Finite element approximation of a fourth order nonlinear PDE

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Outline

- Introduction
- Continuous problem
- Finite element approximation
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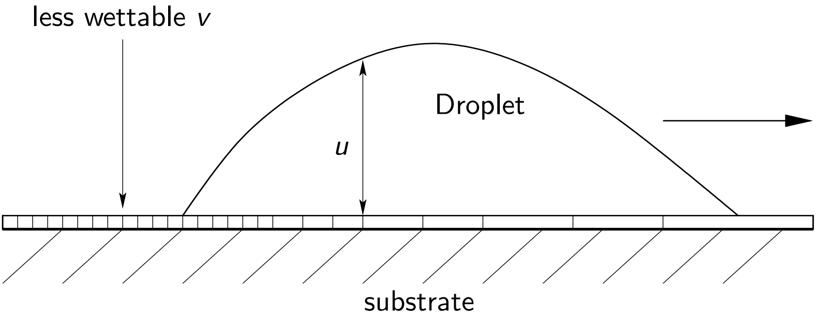
- Conclusions

Introduction

- A moving droplet with chemical on a partially wettable substrate



- u: film thickness,
- v: concentrate of the adsorbate.

Introduction

- Numerous Applications
  - Industrial,
  - Biomedical,
  - Environmental,
  - Petrochemical $\cdots$

Thin Film Equation

Find $u(x,t), w(x,t) \in \mathbb{R}$ s.t.

$$\begin{aligned}
\tfrac{\partial u}{\partial t} - \tfrac{1}{3}\,\nabla\cdot(u^3\,\nabla w) = 0 &\quad \text{in } \ \Omega \times (0,T)\,;\\
w = -c\,\Delta u &\quad \text{in } \ \Omega \times (0,T)\,;\\
u(x,0) = u^0(x) \geq 0 &\quad \forall\ x \in \Omega\,;\\
\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 &\quad \text{on } \partial\Omega \times (0,T)\,, \qquad \nu \text{ is normal to } \partial\Omega\,.
\end{aligned}$$

- 4^{th} order Degenerate Nonlinear Parabolic PDE.
- No Maximum/Comparison Principle.
- Degeneracy yields that $u^0 \geq 0 \Rightarrow u \geq 0$.
- Huge literature on their analysis since the solution concept in one space dimension by Bernis & Friedman (90).
- No Uniqueness for Bernis & Friedman solution concept.
- No Proof of Uniqueness for stronger solution concept.

Model problem

- Evolution of the film thickness

$$\frac{\partial u}{\partial t} - \frac{1}{3}\nabla\cdot(u^3\,\nabla w) = 0$$
$$w = -\Delta u + \phi(u,v)\,.$$

  - $-\Delta u$ is the curvature pressure,
  - $\phi(u,v)$ is the disjoining pressure comprising molecular interactions between the film surface and the substrate:

$$\phi(u,v) = -\delta(1-\tfrac{v}{g})u^{-\nu} + a\,u^{-3}\,,$$

    $a \geq 0$ is the Hamaker constant, $\delta \geq 0$ represents the effect of repulsive Van Der Waals forces, $\nu > 7$ and $g > 1$.

Evolution of the chemical concentration of the adsorbate

- Reaction–diffusion equation

$$\frac{\partial v}{\partial t} = \gamma \Delta v + \Psi(u, v)\,,$$

  - $\gamma > 0$ is the diffusion coefficient,
  - $\Psi(u, v)$ describes adsorption of the substrate,

$$\Psi(u, v) = r\,\Theta(u - \lambda)(1 - v)\,,$$

  $r \geq 0\,, \Theta$ is a cut–off function and $\lambda > 0\,.$

The model problem

Find $v(x,t)\,, u(x,t)\,, w(x,t) \in \mathbb{R}$, s.t.

$$\begin{aligned}
&\tfrac{\partial v}{\partial t} = \gamma \Delta v + \Psi(u,v) && \text{in } \Omega_T \equiv \Omega \times (0,T)\,,\\
&\tfrac{\partial u}{\partial t} = \nabla\cdot(u^3\,\nabla w) && \text{in } \Omega_T\,,\\
&w = -\Delta u + \phi(u,v) && \text{in } \Omega_T\,,\\
&u(x,0) = u^0(x) > 0\,, v(x,0) = v^0(x) \geq 0 && \forall x \in \Omega\,,\\
&\tfrac{\partial v}{\partial \nu} = \tfrac{\partial u}{\partial \nu} = u^3\,\tfrac{\partial w}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0,T)\,.
\end{aligned}$$

The continuous setting

- There is a unique v solution to the diffusion–reaction equation.

