Recent advances in the use of PGD for the solution of engineering problems

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Challenges

• **Computational cost**
  – CPU power
  – Memory

• **Uncertainty in parameters**
  – Material parameters
  – Geometrical parameters
  – Process parameters

• **Availability and ease of use for « downstream » users**
  – Parameter optimization
  – Design in general

• ...
High dimensionality problems
Parametric models

Classical simulation: \[ \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = S \]

\[ T(t, x) \]

2D problem
Parametric models

Classical simulation:  \[
\frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = S
\]

\[T(t, x)\]  
2D problem

Challenging simulation:
Parametric models

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\[ T(t, x) \] 2D problem

Challenging simulation:
\[ \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = S \]

\[ T(t, x, k, S) \] 4D problem

Multidimensional solver: PGD
Parametric models

$T(t, x, k, S)$

Allows the \textit{a priori} solution of a single problem instead the many solutions required by usual strategies in: \textit{optimization, inverse problems, real-time simulation} & \textit{process control}.
Parametric models

\[ T(t, x, k, S) \]

**Offline step:**
Unique solution of a multidimensional parametric model.

**Online step:**
Evaluation of the solution

iPGD
Introducing separated representations

**MESH**

\[ u(x_1, \ldots, x_{DIM}) \sim N^{DIM} \]

**Separated representation**

\[ u(x_1, \ldots, x_{DIM}) \approx \sum_{j=1}^{n} F_{j,1}(x_1) \cdots F_{j,DIM}(x_{DIM}) \]

\[ \text{dof} = n \times N \times DIM \]

+ affine decomposition of the operators
Space-separated non-incremental representation

\[
u(x, y, z, t, p) = \sum_{j=1}^{n} F_{j,1}(x) \cdot F_{j,2}(y) \cdot F_{j,3}(z) \cdot F_{j,4}(t) \cdot F_{j,5}(p)
\]

\[
u(x, y, z, t, p) = \sum_{j=1}^{n} F_{j,1}(x, y) \cdot F_{j,2}(z) \cdot F_{j,3}(t) \cdot F_{j,4}(p)
\]

\[
u(x, y, z, t, p) = \sum_{j=1}^{n} F_{j,1}(x, y, z) \cdot F_{j,2}(t) \cdot F_{j,3}(p)
\]
\[ \mathcal{M}(u) = \mathcal{A} \Rightarrow \int_{\Omega_x \times \Omega_y} u^* (\mathcal{M}(u) - \mathcal{A}) \, dx \, dy = 0 \]

**Mode n+1?**

\[ u(x, y) = \sum_{j=1}^{n} X_j(x) \cdot Y_j(y) + R(x) \cdot S(y) \]

\[ u^*(x, y) = R^*(x) \cdot S(y) + R(x) \cdot S^*(y) \]

**R \iff S**

\[ \int_{\Omega_x \times \Omega_y} R \cdot S^* (\mathcal{M}(u(R, S, X_j, Y_j)) - \mathcal{A}) \, dx \, dy = 0 \]

**S \iff R**

\[ \int_{\Omega_x \times \Omega_y} R^* \cdot S (\mathcal{M}(u(R, S, X_j, Y_j)) - \mathcal{A}) \, dx \, dy = 0 \]

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\[ \mathcal{M}(u) = \mathcal{A} \Rightarrow \int_{\Omega_x \times \Omega_y} u^* \left( \mathcal{M}(u) - \mathcal{A} \right) \, dx \, dy = 0 \]

NO CONVERGENCE WHEN THE OPERATOR IS DOMINATED BY THE TIME DERIVATIVE

\[ c_t \rho_{bet} \frac{\partial u}{\partial t} = \text{div} \left[ \lambda_t \text{grad}(u) \right] + q_t \]

\( \rightarrow \) RESIDUAL MINIMIZATION

Applications

Some examples

Thin & extruded structures simulation

Microstructure & homogenization

Control & optimization

Mobile platforms & real time simulations (iPGD)
Thin & extruded geometries

3D Solution at 2D cost

Ply orientations as new coordinates

A posteriori evaluation
Thin & extruded geometries

Heat transfer in civil engineering
Thermo-elasticity of complex plates

« Intractable » simulations run on a simple laptop.
Microstructure simulation & homogenization

Complex microstructure

 Simulation for any thermal conductivity at each « pixel » of a small patch. Opens the door to recursive numerical homogenization.

Nonlinear & history dependent cases work as well.
Shape optimization

A priori simulation for any value of the design parameters

A posteriori optimization from the general solution.

Issue: design parameters & parametrization of the geometry
Off-line calculation

\[ u(x, y, \vartheta_1, \vartheta_2) \approx \sum_{i}^{N} F_{i}(x, y) \cdot \Theta_{i}^{1}(\vartheta_1) \cdot \Theta_{i}^{2}(\vartheta_2) \]

Online optimization

Determination of optimal values

Identification of the heater temperatures in a failure scenario

Reconfiguration
Real time simulations

\[ u(x, P, s) \approx \sum_{i}^{N} X_i(x) \cdot P_i(p) \cdot S_i(s) \]

700 Hz using matlab
Fast « time parallel » simulations

1D heat transfer problem

Introduction of the initial & boundary conditions on a coarse mesh as additional coordinates

Computation over a small time interval

\[ u(t, x, k, p^1, p^2, \ldots, p^{Nc}) = \sum_{i=1}^{N} T_i(t) \cdot X_i(x) \cdot K_i(k) \cdot P_i^1(p) \cdot P_i^2(p) \cdots P_i^{Nc}(p) \]

A posteriori

Stitching of time intervals
Energy methods for Cauchy problems of evolutions equations

\[ ((\tilde{u}_0)_i, (\tilde{u}'_0)_i) \rightarrow (u_N)_i? \]

\[ u(t, x, u^0(x), u_0, u_N) \xrightarrow{t = \Delta, x = 0} u'(u_N) \]

\[ u'(u_N) = \tilde{u}'_0 \rightarrow u_N \]
Inverse problems

(a) slope = 1  
(b) slope = 5
Inverse problems

(a) $k = 1 / \text{slope} = 5$

(b) $k = 1 / \text{slope} = 15$
Inverse problems

(a) $k = 0.5 / 0$

(b) $k = 1 / 0$
Conclusions & Perspectives

• The PGD is an efficient & promising tool for the solution of many engineering problem.

• The space-variable separation allows the solution of very big problems, even for complex domains

• The a priori computation / a posteriori evaluation allows the fast solution of complex problems on light computing platforms: real time simulation, optimization, control,…

• Optimizing the convergence of the method and reducing the number of modes are key issues for real applications and is still a lock for some applications.