \ \forall x, y \in X,$$

then by contraction mapping theorem (Banach) we know that

(a) $Fix\,(T) = \{x^*\}$;

(b) $x_n = T^n x_0$ (Picard iteration) converges to x^* for all $x_0 \in X$;

(c) both the *a priori* and the *a posteriori* error estimates

$$d(x_n, x^*) \leq \frac{a^n}{1-a} \cdot d(x_0, x_1)\,, \;\; n = 0, 1, 2, \ldots, \tag{1}$$

$$d(x_n, x^*) \leq \frac{a}{1-a} \cdot d(x_{n-1}, x_n)\,, \;\; n = 1, 2, \ldots \tag{2}$$

hold.

Remark. (1): the errors $d(x_n, x^*)$ are decreasing as rapidly as the terms of geometric progression with ratio a, that is, $\{x_n\}_{n=0}^{\infty}$ converges to x^* at least as rapidly as the geometric series. The convergence is however *linear*, as shown by

$$d(x_n, x^*) \leq a \cdot d(x_{n-1}, x^*)\,, \;\; n = 1, 2, \ldots$$

If T satisfies a weaker contractive condition, e.g., T is *nonexpansive*, then Picard iteration does not converge, generally, or even if it converges, its limit is not a fixed point of T. More general iterative procedures are needed.

Example. Let $X = R$ with the usual norm, $K = \left[\frac{1}{2}, 2\right]$ and $T : K \to K$ be the function given by $Tx = \frac{1}{x}$, for all x in K. Then:

(a) T is Lipschitzian with constant $L = 4$;

(b) T is strictly pseudocontractive;

(c) $Fix\,(T) = \{1\}$, where $Fix\,(T) = \{x \in K |\ Tx = x\}$;

(d) The Picard iteration associated to T does not converge to the fixed point of T, for any $x_0 \in K \setminus \{1\}$;

(e) The Krasnoselskij iteration associated to T converges to the fixed point $p = 1$, for any $x_0 \in K$ and $\lambda \in (0, 1/16)$;

(f) The Mann iteration associated to T with $\alpha_n = \frac{n}{2n+1}$, $n \geq 0$ and $x_0 = 2$ converges to 1, the unique fixed point of T.

Let X be a normed linear space and $T : X \to X$ a given operator.

Picard iteration: $\{x_n\}_{n=0}^{\infty}$ is defined by $x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots;$

Krasnoselskij iteration: $x_0 \in X$ and $\lambda \in [0, 1]$, $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n\,, \quad n = 0, 1, 2, \ldots$$

Mann iteration: $y_0 \in X$ and $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1]$, $\{y_n\}_{n=0}^{\infty}$ defined by

$$y_{n+1} = (1 - \alpha_n)y_n + \alpha_n Ty_n\,, \quad n = 0, 1, 2, \ldots$$

Ishikawa iteration: $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_{n+1} & = & (1 - \alpha_n)x_n + \alpha_n Ty_n\,, \quad n = 0, 1, 2, \ldots \\ y_n & = & (1 - \beta_n)x_n + \beta_n Tx_n\,, \quad n = 0, 1, 2, \ldots\,, \end{cases}$$

$\{\alpha_n\}$ and $\{\beta_n\} \subset [0, 1]$, and $x_0 \in X$ arbitrary

Halpern iteration: $y_0 \in X$, $u \in X$ (fixed) and $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$, $\{y_n\}_{n=0}^{\infty}$ defined by

$$y_{n+1} = (1-\alpha_n)u + \alpha_n T y_n\,, \quad n = 0,1,2,\ldots$$

The problem of studying the rate of convergence of fixed point iterative methods arises in two different contexts:

1. For large classes of operators (quasi-contractive type operators) not only Picard iteration, but also the Mann and Ishikawa iterations can be used to approximate the fixed points.

In such situations, it is of theoretical and practical importance to compare these methods in order to establish, if possible, which one converges faster.

2. For a certain fixed point iterative method (Picard, Krasnoselskij, Mann, Ishikawa etc.) we do not know an *analytical* error estimate of the form (1) or (2).

In this case we can try an *empirical* study of the rate of convergence.

1. Theoretical approach.

Definition 1. Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to a, respectively b. Assume there exists

$$l = \lim_{n\to\infty} \frac{|\, a_n - a\,|}{|\, b_n - b\,|}. \tag{3}$$

1) If $l = 0$, then it is said that the sequence $\{a_n\}_{n=0}^{\infty}$ converges to a *faster* than the sequence $\{b_n\}_{n=0}^{\infty}$ to b;

2) If $0 < l < \infty$, then we say that the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ *have the same rate of convergence*.