- If $v^0 \in W^{2-\frac{2}{p}}(\Omega) \cap H^2(\Omega)$ for some $p > d$, and $0 \leq v^0 \leq 1$,

$$0 \leq v \leq 1 \quad \text{ and } v \in L^2(0, T; W^{1,\infty}(\Omega))\,.$$

The continuous setting

- Let us introduce

$$\Phi(u,v) = \Phi^+(u,v) + \Phi^-(u) \text{ where}$$
$$\Phi^+(u,v) := \tfrac{\delta}{\nu-1}(1-\tfrac{v}{g})u^{1-\nu}\,, \Phi^-(u) := -\tfrac{a}{2}u^{-2}\,.$$

Note that $\quad \Phi_u(u,v) = \phi(u,v)\,.$

- Formally, it holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\tfrac{1}{2}|\nabla u|_0^2 + (\Phi(u,v),1)] &= (\nabla u, \nabla \partial_t u) + (\Phi_u(u,v), \partial_t u) + (\Phi_v(u), \partial_t v) \\ &= (w, \partial_t u) + (\Phi_v(u), \partial_t v) \\ &= -(u^3 \nabla w, \nabla w) + (\Phi_v(u), \partial_t v)\,.\end{aligned}$$

And,

$$\begin{aligned}(\Phi_v(u), \partial_t v) &\leq -\gamma(\phi_v(u)\nabla u, \nabla v) \\ &\leq C[(\phi_u^+(u,v)\nabla u, \nabla u) + (\Phi^+(u,v)\nabla v, \nabla v)]\,.\end{aligned}$$

The continuous setting

- Let $G(u)$ where $u^3\nabla[G'(u)] = \nabla u$

$$\frac{\mathrm{d}}{\mathrm{d}t}(G(u),1) + (u^3\nabla w, \nabla G'(u)) = 0\,,$$
$$(\nabla w, \nabla u) = |\Delta u|_0^2 - (\phi(u,v), \Delta u)\,,$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(G(u),1) + |\Delta u|_0^2 + (\phi_u^+(u,v)\nabla u, \nabla u) = -([\phi^-(u)]'\nabla u, \nabla u) - (\phi_v(u)\nabla u, \nabla v)\,.$$

$$(\Phi_v(u), \partial_t v) \quad \le C[(\phi_u^+(u,v)\nabla u, \nabla u) + (\Phi^+(u,v)\nabla v, \nabla v)]\,.$$

The continuous setting

- Energy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\tfrac{1}{2}\,|\nabla u|^2 + \Phi(u,v) + G(u)\right] + \int_\Omega u^3|\nabla w|^2$$
$$- \int_\Omega \Phi_v(u,v)\Psi(u,v) + \int_\Omega |\Delta u|^2 + \int_\Omega \phi_u^+(u,v)|\nabla u|^2$$
$$\leq C\left[1 + |\nabla u|_0^2 + |\nabla v|_{0,\infty}^2(\Phi(u,v),1)\right] \leq C\left[1 + |\nabla u|_0^2\right] .$$

The discret setting

- Finite element approximation
  $\mathcal{T}^h$ a Quasi-Uniform partitioning of Ω^h a polyhedral approximation of Ω into disjoint
  open simplices κ with $h_\kappa := \mathrm{diam}(\kappa)$, $h := \max_{\kappa\in\mathcal{T}^h} h_\kappa$;

  i.e. $\overline{\Omega} = \bigcup_{\kappa\in\mathcal{T}^h} \overline{\kappa}, \quad C_1\, h^d \leq \underline{m}(\kappa) \leq C_2\, h^d \quad \forall\ \kappa \in \mathcal{T}^h.$

$$S^h := \{\chi \in C(\overline{\Omega}) : \chi\,|_\kappa \text{ is linear } \forall\ \kappa \in \mathcal{T}^h\}\,.$$

The discrete setting

- $\pi^h : C(\overline{\Omega}) \to S^h$, the Interpolation Operator, s.t.

$$(\pi^h \eta)(p_j) = \eta(p_j) \qquad \forall\, j \in J,$$

where $\{p_j\}_{j \in J}$ are the nodes (vertices) of $\mathcal{T}^h$.

