# Interpolation and reduced basis for scenario dependent functions

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# Outline

1. Motivation
2. Introduction and motivation: Taylor and Ito-Taylor formulas
   - Taylor
   - Ito-Taylor
3. Cubature on $C^k$ space
   - Background on cubature on the Wiener space
4. Conclusion and perspectives

# Motivation: control and inversion

$$dZ_t/dt = \mathcal{L}(t, Y_t, Z_t; \mu), \ t \in [0, 1]$$

- $Z_t$ the state of the system;
- $Y_t$= parameter; $Y \in E$, with $E$ = Sobolev or $C^k$; chosen at will (control) or belongs to a large set (scenario analysis);
- $\mu \in \mathbb{R}^d$ a quantity (or function) to be fitted; real system corresponds to $\bar{\mu}$;
- measurements / observations : $O(Z_T) = \mathcal{F}_\mu[Y] \in \mathbb{R}^a$, $a << d$

Goal: invert the (nonlinear) mapping $\mu \mapsto \Big(Y \mapsto \mathcal{F}_\mu[Y]\Big)$

Idea: construct a *discriminant family* $Y^k$, $k = 1, ..., K$ such that if $\mathcal{F}_\mu[Y^k] = \mathcal{F}_{\bar{\mu}}[Y^k]$ for all $k$ then $\mu = \bar{\mu}$. Similar perspective (quantum inversion) : Y. Maday and J. Salomon 2009; Leghtas, Rouchon, GT 2011 : local cx of $\mu \mapsto \sum_k (\mathcal{F}_\mu[Y^k] - \mathcal{F}_{\bar{\mu}}[Y^k])^2$

# Motivation: control and inversion

More generally want that if $\left(\mathcal{F}_\mu[Y^k] - \mathcal{F}_{\bar{\mu}}[Y^k]\right)_{k=1}^{K} \in \mathbb{R}^K$ is small then $\mathcal{F}_\mu[\cdot] - \mathcal{F}_{\bar{\mu}}[\cdot]$ is small is some operator norm.

Suppose we use the classical $L^2(E;\mathbb{R})$ norm:

$$\int_E |\mathcal{F}_\mu[Y] - \mathcal{F}_{\bar{\mu}}[Y]|^2 dY$$

Questions: what is that (integral over space of scenarios), how to compute, does this give hints for $Y^k$ ?

Remark: not far from "magic points" idea in reduced basis.

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# Taylor formula

Let $dX(t)/dt = a(t, X(t))$ with integral formulation $X_t = X_0 + \int_0^t a(s, X_s)ds$ and

$$\frac{df(X_t)}{dt} = a(t, X_t)\frac{\partial}{\partial x}f(X_t). \tag{1}$$

Then for $L = a\frac{\partial}{\partial x}$:

$$f(X_t) = f(X_0) + \int_0^t Lf(X_{s_1})ds_1. \tag{2}$$

# Taylor formula

Iterating :

$$f(X_t) = f(X_0) + \int_0^t L\Big\{ f(X_0) + \int_0^{s_1} Lf(X_{s_2}) ds_2 \Big\} ds_1 \quad (3)$$

$$= f(X_0) + Lf(X_0) \int_0^t ds_1 + L^2 f(X_0) \int_0^1 \int_0^{s_1} ds_2 ds_1 + ... (4)$$

thus

$$f(X_t) = f(X_0) + \int_0^t Lf(X_s) ds = ...$$

$$= f(X_0) + \sum_{k=1}^{n} \frac{t^k}{k!} L^k f(X_0) + \int_0^t \int_0^{s_1} ... \int_0^{s_n} L^{n+1} f(X_{s_{n+1}}) ds_{n+1} ... ds_1$$

# Ito-Taylor formula

For the SDE $dX_t = a(t, X_t)dt + b(t, X_t) \circ dW_t$:

$$f(X_t) = f(X_0) + \int_0^t a(X_s)\frac{\partial}{\partial x}f(X_s)ds + \int_0^t b\frac{\partial}{\partial x}f(X_s) \circ dW_s$$

Denote $L^0 = a(X_t)\frac{\partial}{\partial x}$ and $L^1 = b\frac{\partial}{\partial x}$ thus

$$f(X_t) = f(X_0) + \int_0^t L^0 f(X_{s_1})ds_1 + \int_0^t L^1 f(X_{s_1}) \circ dW_{s_1}$$

$$= f(X_0) + L^1 f(X_0)\int_0^t \circ dW_{s_1} + L^0 f(X_0)\int_0^t ds_1 +$$

$$L^1 L^1 f(X_0)\int_0^t \int_0^{s_1} \circ dW_{s_2} \circ dW_{s_1} + (\text{terms of order} \geq 3/2)$$

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# Cubature on Wiener space

Suppose want to compute $\mathbb{E}(O(X_T))$ (rq: this is related to a PDE solution). This is an integral over the Wiener space of all continuous functions (null in zero). Any quadrature (cubature) formulas ?