Remarks.

1) If $l = \infty$, then the sequence $\{b_n\}_{n=0}^{\infty}$ converges *faster* than $\{a_n\}_{n=0}^{\infty}$, that is $b_n - b = o(a_n - a)$.

The concept introduced by Definition 1 allows us to compare the rate of convergence of two sequences, and will be useful in the sequel.

2) The concept of rate of convergence given by Definition 1 is a relative one, while in literature there exist concepts of absolute rate of convergence. However, in the presence of an error estimate of the form (1) or (2), the concept given by Definition 1 is much more suitable.

Indeed, the estimate (1) shows that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to x^* faster than any sequence $\{\theta^n\}$ to zero, where $0 < \theta < a$.

Suppose that for two fixed point iterations $\{x_n\}_{n=0}^{\infty}$, and $\{y_n\}_{n=0}^{\infty}$, converging to the same fixed point x^*, the following a priori error estimates

$$d(x_n, x^*) \leq a_n\,, \quad n = 0, 1, 2, \ldots \tag{4}$$

and

$$d(y_n, x^*) \leq b_n\,, \quad n = 0, 1, 2, \ldots \tag{5}$$

are available, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive real numbers (converging to zero). Then, in view of Definition 1, the following concept appears to be very natural.

Definition 2. If $\{a_n\}_{n=0}^{\infty}$ converges faster then $\{b_n\}_{n=0}^{\infty}$, then we shall say that the fixed point iteration $\{x_n\}_{n=0}^{\infty}$ *converges faster to* x^* than the fixed point iteration $\{y_n\}_{n=0}^{\infty}$ or, simply, that $\{x_n\}_{n=0}^{\infty}$ *is better* than $\{y_n\}_{n=0}^{\infty}$.

Theorem 1. Let E be an arbitrary Banach space, K a closed convex subset of E, and $T : K \to K$ an operator satisfying Zamfirescu's conditions, i.e., there exist the real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b$, $c < 1/2$ such that for each pair x, y in X, at least one of the following is true:
(z_1) $d(Tx, Ty) \leq a\, d(x, y)$;
(z_2) $d(Tx, Ty) \leq b\Big[d(x, Tx) + d(y, Ty)\Big]$;
(z_3) $d(Tx, Ty) \leq c\Big[d(x, Ty) + d(y, Tx)\Big]$.

Let $\{y_n\}_{n=0}^{\infty}$ be the Mann iteration defined by $y_0 \in K$, with $\{\alpha_n\} \subset [0, 1]$ satisfying

$$(i) \qquad \sum_{n=0}^{\infty} \alpha_n = \infty\,.$$

Then $\{y_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T and, moreover, Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_0 \in K$ converges faster than the Mann iteration.

Berinde, V., *Picard iteration converges faster than the Mann iteration for the class of quasi-contractive operators*, Fixed Point Theory and Applications 2004, No. 2, 97-105

Other related results:

Berinde, V., *Comparing Krasnoselskij and Mann iterations for Lipschitzian generalized pseudocontractive operators*. In: Proceed. of Int. Conf. On Fixed Point Theory, Univ. of Valencia, 19-26 July 2003, Yokohama Publishers (2004), pp. 15-26

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2. Empirical approach.

A special software package, call FIXPOINT, was designed by Andrei BOZÂNTAN (as a MSc Dissertation thesis). The execution of the program FIXPOINT for some input data had lead to the following observations:

1) The Krasnoselskij iteration converges to $p = 1$ for any $\lambda \in (0, 1)$ and any initial guess x_0 (recall that the Picard iteration does not converge for any initial value $x_0 \in [1/2, 2]$ different from the fixed point). The convergence is slow for λ close enough to 0 (that is, for Krasnoselskij iterations close enough to the Picard iteration) or close enough to 1. The closer to 1/2, the middle point of the interval $(0, 1)$, λ is, the faster it converges.

For $\lambda = 0.5$ the Krasnoselskij iteration converges very fast to $p = 1$, the unique fixed point of T. For example, starting with $x_0 = 1.5$, only 4 iterations are needed in order to obtain p with 6 exact digits: $x_1 = 1.08335$, $x_2 = 1.00325$, $x_3 = 1.000053$, $x_4 = 1$.