- Numerical Integration - vertex sampling

$$(\eta_1, \eta_2)^h := \int_\Omega \pi^h[\,\eta_1(x)\,\eta_2(x)\,]\,\mathrm{d}x \qquad \forall\, \eta_1,\, \eta_2 \in C(\overline{\Omega}).$$

- Let $0 \equiv t_0 < t_1 < \ldots < t_{N-1} < t_N \equiv T$ be a partitioning of $[0, T]$ into variable time steps $\tau_n := t_n - t_{n-1}$, $n = 1 \to N$.

The discrete setting

- Let us introduce $\Xi(\chi) \approx \chi \mathcal{I}$ for all $\chi \in S^h_{>0}$ s.t.

$$\Xi(\cdot) : S^h_{>0} \to [L^\infty(\Omega)]^{d \times d}$$
$$\Xi(\chi) \text{ is symmetric and positive semidefinite,}$$
$$[\Xi(\chi)]^3 \, \nabla \pi^h [G'(\chi)] = \nabla \chi.$$

[Grün & Rumpf (00)]

The finite element scheme

- For $n \geq 1$ find $\{V_h^n, U_h^n, W_h^n\} \in [S^h]^3$ such that for all $\chi \in S^h$

$$\left(\frac{V_h^n - V_h^{n-1}}{\tau_n}, \chi\right)^h + \gamma\,(\nabla V_h^n, \nabla\chi) = (\Psi(U_h^{n-1}, V_h^n), \chi)^h\,,$$

$$\left(\frac{U_h^n - U_h^{n-1}}{\tau_n}, \chi\right)^h + ([\Xi(U_h^n)]^3\,\nabla W_h^n, \nabla\chi) = 0\,,$$

$$(\nabla U_h^n, \nabla\chi) + (\phi^+(U_h^n, V_h^n) + \phi^-(U_h^{n-1}), \chi)^h = (W_h^n, \chi)^h\,.$$

- Remarks
  - $\Psi(x, \cdot)$ is approximated explicitly and $\Psi(\cdot, x)$ implicitly. ϕ^+ is approximated implicitly and ϕ^- explicitly.
  - The first equation is independent of $U_h^n \Rightarrow$ solving the system for V_h^n and then for $\{U_h^n, W_h^n\}$.

The diffusion–reaction equation

- There exists a unique solution $V_h^n \in S^h$ for $n \geq 1$. Furthermore, $V_h^n \in [0,1]$.
- Let $V_h^0 \in S^h$. Then for all $h > 0$, for all time partitions $\{\tau_n\}_{n=1}^N$ and for given $U_h^{n-1} \in S_{>0}^h$ the solution $\{V_h^n\}_{n=1}^N$ is such that

$$\sum_{n=1}^N \tau_n \left| \frac{V_h^n - V_h^{n-1}}{\tau_n} \right|_h^2 + \gamma \max_{1 \leq n \leq N} |V_h^n|_1^2 + \max_{1 \leq n \leq N} |V_h^n|_h^2 + \sum_{n=1}^N \tau_n |V_h^n|_1^2$$

$$+ \gamma \sum_{n=1}^N |V_h^n - V_h^{n-1}|_1^2 + \frac{1}{2} \sum_{n=1}^N |V_h^n - V_h^{n-1}|_h^2 + \sum_{n=1}^N \tau_n |\mathcal{G}[\tfrac{V_h^n - V_h^{n-1}}{\tau_n}]|_{1,q}^2$$

$$\leq C(T),$$

where $q = 2$ if $d = 1$ and $q \in (1,2)$ if $d = 2$.