Remark: by Wong-Zakai thm. Stratonovich formulation is limit of Riemann-Stieltjes style integrals on (bounded variation) paths. Solutions of $dx_n(t) = a(t, x_n(t))dt + b(t, x_n(t))d\omega_n(t, \omega)$ converge to that of $dX_t = a(t, X_t)dt + b(t, X_t) \circ dW_t$ if curves $\omega_n(t, \omega)$ converge to $W_t = W(t, \omega)$
This is seen directly in the computation above because both give formally the same algebraic calculus.

# Cubature on Wiener space

Lyons-Victoir (Proc. R. Soc. Lond. 2004): use curves $\omega_k(t)$ that match all averages of order $\leq m$ : $\mathbb{E}(\int_0^1 \circ dW_{s_1})$, $\mathbb{E}(\int_0^1 ds_1)$, $\mathbb{E}(\int_0^1 \int_0^{s_1} \circ dW_{s_2} \circ dW_{s_1})$:

$$\sum_k \mu_k \int_0^1 d\omega_k(s_1) = \mathbb{E}(\int_0^1 \circ dW_{s_1}) = 0$$

$$\sum_k \mu_k \int_0^1 ds_1 = \mathbb{E}(\int_0^1 ds_1) = 1$$

$$\sum_k \mu_k \int_0^1 d\omega_k(s_1) \int_0^{s_1} d\omega_k(s_2) = \mathbb{E}(\int_0^1 \int_0^{s_1} \circ dW_{s_2} \circ dW_{s_1}) = 1/2$$

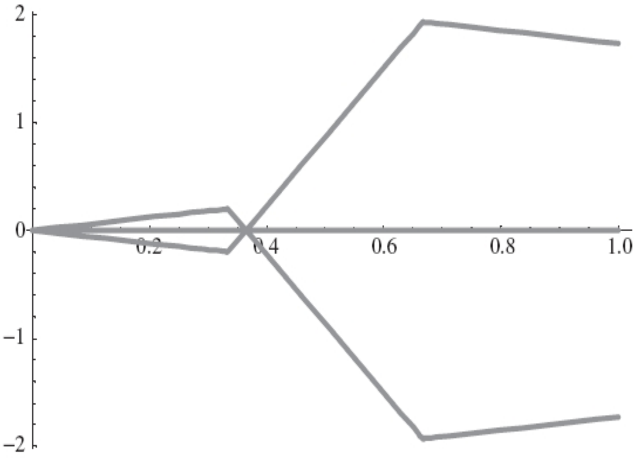


Figure: The 3 paths for a $m = 5$ order cubature paths for one Brownian motion (Lyons-Victoir 2004).

# Cubature on $C^k$

Question: suppose now we want to compute $\mathbb{E}(O(Z_T))$ with $dZ_t/dt = \mathcal{L}(t, Z_t, Y_t)$ (ODE, PDE, ...) which involves a $C^1$ path (scenario) $Y_t$ ($Y_0 = 0$). Applications: control and inversion in quantum physics, finance.

Integral over $\{f \in C^1([0,1]), f(0) = f'(0) = 0\}$. Cubature ?

• use Wiener space to generate the $C^1$ paths by setting $dY_t/dt = W_t$ with solution $Y_t = \int_0^t W_s ds$.
• compute the Ito-Taylor expansion: it always starts with $\int_0^t \int_0^s d\sigma ds$: can use Lyons-Victoir cubature of order $m$ but will have many useless moments (not used: $\mathbb{E}(\int_0^t \circ dW_{s_1})$, $\mathbb{E}(\int_0^t \int_0^{s_1} ds_2 \circ dW_{s_1})$, ... more than half) thus suboptimal.
Problem is worsened with increasing regularity of $Y_t$ ($C^k$ class).

# Cubature on $C^k$: conclusion

Conclusion:

• if in a hurry :-) can use Lyons-Victoir paths (integrate them once for $C^1$: $\int_0^t d\omega_k(s)ds$ or $k$ times for $C^k$) but will lose possibly $k+1$ orders in the Ito-Taylor formula ; non-intuitive behavior (lose precision) with increasing regularity ...

• otherwise use new cubature $C^1$ paths matching the "good" moments.

• Thm.: under hypothesis on $T$ (final time) and $\mathcal{F}$ there exist $Y^k$ $k = 1, ..., K$ such that $\mu \mapsto \left(\mathcal{F}_\mu[Y^k]\right)_k \in \mathbb{R}^K$ is injective (inversion has unique solution).

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# Conclusion and perspectives

- Brownian bridge and $H_0^1([a, b])$ functions
- interpolation based on the same idea
- relationship with quantization approaches...
- what if $\omega_k$ are imposed and only $\lambda_k$ are free ?
- ... and finally reduced basis (some related work: Litterer and Lyons), a priori n-width estimates