For the same value of λ and $x_0 = 2$, again only 4 iterations are needed to obtain p with the same precision, even though the initial guess is far away from the fixed point: $x_1 = 1.25$, $x_2 = 1.025$, $x_3 = 1.0003$ and $x_4 = 1$;

2) The speed of Mann and Ishikawa iterations also depends on the position of $\{\alpha_n\}$ and $\{\beta_n\}$ in the interval $(0, 1)$.

If we take $x_0 = 1.5$, $\alpha_n = 1/(n+1)$, $\beta_n = 1/(n+2)$, then the Mann and Ishikawa iterations converge (slowly) to $p = 1$: after $n = 35$ iterations we get $x_{35} = 1.000155$ for both Mann and Ishikawa iterations.

For $\alpha_n = 1/\sqrt[3]{n+1}$, $\beta_n = 1/\sqrt[4]{n+2}$ we obtain the fixed point with 6 exact digits performing 8 iterations (the Mann scheme) and, respectively, 9 iterations (the Ishikawa iteration). Notice that in this case both Mann and Ishikawa iterations converge not monotonically to $p = 1$.

Conditions like $\alpha_n \to 0$ (as $n \to \infty$) or/and $\beta_n \to 0$ (as $n \to \infty$) are usually involved in many convergence theorems presented in this book. The next results show that these conditions are in general not necessary for the convergence of Mann and Ishikawa iterations.

Indeed, taking

$$x_0 = 2,\ \alpha_n = \frac{n}{2n+1} \nearrow \frac{1}{2},\ \beta_n = \frac{n+1}{2n} \searrow 1/2,$$

we obtain the following results.

For the Mann iteration: $x_1 = 2$, $x_2 = 1.5$, $x_3 = 1.166$, $x_4 = 1.034$, $x_5 = 1.0042$, $x_6 = 1.00397$, $x_7 = 1.000031$, $x_8 = 1.000002$ and $x_9 = 1$.

For the Ishikawa iteration: $x_1 = x_2 = 2$, $x_3 = 1.357$, $x_4 = 1.120$, $x_5 = 1.0289$, $x_6 = 1.0047$, $x_7 = 1.0057$, $x_8 = 1.000054$, $x_9 = 1.00004$ and $x_{10} = 1$.

For all combinations of x_0, λ, α_n and β_n, we notice the following decreasing (with respect to their speed of convergence) chain of iterative methods: Krasnoselskij, Mann, Ishikawa. Consequently, if for a certain operator in the same class, all these methods converge, then we shall use the fastest one (empirically deduced).

The next example presents a function with two repulsive fixed points with respect to the Picard iteration.

Example. Let $K = [0,1]$ and $T : K \to K$ given by $Tx = (1-x)^6$.

Then T has $p_1 \approx 0.2219$ and $p_1 \approx 2.1347$ as fixed points (obtained with Maple).

Here there are some numerical results obtained by running the new version of the program FIXPOINT, to support the previous assertions.

Krasnoselskij iteration: for $x_0 = 2$ and $\lambda = 0.5$, we obtain $x_1 = 1.5$, $x_2 = 0.757$, $x_3 = 0.379$, $x_4 = 0.2181$, $x_5 = 0.2232$ and $x_6 = 0.2214$;

Mann iteration: for $x_0 = 2$ and $\alpha_n = 1/(n+1)$, we obtain $x_1 = 1.0$, $x_2 = 0.5$, $x_3 = 0.338$, $x_4 = 0.2748$, $x_5 = 0.2489$ and $x_6 = 0.2378$;

Ishikawa iteration: for $x_0 = 2$ $\alpha_n = 1/(n+1)$ and $\beta_n = 1/(n+2)$, we obtain $x_1 = 0.01$, $x_2 = 0.55$, $x_3 = 0.346$, $x_4 = 0.2851$, $x_5 = 0.2527$ and $x_6 = 0.2392$;

The previous numerical results suggest that Krasnoselskij iteration converges faster than both Mann and Ishikawa iterations. This fact is more clearer illustrated if we choose $x_0 = p_2$, the repulsive fixed point of T: after 20 iterations, Krasnoselskij method gives $x_{20} = 0.2219$, while Mann and Ishikawa iteration procedures give $x_{20} = 0.6346$ and $x_{20} = 0.6347$, respectively. The convergence of Mann and Ishikawa iteration procedures is indeed very slow in this case: after 500 iterations we get $x_{500} = 0.222$ for both methods.

Note that for $x_0 \in \{-2, 3, 4\}$ and the previous values of the parameters λ, α_n and β_n, all the three iteration procedures: Krasnoselskij, Mann and Ishikawa, converge to 1, which is not a fixed point of T.

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