The diffusion–reaction equation

- Proof:
  - Choosing $\chi = V_h^n - V_h^{n-1}$ yields

$$\tau_n \left| \frac{V_h^n - V_h^{n-1}}{\tau_n} \right|_h^2 + \gamma |V_h^n|_1^2 + \gamma |V_h^n - V_h^{n-1}|_1^2 \leq \gamma |V_h^{n-1}|_1^2 + C\,\tau_n\,.$$

  - Choosing $\chi = V_h^n$ yields

$$\tfrac{1}{2}|V_h^n|_h^2 + \tfrac{1}{2}|V_h^n - V_h^{n-1}|_h^2 + \tau_n \gamma |V_h^n|_1^2 \leq \tfrac{1}{2}|V_h^{n-1}|_h^2 + C\,\tau_n\,.$$

  - For any $\eta \in W^{1,q'}(\Omega)$

$$(\nabla \mathcal{G}[\tfrac{V_h^n - V_h^{n-1}}{\tau_n}], \nabla \eta) = (\tfrac{V_h^n - V_h^{n-1}}{\tau_n}, Q^h \eta)^h \leq C\left[\gamma |\nabla V_h^n|_0 + 1\right] |\eta|_{1,q'}\,.$$

The thin film equation

- **Theorem** Let $U_h^{n-1} \in S^h_{>0}$ and $V_h^n \in S^h$. Then for all h, $\tau_n > 0$ there exists a solution $\{V_h^n, U_h^n, W_h^n\} \in S^h \times S^h_{>0} \times S^h$.

- Proof based on:
  - a regularization procedure,
  - a fixed point argument.

Discrete energy inequality

- For all $h > 0$ on assuming $\sum_{n=1}^{N} \tau_n |V_h^n|_{1,\infty}^2 \leq C$ a solution $\{V_h^n, U_h^n, W_h^n\}$ is such that

$$\begin{aligned}
&\mathcal{E}(U_h^n, V_h^n) + (G(U_h^n), 1)^h + \tfrac{1}{2}|U_h^n - U_h^{n-1}|_1^2 + \tau_n |[\Xi(U_h^n)]^{\frac{3}{2}} \nabla W_h^n|_0^2 \\
&\qquad + \tau_n (\nabla \pi^h \phi^+(U_h^n, V_h^n), \nabla U_h^n) + \tfrac{c\,\tau_n}{3} |\Delta^h U_h^n|_h^2 \\
&\quad \leq \mathcal{E}(U_h^{n-1}, V_h^{n-1}) + (G(U_h^{n-1}), 1)^h \\
&\qquad + C\,\tau_n \big[(\nabla \pi^h \phi^+(U_h^{n-1}, V_h^{n-1}), \nabla U_h^{n-1}) + |U_h^n|_1^2 + |U_h^{n-1}|_1^2 \big] ,
\end{aligned}$$

where

$$\mathcal{E}(U_h^n, V_h^n) = \tfrac{1}{2}|U_h^n|_1^2 + (\Phi(U_h^n, V_h^n), 1)^h .$$

Discrete energy inequality

- Proof:
  - Choosing $\chi = U_h^n - U_h^{n-1}$ in the W–eq and $\chi = W_h^n$ in the thin film eq yields

$$\mathcal{E}(U_h^n, V_h^n) + \tfrac{1}{2}|U_h^n - U_h^{n-1}|_1^2 + \tau_n|[\Xi(U_h^n)]^{\frac{3}{2}}\nabla W_h^n|_0^2 \\ \leq \mathcal{E}(U_h^{n-1}, V_h^{n-1}) - \gamma\tau_n(\nabla\pi^h[\Phi_v(U_h^{n-1}, V_h^{n-1})], \nabla V_h^n)\,.$$

  - Choosing $\chi = -\Delta^h U_h^n$ in the W–eq and $\chi = \pi^h[G'(U_h^n)]$ in the thin film eq yields

$$(G(U_h^n) - G(U_h^{n-1}), 1)^h + \tau_n|\Delta^h U_h^n|_h^2 + \tau_n(\nabla[\pi^h[\phi^+(U_h^n, V_h^n)]], \nabla U_h^n) \\ \leq\; -\tau_n(\nabla[\pi^h[\phi^-(U_h^{n-1})]], \nabla U_h^n)\,.$$

  - Hence,

$$\mathcal{E}(U_h^n, V_h^n) + (G(U_h^n), 1)^h + \tfrac{1}{2}|U_h^n - U_h^{n-1}|_1^2 + \tau_n|\Delta^h U_h^n|_h^2 \\ + \tau_n(\nabla[\pi^h\phi^+(U_h^n, V_h^n)]], \nabla U_h^n) + \tau_n([\Xi(U_h^n)]^3\nabla W_h^n, \nabla W_h^n) \\ \leq \mathcal{E}(U_h^{n-1}, V_h^{n-1}) + (G(U_h^{n-1}), 1)^h + C(U_h^n, U_h^{n-1}, V_h^n) \\ + C\tau_n|V_h^n|_{1,\infty}^2((U_h^n)^{1-\nu}, 1)^h\,.$$

Stability estimate

- **Theorem:** Let $a, \delta > 0$. Let $\{V_h^0, U_h^0\} \in S^h \times S^h_{>0}$. Then for all $h > 0$ a solution $\{V_h^n, U_h^n, W_h^n\}_{n=1}^N$ is such that $⨍ U_h^n = ⨍ U_h^0$ for $n = 1 \to N$ and if $\tau_n \leq \frac{5}{4}\omega\,\tau_{n-1}$, then

$$\max_{n=1\to N} \|U_h^n\|_1^2 + \max_{n=1\to N} (\Phi(U_h^n, V_h^n), 1)^h + c\sum_{n=1}^N \|U_h^n - U_h^{n-1}\|_1^2$$

$$+ \max_{n=1\to N} (G(U_h^n), 1)^h + c\sum_{n=1}^N \tau_n |[\Xi(U_h^n)]^{\frac{3}{2}} \nabla W_h^n|_0^2$$

$$+ c\sum_{n=1}^N \tau_n |\Delta^h U_h^n|_h^2 + c\sum_{n=1}^N \tau_n (\nabla \pi^h[\phi^+(U_h^n, V_h^n)], \nabla U_h^n) \leq C\,.$$

Convergence

- Some notations

$$V_h(t) := \tfrac{t-t_{n-1}}{\tau_n}\, V_h^n + \tfrac{t_n-t}{\tau_n}\, V_h^{n-1} \qquad t \in [t_{n-1}, t_n] \quad n \geq 1$$

and $$V_h^+(t) := V_h^n, \qquad V_h^-(t) := V_h^{n-1} \quad t \in (t_{n-1}, t_n] \quad n \geq 1\,.$$

Convergence

- Find $\{V_h\,,U_h\,,W_h^+\} \in [C([0,T];S^h)]^2 \times L^\infty(0,T;S^h)$ s.t. $\forall \chi \in L^\infty(0,T;S^h)$

$$\int_0^T \left(\tfrac{\partial V_h}{\partial t},\chi\right)^h + \gamma(\nabla V_h^+,\nabla\chi) = \int_0^T (\Psi(U_h^-,V_h^+),\chi)^h \,\mathrm{d}t\,,$$

$$\int_0^T \left[\left(\tfrac{\partial U_h}{\partial t},\chi\right)^h + \tfrac{1}{3}\,([\Xi(U_h^+)]^3\,\nabla W_h^+,\nabla\chi)\right]\mathrm{d}t = 0\,.$$

where for $a.a.\ t \in (0,T)$ and for all $z^h \in S^h$

$$(W_h^+(t,\cdot),z^h)^h = (\nabla U_h^+(t,\cdot),\nabla z^h) + (\phi^+(U_h^+(t,\cdot),V_h^+(t,\cdot)) + \phi^-(U_h^-(t,\cdot)),z^h)^h\,.$$

Convergence

- **Lemma:** Let $\delta > 0$, and $v^0 \in S^h$. Let $\tau\, h^{-d\,(1-\frac{2}{p})} \longrightarrow 0$ as $h \longrightarrow 0$, where $p = 2$ if $d = 1$, and $p > 2$ if $d = 2$.
  There exists a subsequence of $\{V_h\,, U_h\,, W_h\}_h$ and functions

  $$v \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; L^2(\Omega)) \cap H^1(0, T; (W^{1,q}(\Omega))')\,,$$
  $$u \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; (W^{1,q}(\Omega))')\,,$$

  for any $q > 2$

Convergence

such that as $h \to 0$

$$
\begin{aligned}
V_h^{\pm} &\to v && \text{weakly in } L^2(0,T;L^2(\Omega)) \text{ and } L^2(0,T;H^1(\Omega))\,,\\
\mathcal{G}\frac{\partial V_h}{\partial t} &\to \mathcal{G}\frac{\partial v}{\partial t} && \text{weakly in } L^2(0,T;W^{1,q}(\Omega))\,,\\
\mathcal{G}\frac{\partial U_h}{\partial t} &\to \mathcal{G}\frac{\partial u}{\partial t} && \text{weakly in } L^2(0,T;W^{1,q}(\Omega))\,,\\
U_h^{\pm}(t,\cdot) &\to u(t,\cdot) && \text{strongly in } \mathcal{C}^{0,\gamma}(\overline{\Omega}))\,, \gamma \in (0,\tfrac{1}{2})\,,
\end{aligned}
$$

and for a.a $t \in (0,T)$

$$
u(t,\cdot) \in \mathcal{C}^{0,\gamma}(\overline{\Omega}) \quad \text{with } u(t,x) \geq \xi(t) > 0 \quad \forall x \in \overline{\Omega}\,.
$$

$$
\begin{aligned}
W_h^{+}(t,\cdot) &\to w(t,\cdot) && \text{weakly in } H^1(\Omega)\,,\\
[\Xi(U_h)]^{\frac{3}{2}}\nabla W_h^{+} &\to u^{\frac{3}{2}}\nabla w && \text{weakly in } L^2(\Omega_T)\,,\\
\Theta(U_h^{+}) &\to \Theta(u) && \text{strongly in } L^2(0,T;H^1(\Omega))\,.
\end{aligned}
$$

Convergence: Idea of the proof

The sability bound leads for a.a $t \in (0, T)$ to

$$|\Delta^h U_h^+(t,\cdot)|_0 \leq C(t)\,,$$

then for all $\eta \in W^{1,p}(\Omega)$

$$\begin{aligned}\int_0^T (\Delta^h U_h^+, \eta)\,\mathrm{d}t &= \int_0^T (\Delta^h U_h^+, (I - \pi^h)\eta)\,\mathrm{d}t \\ &\quad + \int_0^T [(\Delta^h U_h^+, \pi^h \eta) - (\Delta^h U_h^+, \pi^h \eta)^h]\,\mathrm{d}t \\ &\quad + \int_0^T (\nabla U_h^+, \nabla (I - \pi^h)\eta)\,\mathrm{d}t - \int_0^T (\nabla U_h^+, \nabla \eta)\,\mathrm{d}t \\ &\longrightarrow -\int_0^T (\nabla u, \nabla \eta)\,\mathrm{d}t\,, \quad \text{as } h \to 0\,.\end{aligned}$$

denseness of $W^{1,p}(\Omega)$ in $L^2(\Omega)$ yields

$$\Delta^h U_h^+(t,\cdot) \longrightarrow \Delta u(t,\cdot) \quad \text{weakly in } L^2(\Omega)\,.$$

Convergence: Idea of the proof

As $h \to 0$,

$$W_h^+(t,\cdot) \equiv -c\,\Delta^h U_h^+(t,\cdot) + \pi^h[\phi^+(U_h^+, V_h^+) + \phi^-(U_h^-)](t,\cdot)$$
$$\longrightarrow -c\,\Delta u(t,\cdot) + \phi(u(t,\cdot), v(t,\cdot)) \quad \text{weakly in } L^2(\Omega)\,.$$

Furthermore,

$$c|\nabla W_h^+(t,\cdot)|_0^2 \leq |\Xi(U_h^+(t,\cdot))^{\frac{3}{2}} \nabla W_h^+(t,\cdot)|_0^2 \leq C(t)\,,$$

hence,

$$W_h^+(t,\cdot) \longrightarrow -c\,\Delta u(t,\cdot) + \phi(u(t,\cdot), v(t,\cdot)) \quad \text{weakly in } H^1(\Omega)\,.$$

Finally, there exists $z \in L^2(\Omega_T)$ s.t. for all $\eta \in L^2(0,T; H^1(\Omega))$

$$\int_0^T (\Xi(U_h^+)^{\frac{3}{2}} \nabla W_h^+, \nabla \eta)\,\mathrm{d}t \longrightarrow \int_0^T (z, \nabla \eta)\,\mathrm{d}t \quad \text{as } h \to 0\,.$$

Convergence

For all $\eta \in L^2(0,T;W^{1,q'}(\Omega))$

$$\left|\int_0^T (\Xi(U_h^+)^{\frac{3}{2}}\nabla W_h^+, \nabla(I-\pi^h)\eta)\right| \le C\,\|(I-\pi^h)\eta\|_{L^2(W^{1,q})}\,,$$

and for all $\widetilde{\eta} \in H^1(0,T;W^{1,\infty}(\Omega))$

$$C\left|\int_0^T ((\Xi(U_h^+)^{\frac{3}{2}} - u^{\frac{3}{2}}\Xi(U_h^+))\nabla W_h^+, \nabla\eta)\right| \le \|\Xi(U_h^+)^{\frac{3}{2}} - u^{\frac{3}{2}}\|_{L^2(\Omega_T)} + \|\eta-\widetilde{\eta}\|_{L^2(W^{1,q'})}\,.$$

Since $H^1(0,T;W^{1,\infty}(\Omega))$ is dense in $L^2(0,T;W^{1,q'}(\Omega))$

$$\int_0^T (\Xi(U_h)^3\nabla W_h^+, \nabla(\pi^h\eta))\,\mathrm{d}t \longrightarrow \int_0^T (u^3\nabla w, \nabla\eta)\,\mathrm{d}t\,.$$

Convergence

- **Theorem:** For all $\eta \in L^2(0,T;W^{1,q'}(\Omega))$

$$\int_0^T < \frac{\partial v}{\partial t}, \eta >_{q'} \, \mathrm{d}t + \gamma \int_{\Omega_T} \nabla v \cdot \nabla \eta \, \mathrm{d}x \, \mathrm{d}t = \int_{\Omega_T} \Psi(u,v)\, \eta \, \mathrm{d}x \, \mathrm{d}t \,,$$

$$\int_0^T < \frac{\partial u}{\partial t}, \eta >_{q'} \, \mathrm{d}t + \frac{1}{3} \int_{\Omega_T} u^3 \nabla w \cdot \nabla \eta \, \mathrm{d}x \, \mathrm{d}t = 0 \,.$$

where for a.a $t \in (0,T)$

$$\int_{\Omega} [w(t,\cdot)\xi - \nabla u(t,\cdot) \cdot \nabla \xi - \phi(u(t,\cdot), v(t,\cdot))\xi] = 0 \,, \quad \forall \xi \in H^1(\Omega).$$

Hence, we have existence to a solution of the system of PDE modelling chemically driven droplets.

Conclusions

- Summary
  - Finite element scheme,
  - Convergence of the scheme,
  - Existence to a solution of the model problem.
- Outlook
  - Extension of the techniques to surfactant droplets model.

Iterative algorithm

Given $U_h^{n-1} \in S^h_{>0}$ and $V_h^{n,0} \in S^h$ for $k \geq 1$ find $V_h^{n,k} \in S^h$ s.t.

$$\left(\frac{V_h^{n,k} - V_h^{n-1}}{\tau_n}, \chi\right)^h + \gamma\,(\nabla V_h^{n,k}, \nabla \chi) = (\Psi(U_h^{n-1}, V_h^{n,k-1}), \chi)^h \quad \forall \chi \in S^h\,.$$

Then, given $\{U_h^{n,0}, W_h^{n,0}\} \in S^h_{>0} \times S^h$, for $k \geq 1$ find $\{U_h^{n,k}, W_h^{n,k}\} \in S^h \times S^h$ s.t.

$$\left(\frac{U_h^{n,k} - U_h^{n-1}}{\tau_n}, \chi\right)^h + ([\Xi(U_h^{n,k-1})]^3\,\nabla W_h^{n,k}, \nabla \chi) = 0 \quad \forall \chi \in S^h\,,$$

$$(\nabla U_h^{n,k}, \nabla \chi) + (\phi^+(U_h^{n,k}, V_h^n) + \phi^-(U_h^{n-1}), \chi)^h = (W_h^{n,k}, \chi)^h \;\forall \chi \in S^h\,.$$